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A CLASS OF POSITIVE LINEAR OPERATORS AND APPLICATIONS TO NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. We discuss the class of u_0-positive linear operators relative to two cones and use a comparison theorem for this class to give some short proofs of new fixed point index results for some nonlinear operators that arise from boundary value problems. In particular, for some types of boundary conditions, especially nonlocal ones, we obtain a new existence result for multiple positive solutions under conditions which depend solely on the positive eigenvalue of a linear operator. We also treat some problems where the nonlinearity $f(t,u)$ is singular at $u=0$.

1. Introduction

In a recent paper [35] we introduced a class of positive linear operators we called *u_0-positive relative to two cones $K_0 \subset K_1$*. This is a modification of a concept due to Krasnosel'skiĭ [12], [14]. The motivation was the example of linear operators that arise when studying nonlinear boundary value problems. Using the new concept enabled us to prove, in a simple manner, a new fixed point index result which has applications to nonlinear boundary value problems. In the present paper we continue this study.

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We extend from [35] a comparison result for the eigenvalues of these operators. We also give a new proof of a result on the existence of a positive eigenvalue with a positive eigenfunction under an hypothesis that is sharp. This result is known [5], but our proof is shorter than the previous one. We also prove uniqueness of the eigenvalue when the operator satisfies the u_0-positivity notion.

We then discuss the linear operators that arise in the space $C[0,1]$ when studying boundary value problems (BVPs) via the Hammerstein integral equation that involves the Green's function of the BVP. We discuss when the related linear operator L is u_0-positive relative to two cones. We show how the principal eigenvalue (the spectral radius $r(L)$) can be estimated from above and below and give examples to illustrate the simple use of the result.

Much of our work concentrates on the case when the Green's function $G(t,s)$ of the problem satisfies the strong positivity condition that there exist a non-negative measurable function Φ with $\Phi(s) > 0$ for almost all $s \in (0,1)$, and a constant $c_0 > 0$ such that

$$c_0\Phi(s) \leq G(t,s) \leq \Phi(s), \quad \text{for } 0 \leq t,s \leq 1.$$

This is satisfied by many BVPs, for example, the second order problem

$$u''(t) + f(t,u(t)) = 0, \quad t \in (0,1),$$

with boundary conditions (BCs)

$$\alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0,$$

when $\alpha\gamma + \alpha\delta + \beta\gamma > 0$ provided that $\beta > 0$, $\delta > 0$. The condition does not hold if one BC is $u(0) = 0$ or $u(1) = 0$ but the condition is also satisfied by some nonlocal BCs, for example, the 'three-point' BC $u'(0) = 0$, $u(1) = \beta u(\eta)$, with $0 < \beta < 1$ or the problem with BCs $u(0) = \alpha u(\xi)$, $u(1) = \beta u(\eta)$ (with $\alpha > 0$, $\beta > 0$ and suitably bounded above). The condition can also be satisfied by higher order problems. Some other second order equations and BCs are discussed in [8], [9], [16].

Under this strong positivity condition on G, with a very simple proof, using the u_0-positive concept, we prove some new fixed point index results. These are then used to prove a new result on the existence of multiple positive solutions when the nonlinearity has multiple 'crossings of the eigenvalue'. This is apparently the first time that results on existence of multiple solutions that only use 'crossing of the eigenvalue' have been given. This result applies to many BVPs of local and non local type. In particular, we give simple examples that fit the new theory but cannot be handled by previous theory that uses the tool of fixed point index or equivalent methods.

When G satisfies the strong positivity condition, we also treat singular problems, that is problems where the nonlinearity $f(t,u)$ is singular in u at $u = 0$.

We obtain a new result on existence of multiple positive solutions with a rather simple proof.

2. Eigenvalues of positive linear operators

A subset K of a Banach space X is called a cone if K is closed and $x, y \in K$ and $\alpha \geq 0$ imply that $x + y \in K$ and $\alpha x \in K$, and $K \cap (-K) = \{0\}$. We always suppose that $K \neq \{0\}$. A cone defines a partial order by $x \preceq_K y \Leftrightarrow y - x \in K$. A cone is said to be *reproducing* if $X = K - K$ and to be *total* if $X = \overline{K - K}$.

In the space $C[0,1]$ of real-valued continuous functions on $[0,1]$, endowed with the usual supremum norm, $\|u\| := \sup\{|u(t)| : t \in [0,1]\}$, the cone of non-negative functions $P := \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}$ is well known (and easily shown) to be reproducing.

A useful concept due to Krasnosel'skiĭ, [12], [14] is that of a u_0-positive linear operator on a cone. A bounded linear operator $L\colon X \to X$ is said to be *u_0-positive on a cone K*, [11], [14], if there exists $u_0 \in K \setminus \{0\}$, such that for every $u \in K \setminus \{0\}$ there are constants $k_2(u) \geq k_1(u) > 0$ such that

$$k_1(u)u_0 \preceq_K Lu \preceq_K k_2(u)u_0. \tag{2.1}$$

In a recent paper [35] we have given a modification of this definition. We suppose that we have two cones in X, $K_0 \subset K_1$ and we let $\preceq$ denote the partial order defined by the larger cone K_1, that is, $x \preceq y \Leftrightarrow y - x \in K_1$. We say that L is *positive* if $L(K_1) \subset K_1$,

Our modified definition reads as follows.

Definition 2.1. We say that a positive bounded linear operator $L\colon X \to X$ is *u_0-positive relative to the cones* (K_0, K_1), if there exists $u_0 \in K_1 \setminus \{0\}$, such that for every $u \in K_0 \setminus \{0\}$ there are constants $k_2(u) \geq k_1(u) > 0$ such that

$$k_1(u)u_0 \preceq Lu \preceq k_2(u)u_0.$$

This is stronger than requiring that L is positive and is clearly satisfied if L is u_0-positive on K_1 according to (2.1).

The idea behind our modified definition is that we wish to exploit the extra properties satisfied by elements of K_0 but only use the weaker K_1-ordering. Two cones are used, with the same motivation, in some work of J. Mallet–Paret and R.D. Nussbaum [20], [21], where there is a deep discussion of some spectral properties of nonlinear maps that are homogeneous of degree one, which, of course, includes linear operators.

In the recent paper [35] we proved the following comparison theorem which is similar to one of M.S. Keener and C.C. Travis [11], which was itself a sharpening of some results of Krasnosel'skiĭ [12, § 2.5.5]. Some applications of the

M.S. Keener and C.C. Travis theorem to some nonlinear problems were given in [33], [34].

Theorem 2.2 ([35]). *Let $K_0 \subset K_1$ be cones in a Banach space X, and let $\preceq$ denote the partial order of K_1. Suppose that L_1, L_2 are bounded linear operators and that at least one is u_0-positive relative to (K_0, K_1). If there exist*

$$\begin{aligned} &u_1 \in K_0 \setminus \{0\}, \quad \lambda_1 > 0, \quad \textit{such that } \lambda_1 u_1 \preceq L_1 u_1, \\ \textit{and} \quad &u_2 \in K_0 \setminus \{0\}, \quad \lambda_2 > 0, \quad \textit{such that } \lambda_2 u_2 \succeq L_2 u_2, \end{aligned} \tag{2.2}$$

and $L_1 u_j \preceq L_2 u_j$ for $j = 1, 2$, then $\lambda_1 \le \lambda_2$. If, in addition, $L_j(K_1 \setminus \{0\}) \subset K_0 \setminus \{0\}$ and if $\lambda_1 = \lambda_2$ in (2.2), then it follows that u_1 is a (positive) scalar multiple of u_2.

