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NEUMANN PROBLEMS WITH DOUBLE RESONANCE

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Abstract. We study elliptic Neumann problems in which the reaction term at infinity is resonant with respect to any pair $\{\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\}$ of distinct consecutive eigenvalues. Using variational methods combined with Morse theoretic techniques, we show that when the double resonance occurs in a "nonprincipal" spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, $m \geq 1$, we have at least three nontrivial smooth solutions, two of which have constant sign. If the double resonance occurs in the "principal" spectral $[\widehat{\lambda}_0 = 0, \widehat{\lambda}_1]$, then we show that the problem has at least one nontrivial smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We examine the following nonlinear Neumann problem

$$-\Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega. \tag{1.1}$$

The aim of this paper is to prove multiplicity theorems for problem (1.1), when double resonance occurs, namely asymptotically as $|x| \to \infty$ the quotient $\frac{f(z,x)}{x}$ lies in the spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, $m \geq 0$ and we can have complete interaction with both endpoints of the interval (double resonance). Here $\{\widehat{\lambda}_m\}_{m\geq 0}$ is the sequence of distinct eigenvalues of the negative Laplacian with Neumann boundary conditions, denoted henceforth by $-\Delta^N$. We know that $\widehat{\lambda}_0 = 0$ and $\widehat{\lambda}_m \to \infty$ as $m \to \infty$ (see Section 2).

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In the past such problems were investigated, almost exclusively, in the context of Dirichlet equations. We mention the works of H. Berestycki and D. de Figueiredo [3] (who coinced the term "double resonance"), N.P. Cac [4], S. Robinson [17] and J. Su [18]. H. Berestycki and D. de Figueiredo [3] and S. Robinson [17], prove existence theorems using certain generalized Landesman–Lazer conditions (LL-conditions for short).

N.P. Cac [4] proves existence and multiplicity results, establishing the existence of two nontrivial solutions but under restrictive conditions on the data of the problem. J. Su [18] proves a multiplicity result producing three nontrivial solutions, using generalized LL-conditions, similar to the ones employed first in the work of E. Landesman, S. Robinson and A. Rumbos [8]. He assumes that $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and this makes the energy functional of the problem C^2, a fact that permits the author to use the complete arsenal of Morse theory. For the Neumann problem, to the best of our knowledge, the only work dealing with double resonant problems, is the recent one by M. Filippakis and N.S. Papageorgiou [6], where the authors prove two multiplicity theorems establishing respectively two and three nontrivial solutions when $f(z, \cdot\,) \in C^1(\mathbb{R})$ and certain generalized LL-conditions are satisfied. Also for the three solutions theorem, a restriction is imposed on the eigenvalue $\widehat{\lambda}_m$.

In the present work, the reaction term $f(z,x)$ is a Caratheodory function (i.e. for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \to f(z,x)$ is continuous), we do not use LL-type conditions and our hypotheses are in principle easier to verify. Our approach combines variational methods based on the critical point theory, together with techniques from Morse theory.

2. Mathematical background

In this section, for the convenience of the reader, we briefly review the main mathematical tools that we will use in this work. We start with critical point theory. So, let X be a Banach space and X^* its topological dual. By $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the "Cerami condition" (the C-condition for short), if the following is true:

"*Every sequence* $\{x_n\}_{n\geq 1} \subseteq X$ *such that* $\{\varphi(x_n)\}_{n\geq 1}$ *is bounded and*

$$(1 + ||x_n||)\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

Using this compactness-type condition on φ, we can have the following minimax characterization of certain critical values of φ. The result is known in the literature as the "mountain pass theorem".

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ and it satisfies the* C*-condition, $u_0, u_1 \in X, r > 0$, $||u_1 - u_0|| > r$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} \le \inf[\varphi(u) : ||u - u_0|| = r] = \eta_r,$$
$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\} \quad \text{and} \quad c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)),$$

then $c \ge \eta_r$ and c is a critical value of φ.

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(x) \le c\},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad K_\varphi^c = \{u \in K_\varphi : \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \ge 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group with integer coefficients for the pair (Y_1, Y_2) . Recall that for $k < 0$, $H_k(Y_1, Y_2) = 0$. The critical groups of φ at an isolated critical point $u \in K_\varphi^c$, are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \ge 0,$$

where U is a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory, implies that the above definition is independent of the choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $a < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^a) \quad \text{for all } k \ge 0$$

(see T. Bartsch and S. Li [2]).

Since φ satisfies the C-condition, the second deformation theorem is valid (see, for example, L. Gasinski nad N.S. Papageorgiou [7]). Using the second deformation theorem, we see that the definition of critical groups of φ at infinity is independent of the particular level $a < \inf \varphi(K_\varphi)$. If for some integer $m \ge 0 C_m(\varphi, \infty) \ne 0$, then there exists $u \in K_\varphi$ such that $C_m(\varphi, u) \ne 0$.

In the study of problem (1.1), we will use the following two "natural" spaces

$$C_n^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}, \quad H_n^1(\Omega) = \overline{C_n^1(\overline{\Omega})}^{||\cdot||},$$

where $||\cdot||$ denotes the Sobolev norm $||u|| = (||u||_2^2 + ||Du||_2^2)^{1/2}$ for all $u \in H^1(\Omega)$.

The space $C_n^1(\overline{\Omega})$ is an ordered Banach space, with positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Next let us recall some basic facts about the spectrum of $-\Delta^N$. So, let $m \in L^\infty(\Omega)_+$, $m \not\equiv 0$ (the weight function) and consider the following weighted eigenvalue problem

$$\begin{cases} -\Delta u(z) = \widehat{\lambda} m(z) u(z) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $-\Delta^N$, if problem (2.1) admits a nontrivial solution. It is clear that a necessary condition for $\widehat{\lambda} \in \mathbb{R}$ to be an eigenvalue is that $\widehat{\lambda} \geq 0$. To emphasize the dependence on the weight m, we write $\widehat{\lambda}(m)$ and if $m \equiv 1$, then we set $\widehat{\lambda}(1) = \widehat{\lambda}$. Note that $\widehat{\lambda}_0(m) = 0$ is an eigenvalue of (2.1), with corresponding eigenspace $\mathbb{R}$ (the space of constant functions). Moreover, using the spectral theorem for compact operators, we can show that (2.1) has a sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 0}$ of distinct eigenvalues such that $\widehat{\lambda}_k(m) \to \infty$ as $k \to \infty$.

For every integer $k \geq 0$, by $E(\widehat{\lambda}_k(m))$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(m)$. From regularity theory, we have $E(\widehat{\lambda}_k(m)) \subseteq C^1_n(\overline{\Omega})$ and the space $E(\widehat{\lambda}_k(m))$ exhibits the unique continuation property (UCP for short) which says that, if $u \in E(\widehat{\lambda}_k(m))$ vanishes on a set of positive Lebesgue measure, then $u(z) = 0$ for all $z \in \overline{\Omega}$. We set

$$\overline{H}_i = \bigoplus_{k=0}^{i} E(\widehat{\lambda}_k(m)) \quad \text{and} \quad \widehat{H}_i = \overline{H}_i^{\perp} = \overline{\bigoplus_{k\geq i+1} E(\widehat{\lambda}_k(m))}.$$

Using these spaces, we have the following variational characterizations of the eigenvalues:

$$\widehat{\lambda}_0(m) = \min \left[\frac{\|Du\|_2^2}{\int_\Omega m u^2 \, dz} : u \in H^1_n(\Omega),\ u \not\equiv 0 \right] \tag{2.2}$$

and for $i \geq 1$

$$\begin{aligned} \widehat{\lambda}_i(m) &= \max \left[\frac{\|D\overline{u}\|_2^2}{\int_\Omega m \overline{u}^2 \, dz} : \overline{u} \in \overline{H}_i,\ \overline{u} \not\equiv 0 \right] \\ &= \min \left[\frac{\|D\widehat{u}\|_2^2}{\int_\Omega m \widehat{u}^2 \, dz} : \widehat{u} \in \widehat{H}_{i-1},\ \widehat{u} \not\equiv 0 \right]. \end{aligned} \tag{2.3}$$

In (2.2) and (2.3) the min and max are realized in the corresponding eigenspace $E(\widehat{\lambda}_i(m))$.

The next lemmata are consequences of (2.2), (2.3) and of the UCP.

Lemma 2.2. *If $m, m' \in L^\infty(\Omega)_+ \setminus \{0\}$, $m(z) \leq m'(z)$, for almost all $z \in \Omega$ and $m \neq m'$, then $\widehat{\lambda}_k(m') < \widehat{\lambda}_k(m)$ for all $k \geq 0$.*

Lemma 2.3. *If $\vartheta, \widehat{\vartheta} \in L^\infty(\Omega)$, $m \geq -1$, $\ell \geq 0$, $\vartheta(z) \leq \widehat{\lambda}_{m+1}$, $\widehat{\vartheta}(z) \geq \widehat{\lambda}_\ell$ almost everywhere in Ω, $\vartheta \neq \widehat{\lambda}_m$, $\widehat{\vartheta} \neq \widehat{\lambda}_\ell$, then there exist $\xi_0, \xi_1 > 0$ such that*

$$\begin{aligned} ||Du||_2^2 - \int_\Omega \vartheta u^2\,dz &\geq \xi_0 ||u||^2 \quad \text{for all } u \in \widehat{H}_m, \\ ||Du||_2^2 - \int_\Omega \widehat{\vartheta} u^2\,dz &\leq -\xi_1 ||u||^2 \quad \text{for all } u \in \overline{H}_\ell. \end{aligned}$$

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Also, as already mentioned, by $||\cdot||$ we denote the usual Sobolev norm in the space $H_n^1(\Omega)$, by $||\cdot||_p (1 < p < \infty)$ we denote the norm of the Lebesgue space $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^N)$ and by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Finally by $\widehat{u}_0$ we denote the L^p-normalized positive eigenfunction corresponding to $\widehat{\lambda}_0(m) = 0$. Every eigenfunction to a positive eigenvalue $\widehat{\lambda}_k(m) > 0$, $k \geq 1$ is necessarily nodal (i.e. sign changing).

3. Three solutions theorem

In this section we establish the existence of at least three nontrivial smooth solution when double resonance occurs at any spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$ $m \geq 1$.

The hypotheses on the reaction $f(z,x)$ are:

(H) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, and

- (a) $|f(z,x)| \leq \alpha(z) + c|x|$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ with $\alpha \in L^\infty(\Omega)_+$, $c > 0$;
- (b) there exists an integer $m \geq 1$ such that

$$\widehat{\lambda}_m \leq \liminf_{|x| \to \infty} \frac{f(z,x)}{x} \leq \limsup_{|x| \to \infty} \frac{f(z,x)}{x} \leq \widehat{\lambda}_{m+1}$$

uniformly for almost all $z \in \Omega$ and, if $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\lim_{|x| \to \infty} [f(z,x)x - 2F(z,x)] = \infty \quad \text{uniformly for a.a. } z \in \Omega;$$

- (c) there exists a function $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq 0$, almost everywhere in Ω, $\vartheta \neq 0$ and

$$\limsup_{x \to 0} \frac{2F(z,x)}{x^2} \leq \vartheta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 3.1. Hypothesis (H)(b) implies that we have double resonance in the spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, $m \geq 1$. Also, hypothesis (H)(c) implies that at the origin we have nonuniform nonresonance with respect to the principal eigenvalue $\widehat{\lambda}_0 = 0$. We emphasize that no differentiability conditions are assumed on $f(z,\cdot)$ in contrast to the works of M. Filippakis and N.S. Papageorgiou [16] and J. Su [18].

Example 3.2. The following function $f(x)$ satisfies hypotheses (H) (for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} \widehat{\lambda}_m |x|^{r-2}x - x & \text{if } |x| \leq 1, \\ \widehat{\lambda}_m x - |x|^{q-2}x & \text{if } |x| > 1, \end{cases} \quad \text{with } m \geq 1,\ 1 < q < 2 < r < \infty.$$

Note that f is not C^1.

First we produce two constant sign smooth solutions. For this purpose we choose $\varepsilon \in (0, \widehat{\lambda}_1)$ and introduce the following truncations-perturbations of $f(z, \cdot)$:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(z,x) + \varepsilon x & \text{if } x > 0, \end{cases} \tag{3.1}$$

$$\widehat{f}_-(z,x) = \begin{cases} f(z,x) + \varepsilon x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases} \tag{3.2}$$

Both are Carathéodory functions. Let

$$\widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)\,ds$$

and consider the C^1-functionals $\widehat{\varphi}_\pm : H^1_n(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{2}||Du||_2^2 + \frac{\varepsilon}{2}||u||_2^2 - \int_\Omega \widehat{F}_\pm(z,u(z))\,dz \quad \text{for all } u \in H^1_n(\Omega).$$

Also, let $\varphi : H^1_n(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega F(z,u(z))\,dz \quad \text{for all } u \in H^1_n(\Omega).$$

Evidently $\varphi \in C^1(H^1_n(\Omega))$.

Proposition 3.3. *If hypotheses* (H) *hold, then $\widehat{\varphi}_\pm$ satisfy the* C*-condition.*

Proof. We do the proof for $\widehat{\varphi}_+$, the proof for $\widehat{\varphi}_-$ being similar.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq H^1_n(\Omega)$ such that

$$|\widehat{\varphi}_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0,\ n \geq 1, \tag{3.3}$$

$$(1 + ||u_n||)\widehat{\varphi}'_+(u_n) \to 0 \quad \text{in } H^1_n(\Omega)^* \quad \text{as } n \to \infty. \tag{3.4}$$

From (3.4) we have that for all $h \in H^1_n(\Omega)$,

$$\left| \langle A(u_n), h\rangle + \varepsilon \int_\Omega u_n h\,dz - \int_\Omega \widehat{f}_+(z,u_n)h\,dz \right| \leq \frac{\varepsilon_n ||h||}{1 + ||u_n||}, \tag{3.5}$$

for all $u \in H^1_n(\Omega)$ with $\varepsilon_n \downarrow 0$, where $A \in \mathcal{L}(H^1_n(\Omega), H^1_n(\Omega)^*)$ is defined by

$$\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N}\,dz \quad \text{for all } u, y \in H^1_n(\Omega).$$

In (3.5) we choose $h = -u_n^- \in H_n^1(\Omega)$. Then

$$(3.6)\qquad \|Du_n^-\|_2^2 + \varepsilon\|u_n^-\|_2^2 \le \varepsilon_n \quad \text{for all } n \ge 1 \qquad (\text{see } (3.1)\text{–}(3.2)),$$
$$\Rightarrow\ u_n^- \to 0 \quad \text{in } H_n^1(\Omega) \text{ as } n \to \infty.$$

Claim. $\{u_n^+\}_{n\ge1} \subseteq H_n^1(\Omega)$ is bounded.

We proceed by contradiction. So, suppose that $\|u_n^+\| \to \infty$ and $y_n = u_n^+/\|u_n^+\|$, $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so we may assume that

$$(3.7)\qquad y_n \xrightarrow{w} y \quad \text{in } H_n^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(\Omega).$$

From (3.5) and (3.6) we have

$$(3.8)\qquad \left|\langle A(y_n), h\rangle + \varepsilon\int_\Omega y_n h\,dz - \int_\Omega \frac{\widehat{f}_+(z,u_n^+)}{\|u_n^+\|} h\,dz\right| \le \varepsilon_n'\|h\|,$$

with $\varepsilon_n' \to 0$.