This is typically applied when one of u_j is an eigenfunction of L_j and λ_j is the corresponding eigenvalue.

We begin by proving an extension of this result. We first observe that if a linear operator L is u_0-positive relative to (K_0, K_1) then, with an extra condition, all positive powers of L are also u_0-positive relative to (K_0, K_1). For example, if also $u_0 \in K_0$, and for $u \in K_0 \setminus \{0\}$, we have $k_1(u)u_0 \preceq Lu \preceq k_2(u)u_0$, then $k_1(u_0)u_0 \preceq Lu_0 \preceq k_2(u_0)u_0$. Thus, as $L\colon K_1 \to K_1$ we have

$$k_1(u)\, k_1(u_0)u_0 \preceq L(k_1(u)u_0) \preceq L(L(u)) \preceq L(k_2(u)u_0) = k_2(u)\, k_2(u_0)u_0.$$

This proves that L^2 is u_0-positive relative to (K_0, K_1) and this argument extends to all positive powers.

Alternatively, if L maps $K_0 \setminus \{0\}$ into $K_0 \setminus \{0\}$ then for $u \in K_0 \setminus \{0\}$, we also have $Lu \in K_0 \setminus \{0\}$ and L^2 is then u_0-positive by the argument

$$k_1(Lu)u_0 \preceq L(Lu) \preceq k_2(Lu)u_0.$$

The argument extends to all larger positive powers.

Theorem 2.3. *Let $K_0 \subset K_1$ be cones in a Banach space X, and let $\preceq$ denote the partial order of K_1. Suppose that L_1, L_2 are bounded linear operators and for some $p \in \mathbb{N}$ and some $q \in \mathbb{N}$ at least one of L_1^p, L_2^q is u_0-positive relative to (K_0, K_1), if $p \ne q$ suppose also that $u_0 \in K_0$. Let $\widetilde{p} := \max\{p, q\}$. If there exist*

$$\begin{aligned} &u_1 \in K_0 \setminus \{0\}, \quad \lambda_1 > 0, \quad \textit{such that } \lambda_1 u_1 \preceq L_1 u_1 \\ \textit{and} \quad &u_2 \in K_0 \setminus \{0\}, \quad \lambda_2 > 0, \quad \textit{such that } \lambda_2 u_2 \succeq L_2 u_2, \end{aligned} \tag{2.3}$$

and $L_1^{\widetilde{p}} u_j \preceq L_2^{\widetilde{p}} u_j$ for $j = 1, 2$, then $\lambda_1 \le \lambda_2$. If, in addition, $L_j^{\widetilde{p}}(K_1 \setminus \{0\}) \subset K_0 \setminus \{0\}$ and if $\lambda_1 = \lambda_2$ in (2.3), then it follows that u_1 is a (positive) scalar multiple of u_2.

Proof. We consider the case $p \ne q$, the case $p = q$ is similar. By the argument preceding the statement of the theorem, one of $L_j^{\widetilde{p}}$ is u_0-positive relative to (K_0, K_1). Also it follows that

$$\lambda_1^{\widetilde{p}} u_1 \preceq L_1^{\widetilde{p}} u_1 \quad \text{and} \quad \lambda_2^{\widetilde{p}} u_2 \succeq L_2^{\widetilde{p}} u_2.$$

By Theorem 2.2, $\lambda_1^{\widetilde{p}} \leq \lambda_2^{\widetilde{p}}$, thus $\lambda_1 \leq \lambda_2$. The last part also follows immediately from the previous result. $\square$

We now give a new proof of the existence of a positive eigenvalue. The results are closely related to a result of Krasnosel'skiĭ, Theorem 2.5 of [12], and of D.E. Edmunds, A.J.B. Potter and C.A. Stuart, Theorem 3 of [5]. In [12] it is assumed that L is compact, in [5] it is assumed that L is a k-set contraction, and the Schauder fixed point theorem, and its analogue for k-set contractions, for nonlinear operators is used in their proofs. We use the concept of a condensing operator, which is essentially the same assumption as made in [5]. For a linear operator L it is known, Lemma 6 of [24], (or see Theorem 9.11 of [4]) that L is condensing if and only if it can be written $L = L_1 + L_2$ where L_1 has finite rank (hence compact) and $r_{\text{ess}}(L) \leq r(L_2) \leq \|L_2\|_e < 1$ where $\|\cdot\|_e$ is an equivalent norm on X and $r_{\text{ess}}(L)$ denotes the essential spectral radius of L. Thus, when an equivalent norm is employed, saying L is condensing is equivalent to saying $r_{\text{ess}}(L) < 1$. Although there are several inequivalent definitions of 'essential spectrum', it was shown in [24] that the radius is the same whatever definition is employed.

We use the condensing notion so that there is a well defined fixed point index theory, see [4], [35], the result applies whatever measure of non-compactness is used in the definition of condensing. Note that measures of non-compactness need not be equivalent, see the paper [22]. The important special case is when the map is compact. The papers [5] and [12] use fewer tools, but our new proof is short, and, when we impose a u_0-positivity assumption, we also get uniqueness.

Theorem 2.4. *Let K_1 be a cone in a Banach space X, and let $\preceq$ denote the partial order of K_1. Suppose that L is a bounded linear operator and maps K_1 into K_1. Let there exist $\lambda_0 > 0$ and $v \in X$ such that $Lv \succeq \lambda_0 v$ where $-v \notin K_1$ and v has the form $v = v_1 - v_2$ where $v_1, v_2 \in K_1$. Then, if $\frac{1}{\lambda_0} L \colon X \to X$ is condensing, there exist $\lambda \geq \lambda_0$ and $\varphi \in K_1 \setminus \{0\}$ such that $L\varphi = \lambda\varphi$.*

Furthermore, if K_0 is a cone in X with $K_0 \subset K_1$, and if L maps $K_1 \setminus \{0\}$ into $K_0 \setminus \{0\}$ and is u_0-positive relative to (K_0, K_1), then λ is the unique positive eigenvalue with an eigenfunction φ in K_0. Moreover, any other eigenfunction in K_0 with eigenvalue λ is a scalar multiple of φ.

Note that v is as required if $v \in K_1 \setminus \{0\}$.

Proof. We observe that $v_1 \neq 0$ and $v_1 \succeq v$. Let B_1 be the open unit ball of X and let ∂B_1 be its boundary. We first show that

$$u \neq \frac{1}{\lambda_0} Lu + \sigma v_1 \quad \text{for all } u \in \partial B_1 \cap K_1 \text{ and all } \sigma \geq 0. \tag{2.4}$$

In fact, if not, then there exist $\sigma \geq 0$ and $u \in \partial B_1 \cap K_1$ such that

$$u = \frac{1}{\lambda_0} Lu + \sigma v_1. \tag{2.5}$$

We may suppose that $\sigma > 0$ for otherwise u is the required eigenfunction with eigenvalue λ_0. From (2.5) we have $u \succeq \sigma v_1$ so that

$$Lu \succeq \sigma L v_1 \succeq \sigma L v \succeq \sigma \lambda_0 v,$$

and so, from (2.5) again, $u \succeq \sigma v + \sigma v_1$. Then $Lu \succeq \sigma Lv + \sigma L v_1 \succeq 2\sigma\lambda_0 v$ and hence, using (2.5), we obtain $u \succeq 2\sigma v + \sigma v_1$. Repeating this argument shows that $u \succeq n\sigma v + \sigma v_1$ for every $n \in \mathbb{N}$, and thus $\frac{1}{n\sigma} u - \frac{1}{n} v_1 - v \in K_1$. Letting $n \to \infty$ gives the contradiction $-v \in K_1$. This shows that (2.4) holds. Thus $i_{K_1}(\frac{1}{\lambda_0} L, B_1 \cap K_1) = 0$, by standard properties of fixed point index, see for example, any of [1], [4], [7], [35].

Now, suppose that $Lu \neq \lambda u$ for all $\lambda \geq \lambda_0$ and for all $u \in K_1 \setminus \{0\}$. Then $\frac{1}{\lambda_0} Lu \neq \lambda u$ for all $\lambda \geq 1$ and for all $u \in \partial B_1 \cap K_1$. Standard properties of fixed point index then imply that $i_{K_1}(\frac{1}{\lambda_0} L, B_1 \cap K_1) = 1$. This contradicts $i_{K_1}(\frac{1}{\lambda_0} L, B_1 \cap K_1) = 0$ and we conclude that $Lu = \lambda u$ for some $\lambda \geq \lambda_0$ and some $u \in \partial B_1 \cap K_1$; let $\varphi = u$.

If L is u_0-positive relative to (K_0, K_1) and λ_1, λ_2 are positive eigenvalues with eigenfunctions $\varphi_1, \varphi_2 \in K_1 \setminus \{0\}$ then

$$\lambda_1 \varphi_1 = L\varphi_1 \quad \text{and} \quad \lambda_2 \varphi_2 = L\varphi_2,$$

hence $\varphi_1, \varphi_2 \in K_0$ and then the comparison theorem, Theorem (2.2), gives $\lambda_1 = \lambda_2$ and φ_2 is a scalar multiple of φ_1. □

Remark 2.5. (a) If K_1 is a total cone, L is compact and $r(L) > 0$, (or if L is bounded and $r_{\mathrm{ess}}(L) < r(L)$), then $r(L)$ is an eigenvalue of L with an eigenvector in K_1 by the Krein–Rutman theorem, (or some results of Nussbaum [27]; see also [5], [21]). The unique eigenvalue λ whose existence is asserted in Theorem 2.4 then equals $r(L)$.

(b) The proof applies whenever $\frac{1}{\lambda_0} L$ belongs to a class of mappings for which there is a fixed point index theory satisfying the standard properties, for example the P-compact operators of W.V. Petryshyn, see for example [3], [30], [40]. The result for P-compact operators was previously proved in [5].

Our hypothesis on $(1/\lambda_0)L$ means that $\lambda_0 > r_{\mathrm{ess}}(L)$. In [5] an example is given to show that the result can fail if L is a k-set contraction and $\lambda_0 \leq k$, so the condition is sharp. The example is in the space ℓ^2 and the cone is the natural one, which is reproducing. By Nussbaum's result [27], the real reason for the failure is that $r(L) = r_{\mathrm{ess}}(L) = k$ in that example.

We give another simpler example.

Example 2.6. Let $X = C[0,1]$ and let $P := \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}$, be the usual cone of non-negative functions. Let $k > 0$ and $\varepsilon > 0$ and define a bounded linear operator by $Lu(t) = (k + \varepsilon kt)u(t)$. Let $\widehat{1}$ denote the constant function with value 1. Then we have $L\widehat{1}(t) = (k + \varepsilon kt)\widehat{1}(t) \geq \lambda_0 \widehat{1}(t)$ where $\lambda_0 = k$, but, as is easily checked, L has no positive eigenvalue with an eigenfunction in $P \setminus \{0\}$. In this case $\frac{1}{\lambda_0} Lu(t) = (1 + \varepsilon t)u(t)$ is not condensing. Since ε can be arbitrarily small this shows that the hypothesis is sharp.

There is one result, which is known and has probably been rediscovered many times, but we do not know the original source, which has a similar appearance and requires no condensing or u_0-positivity hypotheses on L and no restriction on K. For completeness we give the simple proof.

Theorem 2.7. *Let L be a bounded linear operator in a Banach space X and let K be a cone in X. Suppose that $L(K) \subset K$ and there exist $\lambda_0 > 0$ and $v \in K \setminus \{0\}$ such that $Lv \succeq_K \lambda_0 v$. Then it follows that $r(L) \geq \lambda_0$.*

Proof. If not, we have $0 \leq r(L) < \lambda_0$. Hence L/λ_0 maps K into K and $r(L/\lambda_0) < 1$. As is well known, from the Neumann series, $(I - L/\lambda_0)^{-1}$ then maps K into K. We have $v \preceq_K L(v/\lambda_0)$ that is $(I - L/\lambda_0)v \preceq_K 0$, hence $v \preceq_K 0$ so that $v = 0$. This contradiction shows that $r(L) \geq \lambda_0$. □

Remark 2.8. In a personal communication, Professor R.D. Nussbaum remarked to this author that the argument actually gives the somewhat more precise statement that $r_K(L)$, the 'cone spectral radius' of $L : K \to K$ satisfies $r_K(L) \geq \lambda_0$. We refer to the papers [21], [28], [27], [29] for more information and some deep results concerning the 'cone spectral radius'.

Theorem 2.7 does not prove that L has an eigenvalue $\lambda \geq \lambda_0$ with eigenfunction in K, see Example 2.6, but if K is a total cone and $r_{\mathrm{ess}}(L) < r(L)$, then, by the extension of the Krein–Rutman theorem due to R.D. Nussbaum, [21], [27], $r(L)$ is an eigenvalue of L.

As may have been anticipated there is a similar result to Theorem 2.4 where we assume conditions on a power of L, L^p for some positive integer p. This could be applicable when the previous result is not, since, for example, there exist linear operators which are not compact but whose square is compact. For the original closely related results see [12] Theorem 2.5 for compact operators, and [5] Theorem 3 for k-set contractive operators and Theorem 4 of [5] for P-compact operators. Our proof is much shorter but we require a u_0-positivity condition.

THEOREM 2.9. *Let $K_0 \subset K_1$ be cones in a Banach space X, and let $\preceq$ denote the partial order of K_1. Let $p \in \mathbb{N}$, and let L be a bounded linear operator such that L^p is u_0-positive relative to (K_0, K_1). Suppose also that $L^p(K_1 \setminus \{0\}) \subset K_0 \setminus \{0\}$ and that there exist $\lambda_0 > 0$ and $v \in X$ such that $L^p v \succeq \lambda_0 v$ where $-v \notin K_1$ and v has the form $v = v_1 - v_2$ where $v_j \in K_1$. Suppose that $\frac{1}{\lambda_0} L^p\colon X \to X$ is condensing. Then there exist a unique $\lambda \geq \lambda_0^{1/p}$ and $u \in K_0 \setminus \{0\}$ such that $Lu = \lambda u$.*

PROOF. By Theorem 2.4 there exist a unique $\nu \geq \lambda_0$ and $u \in K_0 \setminus \{0\}$ such that $L^p u = \nu u$. Then we have

$$L^p(Lu) = L(L^p u) = \nu(Lu),$$

that is, Lu is also an eigenvector of L^p in K_0 with eigenvalue $\nu > 0$. By Theorem 2.2 Lu is a positive scalar multiple of u, that is, $Lu = \lambda u$ and we have $\lambda^p = \nu \geq \lambda_0$. $\square$

3. Some u_0-positive operators

We now investigate a situation that occurs frequently in the study of boundary value problems (BVPs) for ordinary differential equations, such as, for example,

$$u''(t) + g(t)f(t, u(t)) = 0 \quad \text{or} \quad u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1),$$

or more complicated ones, with various kinds of boundary conditions (BCs) of local or nonlocal type, see for example, [38], [39].