In (3.8) we choose $h = y_n - y \in H_n^1(\Omega)$ and pass to the limit as $n \to \infty$. Using (3.7) and hypothesis (H)(a), we obtain

$$(3.9)\qquad \lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0 \ \Rightarrow\ \|Dy_n\|_2 \to \|Dy\|_2$$

(see (3.7)). Also, we know that $Dy_n \xrightarrow{w} Dy$ in $L^2(\Omega,\mathbb{R}^N)$ (see (3.7)). Then by virtue of the Kadec–Klee property of Hilbert spaces, we have

$$(3.10)\qquad y_n \to y \quad \text{in } H_n^1(\Omega) \quad \text{and so} \quad \|y\| = 1, \quad y \ge 0.$$

Hypothesis (H)(a) implies that

$$\left\{\widehat{g}_n(\,\cdot\,) = \frac{\widehat{f}_+(\,\cdot\,,u_n^+(\,\cdot\,))}{\|u_n^+\|}\right\}_{n\ge1} \subseteq L^2(\Omega)$$

is bounded. So, we may assume that

$$(3.11)\qquad \widehat{g}_n \xrightarrow{w} \widehat{g} \quad \text{in } L^2(\Omega) \text{ as } n \to \infty.$$

Using hypotheses (H)(b) and reasoning as in Motreanu, Motreanu and Papageorgiou [11] (see the proof of Proposition 5), we have that

$$(3.12)\qquad \widehat{g} = (\widehat{\xi} + \varepsilon)y \quad \text{with} \quad \widehat{\lambda}_m \le \widehat{\xi}(z) \le \widehat{\lambda}_{m+1} \quad \text{a.e. in } \Omega.$$

We return to (3.8), pass to the limit as $n \to \infty$ and use (3.10)–(3.12). Then

$$(3.13)\qquad \langle A(y), h\rangle = \int_\Omega \widehat{\xi} y h\,dz \quad \text{for all } h \in H_n^1(\Omega),$$
$$\Rightarrow\qquad A(y) = \widehat{\xi} y \qquad \text{in } H_n^1(\Omega)^*,$$
$$\Rightarrow\ -\Delta y(z) = \widehat{\xi}(z) y(z) \qquad \text{a.e. in } \Omega,\ \frac{\partial y}{\partial n} = 0, \text{ on } \partial\Omega,\ y \ne 0,$$

(see Motreanu and Papageorgiou [13] and (3.10)).

From (3.13) and since $\widehat{\xi}(z) \geq \widehat{\lambda}_m$ almost everywhere in Ω with $m \geq 1$, we see that $\widehat{\xi}(z) > 0$ almost everywhere in Ω and so y must be nodal, which contradicts (3.10). Therefore $\{u_n^+\}_{n\geq 1} \subseteq H_n^1(\Omega)$ is bounded and this proves the Claim.

From (3.6) and the Claim, it follows that $\{u_n\}_{n\geq 1} \subseteq H_n^1(\Omega)$ is bounded and so we may assume that $u_n \xrightarrow{w} u$ in $H_n^1(\Omega)$. If in (3.5) we choose $h = u_n - u \in H_n^1(\Omega)$ and pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(u_n), u_n - y\rangle = 0 \;\Rightarrow\; u_n \to u \quad \text{in } H_n^1(\Omega) \quad \text{(as before)}.$$

This proves that $\widehat{\varphi}_+$ satisfies the C-condition. Similarly for $\widehat{\varphi}_-$. □

Proposition 3.4. *If hypotheses* (H) *hold, then* φ *satisfies the* C*-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subseteq H_n^1(\Omega)$ such that

$$|\varphi(u_n)| \leq M_2 \quad \text{for some } M_2 > 0 \text{ and all } n \geq 1 \tag{3.14}$$

and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \quad \text{in } H_n^1(\Omega)^* \text{ as } n \to \infty. \tag{3.15}$$

From (3.15) we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega f(z, u_n) h \, dz\right| \leq \frac{\varepsilon_n ||h||}{1 + ||u_n||}, \tag{3.16}$$

for all $h \in H_n^1(\Omega)$, with $\varepsilon_n \to 0^+$. In (3.16) we choose $h = u_n \in H_n^1(\Omega)$. Then

$$-||Du_n||_2^2 + \int_\Omega f(z, u_n) u_n \, dz \leq \varepsilon_n \quad \text{for all } n \geq 1. \tag{3.17}$$

On the other hand from (3.14), we have

$$||Du_n||_2^2 - \int_\Omega 2F(z, u_n) \, dz \leq 2M_2 \quad \text{for all } n \geq 1. \tag{3.18}$$

Adding (3.17) and (3.18), we obtain

$$\int_\Omega [f(z, u_n)u_n - 2F(z, u_n)] \, dz \leq M_3 \quad \text{for some } M_3 > 0, \quad \text{all } n \geq 1. \tag{3.19}$$

Claim. $\{u_n\}_{n\geq 1} \subseteq H_n^1(\Omega)$ is bounded.

We argue indirectly. So, suppose that $||u_n|| \to \infty$ and set $y_n = u_n/||u_n||$, $n \geq 1$. Then $||y_n|| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H_n^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(\Omega). \tag{3.20}$$

From (3.16), we have

$$\left|\langle A(y_n), h\rangle - \int_\Omega \frac{f(z, u_n)}{||u_n||} h \, dz\right| \leq \frac{\varepsilon_n ||h||}{(1 + ||u_n||)||u_n||}, \quad \text{for all } n \geq 1. \tag{3.21}$$

Note that by virtue of hypothesis (H)(a) $\left\{\frac{f(\,\cdot\,,u_n(\,\cdot\,))}{||u_n||}\right\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded. Hence, if in (3.21) we choose $h = y_n - y \in H^1_n(\Omega)$ and pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \;\Rightarrow\; y_n \to y \quad \text{in } H^1_n(\Omega) \text{ and so } ||y|| = 1. \tag{3.22}$$

Since $\left\{g_n(\,\cdot\,) = \frac{f(\cdot,u_n(\,\cdot\,))}{||u_n||}\right\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded, we may assume that

$$g_n \xrightarrow{w} g \quad \text{in } L^2(\Omega) \quad \text{and} \quad g = \xi y, \quad \widehat{\lambda}_m \leq \xi(z) \leq \widehat{\lambda}_{m+1} \quad \text{a.e. in } \Omega \tag{3.23}$$

(as in the proof of Proposition 3.3). Passing to the limit as $n \to \infty$ in (3.21) and using (3.22) and (3.23), we obtain

$$\begin{aligned} &\langle A(y), h\rangle &&= \int_\Omega \xi y\, dz \quad \text{for all } h \in H^1_n(\Omega),\\ \Rightarrow\quad &A(y) &&= \xi y,\\ \Rightarrow\quad &-\Delta y(z) &&= \xi(z) y(z) \quad \text{a.e. in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.24}$$

First suppose that $\xi \neq \widehat{\lambda}_m$ and $\xi \neq \widehat{\lambda}_{m+1}$ (see (3.23)). Then by virtue of Lemma 2.2, we have

$$\widehat{\lambda}_m(\xi) < \widehat{\lambda}_m(\lambda_m) = 1 \quad \text{and} \quad 1 = \widehat{\lambda}_{m+1}(\widehat{\lambda}_{m+1}) < \widehat{\lambda}_{m+1}(\xi). \tag{3.25}$$

From (3.24) and (3.25) it follows that $y = 0$, which contradicts (3.22).

So, we assume that $\xi(z) = \widehat{\lambda}_m$ or $\xi(z) = \widehat{\lambda}_{m+1}$ almost everywhere in Ω. Then $y \in E(\widehat{\lambda}_m)\setminus\{0\}$ or $y \in E(\widehat{\lambda}_{m+1})\setminus\{0\}$ (see (3.22) and (3.23)) and so by the UCP we have $y(z) \neq 0$ almost everywhere in Ω. Therefore $|u_n(z)| \to \infty$ for almost all $z \in \Omega$ and so by virtue of hypothesis (H)(b) we have

$$\begin{aligned} &f(z,u_n(z))u_n(z) - 2F(z,u_n(z)) \to \infty \quad \text{for a.a. } z \in \Omega,\\ \Rightarrow\quad &\int_\Omega [f(z,u_n)u_n - 2F(z,u_n)]\, dz \to \infty \quad \text{(by Fatou's lemma).} \end{aligned} \tag{3.26}$$

Comparing (3.19) and (3.26), we reach a contradiction. This proves the Claim.