Studying positive solutions of a BVP can be done by finding fixed points, in a suitable cone, of the nonlinear integral operator

$$Nu(t) = \int_0^1 G(t, s)g(s)f(s, u(s))\, ds, \tag{3.1}$$

where the kernel G is the Green's function for the problem. Under mild conditions this defines a compact map N in the space $C[0, 1]$ and, when $G \geq 0$, $g \geq 0$ and $f \geq 0$, the theory of fixed point index in a cone of non-negative functions can be applied to N.

The rather weak conditions that we now impose on G, f, g are similar to ones in the papers [36], [37], [38].

(C_1) The kernel $G \geq 0$ is measurable, and for every $\tau \in [0, 1]$ we have

$$\lim_{t \to \tau} |G(t, s) - G(\tau, s)| = 0 \quad \text{for almost every (a.e.) } s \in [0, 1].$$

(C_2) Suppose that there exist a non-negative measurable function Φ with $\Phi(s) > 0$ for almost every $s \in (0, 1)$ and $c \in P \setminus \{0\}$ such that

$$c(t)\Phi(s) \leq G(t, s) \leq \Phi(s), \quad \text{for } 0 \leq t, s \leq 1. \tag{3.2}$$

For a subinterval $J = [t_0, t_1]$ of $[0, 1]$ let $c_J := \min\{c(t) : t \in J\}$; since $c \in P \setminus \{0\}$, there exists some interval J with $c_J > 0$.

(C_3) The function g is non-negative, $g(s) > 0$ for almost every $s \in (0, 1)$, and satisfies $g\,\Phi \in L^1[0, 1]$.

(C_4) The nonlinearity $f\colon [0, 1] \times [0, \infty) \to [0, \infty)$ satisfies Carathéodory conditions, that is, $f(\,\cdot\,, u)$ is measurable for each fixed $u \geq 0$ and $f(t, \,\cdot\,)$ is continuous for almost every $t \in [0, 1]$ and for each $r > 0$, there exists $\phi_r \in L^\infty[0, 1]$ such that

$$f(t, u) \leq \phi_r(t) \quad \text{for all } u \in [0, r] \text{ and a.e. } t \in [0, 1].$$

Clearly, (C_1) is satisfied if G is continuous. A precursor of condition (C_2) was used in [19]. The function g allows possible pointwise singularities in the nonlinearity at arbitrary points of $[0, 1]$ but it is then convenient to regard g as part of the kernel of the integral operator. The condition (C_2) is frequently satisfied by ordinary differential equations with both local and nonlocal boundary conditions, see, for example, [38] for a quite general situation.

Some fixed point index results, which can be used to prove existence of positive fixed points of N, have been obtained by using the linear operator

$$Lu(t) := \int_0^1 G(t, s)g(s)u(s)\, ds, \tag{3.3}$$

and comparing the behaviour of $\frac{f(t,u)}{u}$ for u near 0 and near ∞ with the principal characteristic value $\mu(L) := 1/r(L)$ of L. It is known that N, L are compact under the above conditions, see, for example, [23] Proposition V.3.1. An example of an existence theorem is: there is at least one positive solution if "the nonlinearity crosses the eigenvalue", that is,

$$\begin{aligned} \text{either} \quad \limsup_{u \to 0+} \frac{f(u)}{u} < \mu(L) \quad &\text{and} \quad \liminf_{u \to \infty} \frac{f(u)}{u} > \mu(L), \\ \text{or} \quad \liminf_{u \to 0+} \frac{f(u)}{u} > \mu(L) \quad &\text{and} \quad \limsup_{u \to \infty} \frac{f(u)}{u} < \mu(L). \end{aligned}$$

Here, for simplicity of exposition, we supposed that f does not depend explicitly on t. For separated boundary conditions see, for example, [6], for some multipoint problems see [41], [36] and for some quite general situations see [37], [38].

Let $P := \{u \in C[0, 1] : u(t) \geq 0\}$ be the standard cone of non-negative continuous functions and let $\preceq$ denote the ordering induced by P.

For an interval $J := [t_0, t_1] \subset [0, 1]$ with $c_J > 0$ we consider the following cones

$$K_c := \{u \in P : u(t) \geq c(t)\|u\|, \text{ for all } t \in [0, 1]\},\\ K_J := \{u \in P : u(t) \geq c_J\|u\|, \text{ for all } t \in J\}. \tag{3.4}$$

These cones, especially the second, have been studied by many authors in the study of existence of multiple positive solutions of boundary value problems. For the first cone we mention [2], [17], [18], for the second see, for example, [7], [36]–[38].

Proposition 3.1. *For every sub-interval $J = [t_0, t_1]$ of $[0, 1]$ with $c_J > 0$, we have $L: P \to K_c \subset K_J$.*

This is essentially known but we give the simple proof for completeness.

Proof. For $u \in P$,

$$\|Lu\| = \max_{t\in[0,1]} Lu(t) \leq \int_0^1 \Phi(s)g(s)u(s)\,ds,$$
$$\text{and} \quad Lu(t) \geq \int_0^1 c(t)\Phi(s)g(s)u(s)\,ds,$$

which proves $Lu(t) \geq c(t)\|Lu\|$. It is easily shown that $K_c \subset K_J$. □

Some sufficient conditions for L to be u_0-positive on P have been given in [12], and in [33], [38] where hypotheses are given which always hold for second order equations with separated BCs. One result is often applicable.

Theorem 3.2 ([38, Corollary 7.5]). *Suppose that G satisfies* (C_1)–(C_3) *with $c(t) > 0$ for $t \in (0, 1)$ and suppose that Φ continuous and $g \in L^1$. Let at least one of the symmetry properties* (a) *or* (b) *that follow be satisfied.*

(a) *$G(t, s) = G(s, t)$, for all t, $s \in [0, 1]$,*
(b) *$G(t, s) = G(1 - s, 1 - t)$, for all t, $s \in [0, 1]$.*

Then L is u_0-positive on P.

It is not clear whether L can be shown to be u_0-positive solely under the conditions (C_1)–(C_3). However, there is a closely related operator defined by

$$L_J u(t) := \int_{t_0}^{t_1} G(t, s)g(s)u(s)\,ds, \tag{3.5}$$

where $J = [t_0, t_1]$ is any sub-interval of $[0, 1]$ for which $c_J > 0$, which played a useful role in [36]. Clearly $L_J u \preceq Lu$ for every $u \in P$, that is, L_J is a minorant of L. Also L_J is compact. In the recent paper [35] we showed that (C_1)–(C_3) imply that L_J is u_0-positive relative to (K_J, P).

This result and some of its consequences were the motivation for our definition of u_0-positive operator relative to two cones. We prove a small extension of that result.

THEOREM 3.3. *Let $J = [t_0, t_1]$ be a sub-interval of $[0,1]$ for which $c_J > 0$ and let L_J be defined on $C[0,1]$ by (3.5). Suppose that* (C_1)–(C_3) *hold. Then L_J is u_0-positive relative to (K_c, P) for $u_0(t) := \int_{t_0}^{t_1} G(t,s)g(s)\,ds$.*

The proof is practically the same as that in [35], for completeness we give the details here.

PROOF. Let $u \in K_c \setminus \{0\}$. Then also $u \in K_J$ and we have

$$L_J u(t) = \int_{t_0}^{t_1} G(t,s)g(s)u(s)\,ds \leq \bigg(\int_{t_0}^{t_1} G(t,s)g(s)\,ds\bigg)\|u\|,$$

and

$$L_J u(t) = \int_{t_0}^{t_1} G(t,s)g(s)u(s)\,ds \geq \bigg(\int_{t_0}^{t_1} G(t,s)g(s)\,ds\bigg)c_J\|u\|.$$

We note that, for $t \in J$, $u_0(t) \geq \int_{t_0}^{t_1} c_J\Phi(s)g(s)\,ds > 0$, so $u_0 \neq 0$. Also (C_1)–(C_3) imply that u_0 is continuous. $\square$

The advantage of this result over the one in [35] is that we can vary J but always use the fixed cone K_c.