By virtue of the Claim, we may assume that $u_n \xrightarrow{w} u$ in $H^1_n(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$. Setting $h = u_n - u$ in (3.16) and passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \;\Rightarrow\; u_n \to u \quad \text{in } H^1_n(\Omega) \quad \text{(as before).}$$

This proves the proposition. □

Proposition 3.5. *If hypotheses* (H) *hold, then* $u = 0$ *is a local minimizer of* $\widehat{\varphi}_{\pm}$ *and of* φ.

Proof. We do the proof for $\widehat{\varphi}_{+}$, the proof for $\widehat{\varphi}_{-}$ and for φ being similar.

By virtue of hypothesis (H)(c), given $\widehat{\varepsilon} > 0$ we can find $\widehat{\delta} = \widehat{\delta}(\widehat{\varepsilon}) > 0$ such that

$$F(z,x) \leq \frac{1}{2}(\vartheta(z) + \widehat{\varepsilon})x^2 \quad \text{for a.a. } z \in \Omega, \text{ and all } |x| \leq \widehat{\delta}. \tag{3.27}$$

Let $u \in C^1_n(\overline{\Omega})$ such that $||u||_{C^1_n(\overline{\Omega})} \leq \widehat{\delta}$. Then

$$\begin{aligned} \widehat{\varphi}_{+}(u) &= \frac{1}{2}||Du||_2^2 + \frac{\varepsilon}{2}||u||_2^2 - \int_\Omega \widehat{F}_{+}(z,u)\,dz \\ &\geq \frac{1}{2}||Du||_2^2 - \int_\Omega F(z,u^{+})\,dz \\ &\geq \frac{1}{2}||Du||_2^2 - \frac{1}{2}\int_\Omega \vartheta u^2\,dz - \frac{\widehat{\varepsilon}}{2}||u||^2 \qquad \text{(see (3.27))} \\ &\geq \frac{\xi_0 - \widehat{\varepsilon}}{2}||u||^2 \qquad \text{(see Lemma 2.3)}. \end{aligned} \tag{3.28}$$

Choosing $\widehat{\varepsilon} \in (0, \xi_0)$, from (3.28) we infer that

$$\begin{aligned} &\widehat{\varphi}_{+}(u) > 0 \quad \text{for all } u \in C^1_n(\overline{\Omega}) \text{ with } 0 < ||u||_{C^1_n(\overline{\Omega})} \leq \widehat{\delta}, \\ &\Rightarrow\ u = 0 \quad \text{is a local } C^1_n(\overline{\Omega})\text{-minimizer of } \widehat{\varphi}_{+} \\ &\Rightarrow\ u = 0 \quad \text{is a local } H^1_n(\Omega)\text{-minimizer of } \widehat{\varphi}_{+} \end{aligned}$$

(see Motreanu, Motreanu and Papageorgiou [12]). □

We may assume that $u = 0$ is an isolated critical point of $\widehat{\varphi}_{+}$. Indeed, otherwise we can find $\{u_n\}_{n\geq 1} \subseteq H^1_n(\Omega) \setminus \{0\}$ such that $u_n \to 0$ in $H^1_n(\Omega)$ and

$$\begin{aligned} \widehat{\varphi}'_{+}(u_n) &= 0, && \text{for all } n \geq 1, \\ \Rightarrow A(u_n) + \varepsilon u_n &= N_{\widehat{f}_{+}}(u_n) && \text{for all } n \geq 1, \end{aligned} \tag{3.29}$$

where $N_{\widehat{f}_{+}}(u)(\cdot) = \widehat{f}_{+}(\cdot, u(\cdot))$ for all $u \in H^1_n(\Omega)$.

Acting on (3.29) with $-u_n^- \in H^1_n(\Omega)$ and using (3.1)–(3.2), we see that $u_n \geq 0$ for all $n \geq 1$ and so (3.29) becomes

$$\begin{aligned} &A(u_n) = N_f(u_n) \quad \text{for all } n \geq 1, \\ &\qquad \text{where } N_f(u)(\cdot) = f(\cdot, u(\cdot)) \text{ for all } u \in H^1_n(\Omega), \\ &\qquad \Rightarrow\ u \in C^1_n(\overline{\Omega}) \quad \text{(by regularity theory) solves problem (1.1)} \end{aligned}$$

(see Motreanu and Papageorgiou [13]).

Therefore we have produced a whole sequence of nonnegative smooth solutions of (1.1) and so we are done.

Since $u = 0$ is an isolated critical point, reasoning as in Motreanu, Motreanu and Papageorgiou [11] (see the proof of Proposition 3.5), we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_+(0) = 0 < \inf[\widehat{\varphi}_+(u) : ||u|| = \rho] = \widehat{\eta}_+. \tag{3.30}$$

Similarly we have that

$$\widehat{\varphi}_-(0) = 0 < \inf[\widehat{\varphi}_-(u) : ||u|| = \rho] = \widehat{\eta}_-. \tag{3.31}$$

Now we are ready to produce two constant sign smooth solutions for problem (1.1).

PROPOSITION 3.6. *If hypotheses* (H) *hold then problem* (1.1) *has at least two constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

PROOF. Let $\xi \in \mathbb{R}$, $\xi > 0$. Then

$$\widehat{\varphi}_+(\xi) = -\int_\Omega F(z,\xi)\,dz \quad \text{(see (3.1)–(3.2))}.$$

From hypothesis (H)(b) it follows that

$$\widehat{\lambda}_m \le \liminf_{\xi\to\infty} \frac{2F(z,\xi)}{\xi^2} \le \limsup_{\xi\to\infty} \frac{2F(z,\xi)}{\xi^2} \le \widehat{\lambda}_{m+1} \quad \text{uniformly for a.a. } z \in \Omega.$$

Hence $-\int_\Omega F(z,\xi)\,dz \to -\infty$ as $\xi \to \infty$ (recall $m \ge 1$). Therefore

$$\widehat{\varphi}_+(\xi) \to -\infty \quad \text{as } \xi \to \infty. \tag{3.32}$$

From (3.30), (3.32) and Proposition 3.3, we see that we can apply Theorem 1.1 (the mountain pass theorem) and obtain $u_0 \in H^1_n(\Omega)$ such that

$$\widehat{\varphi}_+(0) = 0 < \widehat{\eta}_+ \le \widehat{\varphi}_+(u_0), \tag{3.33}$$

$$\widehat{\varphi}'_+(u_0) = 0. \tag{3.34}$$

From (3.33) we have $u_0 \ne 0$. From (3.34) we have

$$A(u_0) + \varepsilon u_0 = N_{\widehat{f}_+}(u_0). \tag{3.35}$$

As before, acting on (3.35) with $-u_0^- \in H^1_n(\Omega)$, we show that $u_0 \ge 0$. So, (3.35) becomes

$$\begin{aligned} &A(u_0) = N_f(u_0), \\ \Rightarrow\ & -\Delta u_0(z) = f(z,u_0(z)) \quad \text{a.e. in } \Omega, \quad \frac{\partial u_0}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ \Rightarrow\ & u_0 \in C_+ \setminus \{0\} \qquad \text{(regularity theory)}. \end{aligned} \tag{3.36}$$

Let $r = ||u_0||_\infty$. Then by virtue of hypothesis (H)(a), we can find $\xi_r > 0$ such that

$$f(z,x) + \xi_r x \geq 0 \quad \text{for all } z \in \Omega \text{ and all } x \in [0,r].$$

Then from (3.36) we have

$$-\Delta u_0(z) \leq \xi_r u_0(z), \quad \text{a.e. in } \Omega, \Rightarrow\ u_0 \in \operatorname{int} C_+ \quad \text{(see Vazquez [19])}.$$

Similarly, working this time with the functional $\widehat{\varphi}_-$ and using (3.31), we obtain a second constant sign smooth solution $v_0 \in \operatorname{int} C_+$. □

Next using Morse theory, we will produce a third nontrivial smooth solution for problem (1.1). To this end, first we prove an auxiliary result which will allow us to compute the critical groups of the functionals $\widehat{\varphi}_\pm$ and φ. We state the result in more generality than what is actually needed here, since we believe that result can be useful in more general settings (like nonlinear equations driven by the p-Laplacian). Our result extends Lemma 2.4 of Perera and Schechter [16] which is formulated in Hilbert spaces.

Lemma 3.7. *If X is a Banach space, $(t,u) \to h_t(u)$ belongs to $C^1([0,1]\times X)$ and it is bounded, the maps $u \to \partial_t h_t(u)$ and $u \to h_t'(u)$ are both locally Lipschitz, h_0 and h_1 both satisfy the C-condition and there exist $\beta \in \mathbb{R}$ and $\delta > 0$ such that*

$$h_t(u) \leq \beta \ \Rightarrow\ (1+||u||)||h_t'(u)||_* \geq \delta \quad \textit{for all } t \in [0,1],$$

then $C_k(h_0,\infty) = C_k(h_1,\infty)$ for all $k \geq 0$.

Proof. Since by hypothesis $h \in C^1([0,1]\times X)$, we know that h_t admits a pseudogradient vector field $v_t(\,\cdot\,)$. Also, by defintion $(t,u) \to v_t(u)$ is locally Lipschitz. For every $(t,u) \in [0,1]\times(X \setminus K_{h_t})$ we have

$$\langle h_t'(u), v_t(u)\rangle \geq ||h_t'(u)||_*^2. \tag{3.37}$$

The map

$$X \setminus K_{h_t} \ni u \to -\frac{|\partial_t h_t(u)|}{||h_t'(u)||_*^2} v_t(u) = w_t(u) \in X$$

is well defined and locally Lipschitz. Since by hypothesis $(t,u) \to h_t(u)$ is bounded, we choose $\eta \leq \beta$ such that

$$h_0^\eta \neq 0 \quad \text{or} \quad h_1^\eta \neq 0.$$

(If no such η can be found, then $C_k(h_0,\infty) = C_k(h_1,\infty) = \delta_{k,0}\mathbb{Z}$ for all $k \geq 0$ and so we are done.) To fix things, we assume that $h_0^\eta \neq 0$ and choose $y \in h_0^\eta$. We consider the following Cauchy problem

$$\frac{d\xi}{dt} = w_t(\xi), \quad t \in [0,1],\ \xi(0) = y. \tag{3.38}$$

Since w_t is locally Lipschitz, this Cauchy problem admits a unique local flow (see L. Gasinski and N.S. Papageorgiou [7, p. 618]). We have

$$\begin{aligned}
\frac{d}{dt}h_t(\xi) &= \left\langle h_t'(\xi), \frac{d\xi}{dt}\right\rangle + \partial_t h_t(\xi) \\
&= \langle h_t'(\xi), w_t(\xi)\rangle + \partial_t h_t(\xi) && \text{(see (3.38))} \\
&\le -|\partial_t h_t(\xi)| + \partial_t h_t(\xi) \le 0 && \text{(see (3.37))} \\
\Rightarrow \qquad & t \to h_t(\xi(t,y)) && \text{is nonincreasing,} \\
\Rightarrow\ & h_t(\xi(t,y)) \le h_0(\xi(0,y)) = h_0(y) \le \eta \le \beta, \\
\Rightarrow \qquad & (1+||\xi(t,y)||)||h_t'(\xi(t,y))||_* \ge \delta && \text{(by hypothesis),} \\
\Rightarrow\ & h_t'(\xi(t,y)) \neq 0.
\end{aligned}$$

This shows that the flow $\xi(\,\cdot\,,y)$ is global on $[0,1]$. Then $\xi(1,y)$ is a homeomorphism between h_0^η and a subset of h_1^η. Reversing the time ($t \to 1-t$), we show that h_1^η is a homeomorphism to a subset of h_0^η. Therefore h_0^η and h_1^η are homotopy equivalent and so

$$\begin{aligned}
H_k(X,h_0^\eta) &= H_k(X,h_1^\eta) \quad \text{for all } k \ge 0, \\
\Rightarrow C_k(h_0,\infty) &= C_k(h_1,\infty) \quad \text{for all } k \ge 0.
\end{aligned}$$ □

Lemma 3.8. *If hypotheses* (H) *hold and* $d_m = \dim \overline{H}_m \ge 2$, *then* $C_k(\varphi,\infty) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \ge 0$.

Proof. Let $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and consider the C^2-functional $\psi\colon H_n^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}||Du||_2^2 - \frac{\mu}{2}||u||_2^2 \quad \text{for all } u \in H_n^1(\Omega).$$

We consider the homotopy $h\colon [0,1] \times H_n^1(\Omega) \to \mathbb{R}$ defined by

$$h(t,u) = (1-t)\varphi(u) + t\psi(u) \quad \text{for all } (t,u) \in [0,1] \times H_n^1(\Omega).$$

Clearly we may assume that K_φ is finite (or otherwise we already have infinitely many nontrivial smooth solutions of (1.1) and so we are done).

Note that $h_0(\,\cdot\,) = h(0,\,\cdot\,) = \varphi$ satisfies the C-condition (see Proposition 3.4) and $h_1(\,\cdot\,) = h(1,\,\cdot\,) = \psi$ which too satisfies the C-condition, since by hypothesis $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$.

Claim. There exist $\beta \in \mathbb{R}$ and $\delta > 0$ such that

$$h(t,u) \le \beta \ \Rightarrow\ (1+||u||)||h_u'(t,u)||_* \ge \delta \quad \text{for all } t \in [0,1].$$

We argue by contradiction. So, suppose that the Claim is not true. Since $h(\,\cdot\,,\,\cdot\,)$ is bounded, we can find $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq H_n^1(\Omega)$ such that

$$\text{(3.39)} \qquad t_n \to t, \quad ||u_n|| \to \infty, \quad h(t_n,u_n) \to -\infty, \quad x_n^* \to 0 \quad \text{in } H_n^1(\Omega)^*,$$

where $x_n^* = (1+||u_n||)h'_u(t_n,u_n)$, $n \geq 1$.

By virtue of the last convergence in (3.39), we have

$$\left|\langle A(u_n),h\rangle - (1-t_n)\int_\Omega f(z,u_n)h\,dz - t_n\mu\int_\Omega u_n h\,dz\right| \leq \frac{\varepsilon_n||h||}{1+||u_n||} \tag{3.40}$$

for all $h \in H_n^1(\Omega)$ with $\varepsilon_n \to 0^+$.

We set $y_n = u_n/||u_n||$, $n \geq 1$. Then $||y_n|| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H_n^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(\Omega). \tag{3.41}$$

From (3.40) we have

$$\begin{aligned}\left|\langle A(y_n),h\rangle - (1-t_n)\int_\Omega \frac{f(z,u_n)}{||u_n||}h\,dz - t_n\mu\int_\Omega y_n h\,dz\right|& \\ \leq \frac{\varepsilon_n||h||}{(1+||u_n||)||u_n||},& \quad \text{for all } n \geq 1.\end{aligned} \tag{3.42}$$

Recall (see the proof of Proposition 3.4), that

$$\frac{f(\,\cdot\,,u_n(\,\cdot\,))}{||u_n||} \xrightarrow{w} g = \xi y \quad \text{in } L^2(\Omega) \text{ with } \widehat{\lambda}_m \leq \xi(z) \leq \widehat{\lambda}_{m+1} \text{ a.e. in } \Omega. \tag{3.43}$$

In (3.42) we choose $h = y_n - y \in H_n^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.41). Then

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0, \;\Rightarrow\; y_n \to y \quad \text{in } H_n^1(\Omega) \quad \text{and so } ||y|| = 1. \tag{3.44}$$

So, if in (3.42) we pass to the limit as $n \to \infty$ and use (3.43) and (3.44), then

$$\begin{aligned} &\langle A(y),h\rangle = (1-t)\int_\Omega \xi y h\,dz + t\mu\int_\Omega yh\,dz && \text{for all } h \in H_n^1(\Omega), \\ \Rightarrow\quad & A(y) = \xi_t y && \text{with } \xi_t = (1-t)\xi + t\mu, \\ \Rightarrow\quad & -\Delta y(z) = \xi_t(z)y(z) && \text{a.e. in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.45}$$

Note that $\widehat{\lambda}_m \leq \xi_t(z) \leq \widehat{\lambda}_{m+1}$ almost everywhere in Ω and for all $t \in [0,1]$. If $t \in (0,1]$, then:

$$\begin{aligned} &\xi_t \neq \widehat{\lambda}_m \quad \text{and} \quad \xi_t \neq \widehat{\lambda}_{m+1}, \\ \Rightarrow\ &\widehat{\lambda}_m(\xi_t) < \widehat{\lambda}_m(\widehat{\lambda}_m) = 1, \quad \widehat{\lambda}_{m+1}(\xi_t) > \widehat{\lambda}_{m+1}(\widehat{\lambda}_{m+1}) = 1 && \text{(see Lemma 2.2)} \\ \Rightarrow\ & y = 0 && \text{(see (3.45))}, \end{aligned}$$

contradicting (3.44).