In the following theorem we shall use a result of R.D. Nussbaum, Lemma 2 on page 226 of [28], which says that if L_n are compact linear operators and $L_n \to L$ in the operator norm then $r(L_n) \to r(L)$.

THEOREM 3.4. *Let* (C_1)–(C_3) *be satisfied with $c(t) > 0$ for $t \in (0,1)$, and let L be defined by* (3.3).

(a) *Suppose there exist $u_1 \in P \setminus \{0\}$ and $\lambda_1 > 0$ such that $\lambda_1 u_1 \preceq Lu_1$ then $r(L) \geq \lambda_1$.*
(b) *Suppose there exist $u_2 \in P \setminus \{0\}$ and $\lambda_2 > 0$ such that $\lambda_2 u_2 \succeq Lu_2$ then $r(L) \leq \lambda_2$.*

PROOF. (a) This is Theorem 2.7, and is also a consequence of Theorem 2.4.

(b) Since $L\colon P \to K_c$, by replacing u_2 by Lu_2 we can, and do, assume that $u_2 \in K_c \setminus \{0\}$.

Let $J = [t_0, t_1]$ be an arbitrary subset of $(0,1)$ and let L_J be defined by (3.5). Then we have

$$L_J c(t) = \int_{t_0}^{t_1} G(t,s)g(s)c(s)\,ds \geq c(t)\int_{t_0}^{t_1} \Phi(s)g(s)c(s)\,ds.$$

This shows that $L_J\, c \succeq \big(\int_{t_0}^{t_1} \Phi(s)g(s)c(s)\,ds\big)\, c$, hence by Theorem 2.4 there exists an eigenvalue λ_J of L_J with $\lambda_J \geq \big(\int_{t_0}^{t_1} \Phi(s)g(s)c(s)\,ds\big) > 0$. Therefore, by the Krein–Rutman theorem, $r(L_J)$ is an eigenvalue of L_J with an eigenfunction $\varphi_J \in P$, hence also $\varphi_J \in K_c$ since $L_J\colon P \to K_c$.

Now we have $L_J u \preceq Lu$, $r(L_J)\varphi_J = L_J\varphi_J$, and $\lambda_2 u_2 \succeq Lu_2$. By Theorem 3.3 we may apply the comparison theorem to obtain $r(L_J) \leq \lambda_2$. We now let $t_0 \to 0+$, $t_1 \to 1-$ and apply the result of Nussbaum to deduce that $r(L) \leq \lambda_2$.$\square$

We give two simple examples of Theorem 3.4, one with a singular term.

EXAMPLE 3.5. Consider the second order BVP

$$u''(t) + g(t)f(u(t)) = 0, \quad u(0) = 0, \quad u(1) = 0.$$

It is well-known that the Green's function is given by

$$G(t,s) := \begin{cases} s(1-t) & \text{if } s \leq t, \\ t(1-s) & \text{if } s > t. \end{cases}$$

First, take $g(t) \equiv 1$. It is well-known that the principal characteristic value of L is $\mu = \pi^2 \approx 9.8696$. Let $v(t) = t(1-t)$. By a direct computation we find constants $\lambda_1 = 1/12$, $\lambda_2 = 5/48$ such that $\lambda_1 v(t) \leq Lv(t) \leq \lambda_2 v(t)$ so we get both upper and lower bounds and these give

$$9.6 \leq \mu \leq 12.$$

We can now take $v_2(t) = Lv(t) = t/12 - t^3/6 + t^4/12$, using Maple we find upper and lower bounds which give (the numbers are rounded to 3 decimal places if not exact)

$$9.836 \leq \mu \leq 10.$$

Similarly, taking $v_3 = Lv_2 = t/120 - t^3/72 + t^5/120 - t^6/360$, we obtain the reasonable approximations

$$9.865 \leq \mu \leq 9.883.$$

If we begin with the 'lucky' choice of $v(t) = \sin(\pi t)$ then, of course, we find immediately that $\mu = \pi^2$.

Secondly, consider a weakly singular case with $g(t) = 1/\sqrt{t(1-t)}$. In this case, as far as I am aware, μ is not well-known. Again taking $v(t) = t(1-t)$, by a computation using Maple, we find both upper and lower bounds (the numbers are rounded to 3 decimal places)

$$4.424 \leq \mu \leq 5.093.$$

The method of the previous example fails, Lv is not simple enough for continuing integration, and there does not appear to be any other simple choice of v to improve these estimates. For better accuracy some numerical method would be superior, usually one needs a method that can handle the singularity in g. For nonsingular problems I use a desktop pc program written in C by my colleague Prof. K.A. Lindsay. In this particular example, it is well known and easy to show that $c(t)s(1-s) \leq G(t,s) \leq s(1-s)$ for $c(t) = \min\{t, 1-t\}$. Therefore the kernel of the linear integral operator $G(t,s)g(s)$ does not in fact have singularities. The numerical program gives $\mu \approx 4.506$.

Obviously it is optimal to use the exact value of μ or a good numerical approximation, but, if μ cannot be easily calculated, then upper and lower estimates of the above type may be useful.

4. A new multiple 'eigenvalue crossing' existence result

The theory of fixed point index has been used to prove existence of multiple positive solutions for the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds,$$

when G, g, f are (at least) non-negative. Conditions are known which give an arbitrary finite number of positive solutions under suitable conditions on f, see for example [15], [37], [38]. We intend to prove that, under a more restrictive version of the set-up of Section 3, there is a multiple existence result involving $\mu(L)$ alone, with the nonlinearity 'crossing the eigenvalue' many times.

The stronger positivity requirement on G is the following condition.

$(\mathrm{C}_2)_0$ Suppose that there exist a non-negative measurable function Φ with $\Phi(s) > 0$ for almost every $s \in (0,1)$, and a constant $0 < c_0 \le 1$ such that

$$c_0\Phi(s) \le G(t,s) \le \Phi(s), \quad \text{for } 0 \le t,s \le 1. \tag{4.1}$$

This condition is satisfied by many problems. For example, for second order equations of the form

$$-u''(t) = g(t)f(t,u(t)), \tag{4.2}$$

with separated boundary conditions

$$\alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0,$$

when $\alpha\gamma + \alpha\delta + \beta\gamma > 0$ and additionally $\beta > 0$, $\delta > 0$, the condition holds. Some other second order equations with periodic or Neumann BCs can also satisfy this condition, see [16]. It is not satisfied by (4.2) when one of the BCs is $u(0) = 0$ or $u(1) = 0$ but can, for example, be satisfied by nonlocal BCs of the form $u(0) = \beta_1[u], u(1) = \beta_2[u]$, where β_j are positive linear functionals on $C[0,1]$. It can also be satisfied by equations of higher order. We shall give some simple specific examples later in the paper.