Suppose that $t = 0$. Then $\xi_0 = \xi$ and we proceed as in the proof of Proposition 3.4 to reach a contradiction using hypothesis (H)(b) and the third convergence in (3.39). This proves the Claim.

Because of the Claim, we can apply Lemma 3.7 and infer that

$$C_k(\varphi,\infty)=C_k(\psi,\infty)\quad\text{for all } k\geq 0. \tag{3.46}$$

Since $\mu\in(\widehat{\lambda}_m,\widehat{\lambda}_{m+1}), u=0$ is the only critical point of ψ. Hence

$$C_k(\psi,\infty)=C_k(\psi,0)\quad\text{for all } k\geq 0. \tag{3.47}$$

Recall that $\psi\in C^2(H^1_n(\Omega))$ and that $\psi''(0)$ is invertible since $\mu\in(\widehat{\lambda}_m,\widehat{\lambda}_{m+1})$. So, $u=0$ is a nondegenerate critical point of ψ with Morse index d_m (since $(\varphi''(0)y,y)<0$ for all $y\in\overline{H}_m$, see Lemma 2.3). Hence

$$C_k(\psi,0)=\delta_{k,d_m}\mathbb{Z}\quad\text{for all } k\geq 0 \tag{3.48}$$

(see Mawhin and Willem [10, p. 188]). From (3.46)–(3.48), we conclude that $C_k(\varphi,\infty)=\delta_{k,d_m}\mathbb{Z}$ for all $k\geq 0$. □

We compute also the critical groups at infinity for the functionals $\widehat{\varphi}_\pm$.

Proposition 3.9. *If hypotheses* (H) *hold, then*

$$C_k(\widehat{\varphi}_+,\infty)=C_k(\widehat{\varphi}_-,\infty)=0,\quad \textit{for all } k\geq 0.$$

Proof. We do the proof for $\widehat{\varphi}_+$, the proof for the functional $\widehat{\varphi}_-$ being similar. Let $\mu\in(\widehat{\lambda}_m,\widehat{\lambda}_{m+1})$ and for $\varepsilon\in(0,\widehat{\lambda}_1)$, we consider the C^1-functional $\psi_+: H^1_n(\Omega)\to\mathbb{R}$ defined by

$$\psi_+(u)=\frac{1}{2}||Du||_2^2+\frac{\varepsilon}{2}||u||_2^2-\frac{\mu+\varepsilon}{2}||u^+||_2^2\quad\text{for all } u\in H^1_n(\Omega).$$

We consider the homotopy $h_+:[0,1]\times H^1_n(\Omega)\to\mathbb{R}$ defined by

$$h_+(t,u)=(1-t)\widehat{\varphi}_+(u)+t\psi_+(u)\quad\text{for all } t\in[0,1] \text{ and all } u\in H^1_n(\Omega).$$

As before, without any loss of generality we may assume that $K_{\widehat{\varphi}_+}$ is finite.

Claim. There exist $\beta\in\mathbb{R}$ and $\delta>0$ such that

$$h_+(t,u)\leq\beta\ \Rightarrow\ (1+||u||)||(h_+)'_u(t,u)||_*\geq\delta\quad\text{for all } t\in[0,1].$$

As before we argue by contradiction and so we assume that we can find $\{t_n\}_{n\geq1}\subseteq[0,1]$ and $\{u_n\}_{n\geq1}\subseteq H^1_n(\Omega)$ such that

$$t_n\to t,\quad ||u_n||\to\infty,\quad h_+(t_n,u_n)\to-\infty,\quad y_n^*\to 0\quad\text{in } H^1_n(\Omega)^*, \tag{3.49}$$

where $y_n^*=(1+||u_n||)(h_+)'_u(t_n,u_n)$, $n\geq1$.

From the last convergence in (3.49), we have

$$\bigg|\langle A(u_n),h\rangle+\varepsilon\int_\Omega u_n h\,dz - (1-t_n)\int_\Omega \widehat{f}_+(z,u_n)h\,dz-t_n(\mu+\varepsilon)\int_\Omega u_n^+h\,dz\bigg|\leq\frac{\varepsilon_n||h||}{1+||u_n||} \tag{3.50}$$

for all $h \in H^1_n(\Omega)$, with $\varepsilon_n \to 0^+$.

In (3.50) we choose $h = -u_n^- \in H^1_n(\Omega)$ and obtain

$$\|Du_n^-\|_2^2 + \varepsilon\|u_n^-\|_2^2 \le \varepsilon_n \quad \text{for all } n \ge 1 \ \Rightarrow\ u_n^- \to 0 \quad \text{in } H^1_n(\Omega). \tag{3.51}$$

Therefore we must have $\|u_n^+\| \to \infty$ (see (3.49)). We set $y_n = u_n^+/\|u_n^+\|$, $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1_n(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(\Omega). \tag{3.52}$$

From (3.50) and (3.51) we have

$$\begin{aligned}\bigg|\langle A(y_n), h\rangle + \varepsilon \int_\Omega y_n h\, dz \\ - (1-t_n)\int_\Omega \frac{\widehat{f}_+(z,u_n^+)}{\|u_n^+\|} h\, dz - t_n(\mu+\varepsilon)\int_\Omega y_n h\, dz\bigg| \le \varepsilon_n'\|h\|,\end{aligned} \tag{3.53}$$

for all $h \in H^1_n(\Omega)$, with $\varepsilon_n' \to 0^+$.

In (3.53) we choose $h = y_n - y$, pass to the limit as $n \to \infty$ and use (3.52). Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \ \Rightarrow\ y_n \to y \quad \text{in } H^1_n(\Omega) \text{ and so } \|y\| = 1,\ y \ge 0. \tag{3.54}$$

Recall that

$$\frac{\widehat{f}_+(\,\cdot\,, u_n^+(\,\cdot\,))}{\|u_n^+\|} \xrightarrow{w} (\xi+\varepsilon)y \tag{3.55}$$

in $L^2(\Omega)$ with $\widehat{\lambda}_m \le \xi(z) \le \widehat{\lambda}_{m+1}$, almost everywhere in Ω (see the proof of Proposition 3.3).

So, if in (3.53) we pass to the limit as $n \to \infty$ and use (3.54) and (3.55), then

$$\begin{aligned}&\langle A(y), h\rangle = \int_\Omega \xi_t y h\, dz \quad \text{for all } h \in H^1_n(\Omega) \text{ with } \xi_t = (1-t)\xi + t\mu, \\ \Rightarrow\quad & A(y) = \xi_t y, \\ \Rightarrow\quad & -\Delta y(z) = \xi_t(z) y(z) \quad \text{a.e. in } \Omega,\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega.\end{aligned} \tag{3.56}$$

Since $\xi_t(z) \in [\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, almost everywhere in Ω, from (3.56) we see that y is nodal or trivial, contradicting (3.54). This proves the Claim.

Because of the Claim, we can apply Lemma 3.7 and have

$$C_k(\widehat{\varphi}_+, \infty) = C_k(\psi_+, \infty) \quad \text{for all } k \ge 0. \tag{3.57}$$

Since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$,$u = 0$ is the only critical point of ψ_+. Hence

$$C_k(\psi_+, \infty) = C_k(\psi_+, 0) \quad \text{for all } k \ge 0. \tag{3.58}$$

Let $\beta \in L^\infty(\Omega)_+, \beta \neq 0$ and consider the homotopy $\widehat{h}_+ \colon [0,1] \times H^1_n(\Omega) \to \mathbb{R}$ defined by

$$\widehat{h}_+(t,u) = \psi_+(u) - t\int_\Omega \beta u\,dz \quad \text{for all } (t,u) \in [0,1] \times H^1_n(\Omega).$$

We claim that

$$(\widehat{h}_+)'_u(t,u) \neq 0 \quad \text{for all } t \in [0,1] \text{ and all } u \neq 0. \tag{3.59}$$

We argue indirectly. So, suppose we can find $t \in [0,1]$ and $u \neq 0$ such that

$$(\widehat{h}_+)'_u(t,u) = 0, \;\Rightarrow\; A(u) + \varepsilon u = (\mu + \varepsilon)u^+ + t\beta. \tag{3.60}$$

On (3.60) we act with $-u_n^- \in H^1_n(\Omega)$ and we obtain $||Du^-||_2^2 + \varepsilon||u^-||_2^2 \leq 0$ and so $u \geq 0$. Hence (3.60) becomes

$$A(u) = \mu u + t\beta, \quad u \geq 0, \quad u \neq 0. \tag{3.61}$$

If $t = 0$, then

$$\begin{aligned} & A(u) = \mu u, \\ \Rightarrow\; & -\Delta u(z) = \mu u(z) \quad \text{a.e. in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Omega, \\ \Rightarrow\; & \qquad u = \text{nodal} \quad (\text{since } \mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})) \end{aligned}$$

which contradicts (3.61). If $t \in (0,1]$, then

$$\begin{aligned} & A(u) = \mu u + t\beta, \\ \Rightarrow\; & -\Delta u(z) = \mu u(z) + t\beta(z) \quad \text{a.e. in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Omega. \end{aligned} \tag{3.62}$$

By regularity theory $u \in C_+ \setminus \{0\}$ and since $\Delta u(z) \leq 0$ almost everywhere in Ω (see (3.62)), from Vazquez [19] we have $u \in \operatorname{int} C_+$. For $y \in C_+$, we set

$$R(y,u)(z) = ||Dy(z)||^2 - \left(Du(z), D\left(\frac{y^2}{u}\right)(z)\right)_{\mathbb{R}^N}.$$

Then from Picone's identity (see Allegretto and Huang [1]), we have

$$\begin{aligned} 0 \leq & \int_\Omega R(y,u)\,dz \\ = & ||Dy||_2^2 - \int_\Omega (-\Delta u)\frac{y^2}{u}\,dz && \text{(by Green's identity)} \\ = & ||Dy||_2^2 - \int_\Omega (\mu u + t\beta)\frac{y^2}{u}\,dz && \text{(see (3.62))} \\ \leq & ||Dy||_2^2 - \mu||y||_2^2 && (\text{since } \beta \geq 0). \end{aligned}$$

We choose $y = \widehat{u}_0$ (recall $\widehat{u}_0 = 1/|\Omega|_N^{1/2} \in \operatorname{int} C_+$ is the L^2-normalized principal eigenfunction of $-\Delta^N$). Then

$$0 \le -\mu \widehat{u}_0^2 |\Omega|_N < 0,$$

a contradiction. Therefore (3.59) holds.

From this and the homotopy invariance of critical groups we have

$$C_k(\psi_+, 0) = C_k(\widetilde{\psi}_+, 0) \quad \text{for all } k \ge 0, \tag{3.63}$$

where $\widetilde{\psi}(u) = \psi_+(u) - \beta u = \widehat{h}_+(1, u)$. But from the previous considerations we have that $K_{\widetilde{\psi}} = \vee$ and so

$$\begin{aligned} & C_k(\widetilde{\psi}_+, 0) = 0 \quad \text{for all } k \ge 0, \\ \Rightarrow\ & C_k(\psi_+, 0) = 0 \quad \text{for all } k \ge 0 \qquad \text{(see (3.63))}, \\ \Rightarrow\ & C_k(\widehat{\varphi}_+, \infty) = 0 \quad \text{for all } k \ge 0 \qquad \text{(see (3.57) and (3.58))}. \end{aligned}$$

Similarly we show that $C_k(\widehat{\varphi}_-, \infty) = 0$ for all $k \ge 0$. □

Next we compute the critical groups of φ at the two constant sign smooth solutions u_0 and v_0 obtained in Proposition 3.6.

Proposition 3.10. *If hypotheses* (H) *hold and* $u_0 \in \operatorname{int} C_+$ *and* $v_0 \in -\operatorname{int} C_+$ *are the two constant sign smooth solutions of* (1.1) *obtained in Proposition* 3.6*, then* $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \ge 0$.

Proof. We do the proof for the pair (φ, u_0), the proof for (φ, v_0) being similar.

Suppose that $\{0, u_0\}$ are the only critical points of $\widehat{\varphi}_+$ (otherwise we already have a third nontrivial solution of (1.1) which in fact is positive and so we are done).

Let $\xi < 0 < \beta < \widehat{\eta}_+$ (see (3.30)) and consider the following triple of sets

$$\widehat{\varphi}_+^{\xi} \subseteq \widehat{\varphi}_+^{\beta} \subseteq H_n^1(\Omega).$$

For this triple, we consider the long exact sequence of singular homology groups

$$\ldots H_k(H_n^1(\Omega), \widehat{\varphi}_+^{\xi}) \xrightarrow{i_*} H_k(H_n^1(\Omega), \widehat{\varphi}_+^{\beta}) \xrightarrow{\partial_*} H_{k-1}(\widehat{\varphi}_+^{\beta}, \widehat{\varphi}_+^{\xi}) \ldots \tag{3.64}$$

Here i_* is the group homomorphism induced by the inclusion

$$(H_n^1(\Omega), \widehat{\varphi}_+^{\xi}) \xhookrightarrow{i} (H_n^1(\Omega), \widehat{\varphi}_+^{\beta})$$

and ∂_* is the boundary homomorphism. From the rank theorem, we have

$$\begin{aligned} \operatorname{rank} H_k(H_n^1(\Omega), \widehat{\varphi}_+^{\beta}) &= \operatorname{rank}(\ker \partial_*) + \operatorname{rank}(\operatorname{im} \partial_*) \\ &= \operatorname{rank}(\operatorname{im} i_*) + \operatorname{rank}(\operatorname{im} \partial_*) \end{aligned} \tag{3.65}$$

(due to the exactness of (3.64)).

From the choice of ξ and by Proposition 3.8 we know that

$$H_k(H^1_n(\Omega),\widehat{\varphi}^{\xi}_+) = C_k(\widehat{\varphi}_+,\infty) = 0 \quad \text{for all } k \geq 0 \;\Rightarrow\; \operatorname{im} i_* = \{0\}. \tag{3.66}$$

Also we have $H_{k-1}(\widehat{\varphi}^{\beta}_+,\widehat{\varphi}^{\xi}_+) = C_{k-1}(\widehat{\varphi}_+,0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see Proposition 3.5). Therefore

$$\operatorname{rank}(\operatorname{im}\partial_*) \leq 1. \tag{3.67}$$

Finally note that from the choice of $\beta > 0$, we have

$$H_k(H^1_n(\Omega),\widehat{\varphi}^{\beta}_+) = C_k(\widehat{\varphi}_+,u_0) \quad \text{for all } k \geq 0. \tag{3.68}$$

Returning to (3.65) and using (3.66)–(3.68), we obtain

$$\operatorname{rank} C_1(\widehat{\varphi}_+,u_0) \leq 1. \tag{3.69}$$

From the proof of Proposition 3.6 we know that u_0 is a critical point of mountain pass type for $\widehat{\varphi}_+$. Hence $C_1(\widehat{\varphi}_+,0) \neq 0$ (see, for example, Chang [5, p. 89]) and this combined with (3.69) implies

$$C_k(\widehat{\varphi}_+,u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \quad 3.70$$