We now assume that (C_1), $(\mathrm{C}_2)_0$, (C_3) and (C_4) hold. When $(\mathrm{C}_2)_0$ holds the relevant cones are

$$\begin{aligned} P &:= \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}, \\ K_{c_0} &:= \{u \in P : u(t) \geq c_0\|u\|,\ t \in [0,1]\}. \end{aligned} \tag{4.3}$$

Clearly all non-negative constant functions belong to K_{c_0}. An important point is that nonzero functions in K_{c_0} are positive on $[0,1]$. We consider the operators

$$Nu(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds, \qquad Lu(t) := \int_0^1 G(t,s)g(s)u(s)\,ds.$$

We may now choose $J = [0,1]$ and the earlier results apply, in particular

$$L = L_J \text{ is } u_0\text{-positive relative to } (K_{c_0}, P). \tag{4.4}$$

For $r > 0$ let B_r be the open ball in $C[0,1]$ with centre 0 and radius r and write $K_r = B_r \cap K_{c_0}$. If $u \in \partial K_r$ (the boundary relative to K_{c_0}) then $\|u\| = r$ and $c_0 r \leq u(t) \leq r$ for $t \in [0,1]$.

Theorem 4.1. *Let $r > 0$ and suppose that $f(s,u) < \mu(L)u$ for all $u \in [c_0 r, r]$ and almost all $s \in [0,1]$. Then $i_{K_{c_0}}(N, K_r) = 1$.*

Proof. We will show that $Nu \neq \sigma u$ for all $\sigma \geq 1$ and all $u \in \partial K_r$ which will prove the result by properties of fixed point index. In fact, if not, there exist $\sigma \geq 1$ and $u \in \partial K_r$ such that $\sigma u = Nu$. Then $c_0 r \leq u(s) \leq r$ for $s \in [0,1]$ so we have

$$\sigma u(t) = Nu(t) \leq \int_0^1 G(t,s)g(s)\mu(L)u(s)\,ds = \mu(L)Lu(t),$$

that is $\sigma r(L)u \preceq Lu$. By the comparison theorem, Theorem 2.4, this can only happen if $\sigma = 1$ and u is a multiple of the eigenfunction φ so $u = \mu(L)Lu$. Then we must have equality above, that is,

$$u(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds = \int_0^1 G(t,s)g(s)\mu(L)u(s)\,ds,$$

which is impossible when $f(s,u(s)) < \mu(L)u(s)$ for all $s \in [0,1]$, since for each t, $G(t,s)g(s) > 0$ for almost every s by $(\mathrm{C}_2)_0$ and (C_3). $\square$

Remark 4.2. The standard result of this type assumes that $f(s,u) < \mu(L)u$ for all $u \in (0,r]$; we can assume less because condition $(C_2)_0$ holds. Taking r small, Theorem 4.1 proves a known result that the index equals one when, for example,

$$\limsup_{u \to 0+} \frac{f(u)}{u} < \mu(L).$$

By choosing r large, Theorem 4.1 proves a result when it is assumed that $f(t,u) < \mu(L)u$ for all sufficiently large u, for example, if

$$\limsup_{u \to \infty} \frac{f(u)}{u} < \mu(L).$$

Previously the index equals one result for r large has been proved using the fact that the cone P is a normal cone, as for example in [36], which property is not needed in the above proof.

The corresponding index zero result is as follows.

Theorem 4.3. *Let $r > 0$ and suppose that $f(s,u) > \mu(L)u$ for all $u \in [c_0 r, r]$ and almost all $s \in [0,1]$. Then $i_{K_{c_0}}(N, K_r) = 0$.*

Proof. Let $e \in K_{c_0} \setminus \{0\}$. We will show that $u \neq Nu + \sigma e$ for all $\sigma \geq 0$ and all $u \in \partial K_r$. Indeed, if not there exist $\sigma \geq 1$ and $u \in \partial K_r$ such that $u = Nu + \sigma e$, hence $u \succeq Nu$ and, similarly to the proof of Theorem 4.1, $Nu \succeq \mu(L)Lu$. Thus $r(L)u \succeq Lu$ and, by Theorem 2.4, this implies that u is a multiple of the eigenfunction φ so $r(L)u = Lu$. Then we get $u = Nu + \sigma e \succeq \mu(L)Lu + \sigma e = u + \sigma e$, thus we must have $\sigma = 0$ and $u = Nu = Lu$. As in Theorem 4.1, this is impossible. □

We now show that very similar arguments prove non-existence results if there is no 'crossing of the eigenvalue'. This is similar to Theorem 4.9 of [35] where a more abstract result is proved.

Theorem 4.4. *Let D be a subset of $C[0,1]$ with $D \cap P \neq \emptyset$.*

(a) *The operator N has no nonzero fixed points in $D \cap P$ if*

$$f(t,u) < \mu(L)u \quad \text{for all } 0 \leq t \leq 1, \ u \in D \cap P.$$

(b) *The operator N has no nonzero fixed points in $D \cap P$ if*

$$f(t,u) > \mu(L)u \quad \text{for all } 0 \leq t \leq 1, \ u \in D \cap P.$$

Proof. (a) If N has a nonzero fixed point $u \in D \cap P$ then $u = Nu \preceq \mu(L)Lu$ and $\varphi = \mu(L)L\varphi$ for an eigenfunction φ. By the comparison theorem, u is a positive multiple of φ and so $\mu(L)Lu = u$, hence $Nu = \mu(L)Lu$. As in the proof of Theorem 4.1 this is impossible.

The proof of (b) is almost identical hence omitted. □

A short proof of part (a) is essentially given by R.D. Nussbaum in Proposition 2 of [26] with a simple argument that does not use any u_0-positivity concept.

We now state and prove the new results on existence of an arbitrary finite number of positive solutions involving 'eigenvalue crossings'. We are also able to give more precise localization of the solutions than is usually possible.

Theorem 4.5. *For $n \in \mathbb{N}$ and $n \geq 2$, let $0 < r_1 < c_0 r_2 < r_2 < c_0 r_3 < r_3 < c_0 r_4 < \ldots < r_n$. Suppose that*

$$\begin{aligned} &f(t,u) < \mu(L)u \quad \text{for } 0 \leq t \leq 1,\ c_0 r_{2j-1} \leq u \leq r_{2j-1},\ j = 1,2,\ldots, \\ &f(t,u) > \mu(L)u \quad \text{for } 0 \leq t \leq 1,\quad c_0 r_{2j} \leq u \leq r_{2j},\quad j = 1,2,\ldots. \end{aligned} \tag{4.5}$$

Then N has at least $n-1$ fixed points u_i in K_{c_0}, and for each $i = 1, \dots, n-1$, $r_i \le \|u_i\| < r_{i+1}$ and $c_0 r_i \le u_i(t) < r_{i+1}$ for all $t \in [0,1]$, and there exist $\tau_{i1}, \tau_{i2} \in [0,1]$ such that $r_i < u_i(\tau_{i1})$ and $u_i(\tau_{i2}) < c_0 r_{i+1}$.

Proof. Theorems 4.1 shows that $i_{K_{c_0}}(N, K_{r_{2j-1}}) = 1$ and Theorem 4.3 shows that $i_{K_{c_0}}(N, K_{r_{2j}}) = 0$. By the additivity property of fixed point index we have

$$i_{K_{c_0}}(N, K_{r_{2j}} \setminus \overline{K}_{r_{2j-1}}) = 0 - 1 = -1,$$

so by the existence property there is a fixed point $u_{2j-1} \in K_{r_{2j}} \setminus \overline{K}_{r_{2j-1}}$, and $r_{2j-1} \le \|u_{2j-1}\| < r_{2j}$. Since we are in the cone K_{c_0} this gives $c_0 r_{2j-1} \le u_{2j-1}(t) < r_{2j}$ for all t in $[0,1]$. But, by the nonexistence result, Theorem 4.4, there is no positive solution satisfying $c_0 r_{2j-1} \le u(t) \le r_{2j-1}$ for all t in $[0,1]$, and no positive solution satisfying $c_0 r_{2j} \le u(t) \le r_{2j}$, for all t, thus there exist τ_{i1}, τ_{i2} such that $r_{2j-1} < u_{2j-1}(\tau_{i1})$ and $u_{2j-1}(\tau_{i2}) < c_0 r_{2j}$.

Similarly we have

$$i_{K_{c_0}}(N, K_{r_{2j+1}} \setminus \overline{K}_{r_{2j}}) = 1 - 0 = 1,$$

hence there is a fixed point $u_{2j} \in K_{r_{2j+1}} \setminus \overline{K}_{r_{2j}}$, and $r_{2j} \le \|u_{2j}\| < r_{2j+1}$. The same localization argument as before applies. There is also a fixed point in K_{r_1} but this may be zero. □

We can also start the process with index equals zero. The result is as follows.

Theorem 4.6. *For $n \in \mathbb{N}$ and $n \ge 2$, let $0 < r_1 < c_0 r_2 < r_2 < c_0 r_3 < r_3 < c_0 r_4 < \dots < r_n$. Suppose that*

$$\begin{aligned} f(t,u) &> \mu(L)u \quad \text{for } 0 \le t \le 1,\ c_0 r_{2j-1} \le u \le r_{2j-1},\ j = 1,2,\dots, \\ f(t,u) &< \mu(L)u \quad \text{for } 0 \le t \le 1, \quad c_0 r_{2j} \le u \le r_{2j}, \quad j = 1,2,\dots. \end{aligned}$$

Then N has at least $n-1$ fixed points u_i in K_{c_0}, and for each $i = 1, \dots, n-1$, $c_0 r_i \le u_i(t) < r_{i+1}$ for all $t \in [0,1]$ and there exists $\tau \in [0,1]$ such that either $r_i < u_i(\tau)$ or $u_i(\tau) < c_0 r_{i+1}$.

The proof is very similar to the above one and is therefore omitted.

Remark 4.7. As far as I am aware these are the first results on existence of multiple solutions that use only 'crossing of the eigenvalue'. These rely on the fact that such hypotheses on f can be made on disjoint intervals which depends on being able to use the cone K_{c_0}.

Since we can let $r_i = c_0 R_i$ the conditions in Theorem 4.5 can read

$$\begin{aligned} f(t,u) &< \mu(L)u \quad \text{for } 0 \le t \le 1,\ R_{2j-1} \le u \le R_{2j-1}/c_0,\ j = 1,2,\dots, \\ f(t,u) &> \mu(L)u \quad \text{for } 0 \le t \le 1, \quad R_{2j} \le u \le R_{2j}/c_0, \quad j = 1,2,\dots. \end{aligned} \tag{4.6}$$

A similar remark applies to Theorem 4.6.

For comparison purposes we state the multiple existence result which applies when (C_2) holds, that is,

$$c(t)\Phi(s) \le G(t,s) \le \Phi(s), \quad \text{for } 0 \le t, s \le 1,$$

where $c(t) > 0$ for $t \in (0,1)$. Let $J = [t_0, t_1]$ be a sub-interval of $(0,1)$. Define constants c_J, m, M_J by

$$\begin{aligned} c_J &:= \min\{c(t) : t \in J\}, \\ m &:= \left(\sup_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds \right)^{-1}, \\ M_J &:= \left(\inf_{t\in J} \int_{t_0}^{t_1} G(t,s)g(s)\,ds \right)^{-1}. \end{aligned} \tag{4.7}$$

It is known that $m \le \mu(L) \le M_J$, this was shown by a direct argument in [36] and by using the comparison theorem in [35].

Theorem 4.8. *For $n \in \mathbb{N}$ and $n \ge 2$, suppose that $0 < r_1 < c_J r_2 < r_2 < c_J r_3 < r_3 < c_J r_4 < \dots < r_n$ and that*

$$\begin{aligned} &f(t,u) < r_{2j-1} m \quad &&\text{for } 0 \le t \le 1, \quad &&0 \le u \le r_{2j-1},\ j = 1,2,\dots, \\ &f(t,u) > c_J r_{2j} M_J \quad &&\text{for } t \in J, \quad &&c_J r_{2j} \le u \le r_{2j}, \quad j = 1,2,\dots \end{aligned} \tag{4.7}$$

Then N has at least $n-1$ fixed points u_i in K_c, and $r_i \le \|u_i\| \le r_{i+1}$ for $1 \le i \le n-1$.

This result can be found in a number of papers, for example, [15], [36]–[38]. There is a version, starting with the index 0 case, similar to Theorem 4.6, we omit the statement of this. Theorem 4.8 can be modified by assuming $f(t,u) < \mu(L)u$ in place of $f(t,u) < mr_1$ for $u \le r_1$ and with $f(t,u) > \mu(L)u$ for all u sufficiently large. These give weaker conditions when these are obtained from assumptions on the existence of limits as $u \to 0$ and as $u \to \infty$.

Remark 4.9. It is implicit in Theorem 4.8, for consistency, that the constants are restricted by the requirements $M_j r_{2j} \le m r_{2j+1}$.

When we have the stronger positivity condition and can use the cone K_{c_0} the conditions in Theorem 4.8 can be modified to read

$$\begin{aligned} &f(t,u) < r_{2j-1} m \quad &&\text{for } 0 \le t \le 1,\ c_0 r_{2j-1} \le u \le r_{2j-1},\ j = 1,2,\dots, \\ &f(t,u) > c_0 r_{2j} M_J \quad &&\text{for } t \in J, \qquad c_0 r_{2j} \le u \le r_{2j}, \quad j = 1,2,\dots. \end{aligned}$$

The proofs are essentially the known ones, for example, [15], [36]–[38], the second uses fixed point index on the open set $\Omega_r := \{u \in P : \min_{t\in J} u(t) < c_0 r\}$, which was introduced by Lan [15].

We give some examples to illustrate the new results.

EXAMPLE 4.10. Consider the BVP of second order

$$u''(t) + f(u(t)) = 0, \quad u'(0) = 0, \quad u(1) + u'(1) = 0. \tag{4.8}$$

It is easy to check that the Green's function for this problem is

$$G(t,s) := \begin{cases} 2-t & \text{if } s \le t, \\ 2-s & \text{if } s > t. \end{cases}$$

Therefore $\Phi(s) = 2 - s$, $c(t) = 1 - t/2$, that is $(\mathrm{C}_2)_0$ holds with $c_0 = 1/2$. By direct calculation $m = 2/3$, $M_{(0,1)} = 1$ and $M_{(0,1)}$ is the smallest M_J for J a sub-interval of $[0,1]$ in this example. From the differential equation, the eigenfunction is of the form $\cos(\omega t)$, hence $\mu(L) = \omega^2$ where ω is the smallest positive solution of the equation $\cos(\omega) = \omega \sin(\omega)$, so that $\mu(L) \approx 0.740174$.

Now take $f(u)$ as follows.

$$f(u) := \begin{cases} 0 & \text{for } 0 \le u \le 1/4, \\ (3/2)(u - 1/4) & \text{for } 1/4 \le u \le 1/2, \\ (3/4)u & \text{for } 1/2 \le u \le 1, \\ 3/4 & \text{for } u \ge 1. \end{cases}$$

Then, taking $r_1 = 1/4, r_2 = 1, r_3 = 5/2$, by Theorem 4.5, the BVP has at least two positive solutions, say u_1, u_2. These satisfy the bounds $1/4 \le \|u_1\| < 1$ and $1/8 \le u_1(t) < 1$ for all $t \in [0,1]$, and $1 \le \|u_2\| < 5/2$ and $1/2 \le u_2(t) < 5/2$ for all $t \in [0,1]$.

Theorem 4.8 can only give the zero solution here since the graph of f lies completely below the line of slope $M_{(0,1)} = 1$. However, an important point of Theorem 4.8 is that it can be applied in cases where $(C_2)_0$ does not hold.

EXAMPLE 4.11. For the BVP $u''(t) + f(u(t)) = 0$, $u'(0) = 0$, $u(1) = 0$, condition (C_2) holds but not $(\mathrm{C}_2)_0$. However condition $(\mathrm{C}_2)_0$ is satisfied when we have a nonlocal BC at 1 such as $u(1) = \beta[u]$ where β is a positive linear functional on $C[0,1]$, thus given by a Riemann–Stieltjes integral $\beta[u] = \int_0^1 u(t)\,dB(t)$ where dB is a (positive) measure. As a simple example we consider the well studied 3-point boundary condition

$$u'(0) = 0, \quad u(1) = \beta u(\eta), \quad \eta \in (0,1), \quad 0 < \beta < 1. \tag{4.9}$$

The Green's function and its properties are known for this problem; it was studied in detail in [32]. We have

$$G(t,s) = \frac{1}{1-\beta}((1-s) - \beta(\eta - s)H(\eta - s)) - (t-s)H(t-s),$$

where H denotes the Heaviside function

$$H(x) := \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then the constants are given by $m = 2(1-\beta)/(1-\beta\eta^2)$, $c(t) = 1-(1-\beta)t/(1-\beta\eta)$, therefore $(\mathrm{C}_2)_0$ is satisfied with $c_0 = \beta(1-\eta)/(1-\beta\eta)$. Formulae for the smallest M are also given in [32].

We take a specific example, let $\eta = 1/2, \beta = 1/4$. Then $c_0 = 1/7$, $m = 8/5$, the smallest possible M is $M_{(0,4/7)} = 56/19 \approx 2.947$, whereas $M_{(0,1)} = 8$. The 'eigenvalue' $\mu(L) = \omega^2$ where ω is the smallest positive root of the equation $\cos(\omega) = \beta\cos(\omega\eta)$ which in this case gives $\mu(L) \approx 1.89471$.

It would now be routine to give examples similar to that above to show there are two (or more) positive solutions for suitable f, and for which known results using the constants m, M can not give existence of positive solutions.

Similarly, the problem with BCs $u(0) = 0$, $u(1) = 0$ does not satisfy the condition $(\mathrm{C}_2)_0$ but the condition is satisfied with the nonlocal BCs $u(0) = \beta_1[u]$, $u(1) = \beta_2[u]$ when β_i are suitable positive linear functionals on $C[0,1]$. Higher order problems, especially with positive non-local BCs can also fit this framework, see [39] for some fourth order problems and [38] for a general situation.

5. Singular problems

We assume that we have a BVP with a nonlinearity of the form $g(t)f(t,u(t))$ that can be studied in the space $C[0,1]$ with a Green's function satisfying (C_1), $(\mathrm{C}_2)_0$ and g satisfying (C_3). When f is continuous on $[0,1]\times[0,\infty)$ solutions of the problem are equivalent to fixed points of the nonlinear integral operator

$$Nu(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds. \tag{5.1}$$

We consider problems where the nonlinearity is allowed to be singular at $u = 0$; many possible singularities in t are taken care of by the term $g(t)$. In place of (C_4) we suppose that $f(t,u)$ is continuous on $[0,1]\times(0,\infty)$ and f is allowed to 'blow-up' as $u \to 0+$.

Our observation is that when $(\mathrm{C}_2)_0$ holds the singularity at $u = 0$ is not a serious problem. Under the above assumptions we have the following result, thus Theorem 4.6 is practically unchanged.

Theorem 5.1. *For $n \in \mathbb{N}$ and $n \geq 2$, let $0 < r_1 < c_0 r_2 < r_2 < c_0 r_3 < r_3 < c_0 r_4 < \ldots < r_n$. Suppose that*

$$\begin{aligned} f(t,u) &> \mu(L)u \quad \text{for } c_0 r_{2j-1} \leq u \leq r_{2j-1},\ j = 1,2,\ldots, \\ f(t,u) &< \mu(L)u \quad \text{for } c_0 r_{2j} \leq u \leq r_{2j},\quad j = 1,2,\ldots. \end{aligned}$$

Then the singular BVP has at least $n-1$ positive solutions u_i in K_{c_0}, and they satisfy $r_i \leq \|u_i\| \leq r_{i+1}$ and $c_0 r_i \leq u_i(t) < r_{i+1}$ for all $t \in [0,1]$.

Remark 5.2. When f is singular at $u = 0$ only the analogue of Theorem 4.6 is appropriate. There is a similar result using m, M_J, as in the comments following Theorem 4.8.

Proof. Define a modification of f by

$$\widetilde{f}(t,u) := f(t, c_0 r_1 + (u - c_0 r_1)^+) \quad \text{where } v^+ := \begin{cases} v & \text{if } v \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{f}$ is continuous on $[0,1] \times [0,\infty)$ and $\widetilde{f}(t,u) = f(t,u)$ for $t \in [0,1]$ and $u \geq c_0 r_1$. Thus $\widetilde{f}$ satisfies the conditions of Theorem 4.6 and so the integral operator $\widetilde{N}u(t) := \int_0^1 G(t,s)g(s)\widetilde{f}(s,u(s))\,ds$ has fixed points u_i with $\|u_i\| \geq r_1$ and these are solutions of the modified BVP (with f replaced by $\widetilde{f}$). Since we are in the cone K_{c_0}, we have $u_i(t) \geq c_0\|u_i\| \geq c_0 r_1$ so that $\widetilde{f}(t,u_i(t)) = f(t,u_i(t))$ and the solutions are solutions of the original BVP. □

Remark 5.3. The reason why the singularity is practically irrelevant is simple: nonzero solutions in K_{c_0} are (strictly) positive on $[0,1]$ so do not interact with the singularity.

In [16], Lan also discusses singular problems when the cone K_{c_0} can be used with a similar method. Lan has results related to Theorem 4.8 with $n = 3$ but does not have results on 'eigenvalue crossings'. He gives applications to some second order equations with periodic BCs and with Neumann BCs. Infante [8], [9] uses similar ideas, related to Theorem 4.8, and treats some problems that arise from a thermostat model, with a nonlocal BC involving a Riemann–Stieltjes integral.

In [31], the equation $u''(t) + \mu a(t) f(t,u(t)) = 0$ ($\mu > 0$ is a parameter) with the BCs $u(0) - \beta u'(0) = 0$, $u(1) = \alpha u(\eta)$ is studied when f is singular in u as $u \to 0$ and $\alpha > 0$, $\beta > 0$. The authors impose an integrability condition that has been employed in a number of papers, see some references in [10], [31]. One or two positive solutions are found in [31] using results similar to the case $n = 3$ of Theorem 4.8. We remark that this is similar to Example 4.2, in fact the Green's function is strictly positive, so the integrability condition is not needed and our Theorem 4.1 (also see Remark 5.2) can give stronger results on that problem. The integrability condition is useful for other problems such as when one BC is $u(0) = 0$. For example, singular problems are studied in [10] using the integrability condition when the BCs are

$$\alpha u(0) - \beta u'(0) = \int_0^1 a(s)u(s)\,ds, \qquad \gamma u(1) + \delta u'(1) = \int_0^1 b(s)u(s)\,ds,$$

with $\alpha\gamma + \alpha\delta + \beta\gamma > 0$ (but not necessarily $\beta > 0$, $\delta > 0$), but, as we remarked earlier, if the nonlocal parts are (strictly) positive then we may well be able to use the cone K_{c_0}.

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