We consider the homotopy $\widetilde{h}_+\colon [0,1]\times H^1_n(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{h}_+(t,u) = (1-t)\varphi(u) + t\widehat{\varphi}_+(u) \quad \text{for all } (t,u) \in [0,1]\times H^1_n(\Omega).$$

Claim. We can always assume that there exists $\rho \in (0,1)$ small such that u_0 is the only critical point of $\widetilde{h}_+(t,\,\cdot\,)$ for all $t \in [0,1]$ in

$$\overline{B}_\rho(u_0) = \{u \in H^1_n(\Omega) : ||u-u_0|| \leq \rho\}.$$

Indeed, if no such small $\rho \in (0,1)$ can be found, then there exist $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq H^1_n(\Omega)$ such that

$$t_n \to t \quad \text{in } [0,1], \quad \widetilde{u}_n \to u_0 \quad \text{in } H^1_n(\Omega), \quad (\widetilde{h}_+)'_u(t_n,u_n) = 0, \quad n \geq 1. \tag{3.71}$$

From the equation in (3.71), we have

$$\begin{aligned} A(\widetilde{u}_n) + t_n\varepsilon\widetilde{u}_n =& (1-t_n)N_f(\widetilde{u}_n) + t_n N_{\widehat{f}_+}(\widetilde{u}_n) && \text{for all } n \geq 1, \\ \Rightarrow \qquad -\Delta\widetilde{u}_n(z) =& f(z,\widetilde{u}^+_n(z)) \\ &+ (1-t_n)f(z,-\widetilde{u}^-_n(z)) + t_n\varepsilon u^-_n(z) && \text{a.e. in } \Omega, \\ \frac{\partial\widetilde{u}_n}{\partial n} =& 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.72}$$

(see (3.1)–(3.2)). From (3.72), regularity theory and Theorem 2 of Lieberman [9], we can find $\tau \in (0,1)$ and $M_4 > 0$ such that

$$\widetilde{u}_n \in C^{1,\tau}(\overline{\Omega}) \quad \text{and} \quad ||\widetilde{u}_n||_{C^{1,\tau}(\overline{\Omega})} \leq M_4 \quad \text{for all } n \geq 1. \tag{3.73}$$

Recalling that $C^{1,\tau}(\overline{\Omega})$ is embedded compactly in $C^1(\overline{\Omega})$, from (3.71) and (3.73) it follows that we have

$$\widetilde{u}_n \to u_0 \quad \text{in } C^1_n(\overline{\Omega}).$$

Since $u_0 \in \text{int}\, C_+$ (see Proposition 3.6), we can find integer $n_0 \geq 1$ such that

$$\begin{aligned} \widetilde{u}_n \in \text{int}\, C_+ \quad & \text{for all } n \geq n_0, \\ \Rightarrow -\Delta \widetilde{u}_n(z) = f(z, \widetilde{u}_n(z)) \quad & \text{a.e. in } \Omega, \\ \frac{\partial \widetilde{u}_n}{\partial n} = 0 \quad & \text{on } \partial\Omega, \\ & \text{for all } n \geq n_0 \text{ (see (3.72))} \end{aligned}$$

and we conclude that $\{\widetilde{u}_n\}_{n \geq n_0} \subseteq \text{int}\, C_+$ are nontrivial smooth distinct solutions of problem (1.1). This establishes the Claim.

Then the Claim and the homotopy invariance of the critical groups, imply

$$C_k(\varphi, u_0) = C_k(\widehat{\varphi}_+, u_0) \quad \text{for all } k \geq 0, \ \Rightarrow\ C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0$$

(see (3.70)). Similarly we show that $C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. □

Now we are ready for the full multiplicity theorem for problem (1.1).

Theorem 3.11. *If hypotheses* (H) *hold, then problem* (1.1) *has at least three nontrivial smooth solutions*

$$u_0 \in \text{int}\, C_+, \quad v_0 \in -\text{int}\, C_+, \quad y_0 \in C^1_n(\overline{\Omega}).$$

Proof. From Proposition 3.6, we already have two nontrivial smooth solutions of constant sign

$$u_0 \in \text{int}\, C_+, \quad v_0 \in -\text{int}\, C_+.$$

From Proposition 3.10 we know that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{3.74}$$

From Proposition 3.5, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{3.75}$$

Finally from Lemma 3.8, we know that $C_{d_m}(\varphi, \infty) \neq 0$. So, we can find $y_0 \in K_\varphi$ such that

$$C_{d_m}(\varphi, y_0) \neq 0 \quad \text{and} \quad d_m \geq 2. \tag{3.76}$$

Comparing (3.76) with (3.74) and (3.75), we see that $y_0 \notin \{0, u_0, v_0\}$. Also $y_0 \in C^1_n(\overline{\Omega})$ (regularity theory) solves (1.1). □

4. Existence theorem

In previous section, we allowed for double resonance in any spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, $m \geq 1$. In this section we investigate what happens if we have double resonance in the "principal" spectral interval $[0, \widehat{\lambda}_1]$. We show that in this case we can still guarantee existence of a nontrivial smooth solution.

The hypotheses on the reaction $f(z,x)$ are the following:

(H$'$) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0$ and

(a) $|f(z,x)| \leq \alpha(z) + c|x|$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$ with $\alpha \in L^\infty(\Omega)_+$, $c > 0$;

(b) $0 \leq \liminf\limits_{|x|\to\infty} \dfrac{f(z,x)}{x} \leq \limsup\limits_{|x|\to\infty} \dfrac{f(z,x)}{x} \leq \widehat{\lambda}_1$ uniformly for almost all $z \in \Omega$ and $\lim\limits_{|x|\to\infty} [f(z,x)x - 2F(z,x)] = \infty$ uniformly for almost all $z \in \Omega$;

(c) there exist an integer $\ell \geq 1$ and functions $\eta, \widehat{\eta} \in L^\infty(\Omega)_+$ such that $\widehat{\lambda}_\ell \leq \eta(z) \leq \widehat{\eta}(z) \leq \widehat{\lambda}_{\ell+1}$ for almost all $z \in \Omega$, $\widehat{\lambda}_\ell \neq \eta$, $\widehat{\lambda}_{\ell+1} \neq \widehat{\eta}$ and

$$\eta(z) \leq \liminf_{x\to 0} \frac{f(z,x)}{x} \leq \limsup_{x\to 0} \frac{f(z,x)}{x} \leq \widehat{\eta}(z)$$

uniformly for almost all $z \in \Omega$.

Example 4.1. The following function satisfies hypotheses (H$'$) (as before, for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} \eta x - c_0 x^3, & \text{if } |x| \leq 1, \\ \widehat{\lambda}_1 x - \dfrac{c_1}{x} & \text{if } |x| > 1, \end{cases}$$

with $c_0 > c_1 > 0, \eta - \widehat{\lambda}_1 = c_0 - c_1$ and $\eta \in (\widehat{\lambda}_\ell, \widehat{\lambda}_{\ell+1})$, $\ell \geq 1$.

As before $\varphi\colon H^1_n(\Omega) \to \mathbb{R}$ is the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega F(z,u(z))\,dz \quad \text{for all } u \in H^1_n(\Omega).$$

We know that $\varphi \in C^1_n(\overline{\Omega}))$. A careful reading of the proof of Proposition 3.4, reveals that it remains valid in the present setting too and so we have:

Proposition 4.2. *If hypotheses* (H$'$) *hold, then* φ *satisfies the* C*-condition.*

Let $\mu \in (\widehat{\lambda}_\ell, \widehat{\lambda}_{\ell+1})$ and let $\psi \in C^2(H^1_n(\Omega))$ be the functional defined by

$$\psi(u) = \frac{1}{2}||Du||_2^2 - \frac{\mu}{2}||u||_2^2 \quad \text{for all } u \in H^1_n(\Omega).$$

We have the following existence theorem.

Theorem 4.3. *If hypotheses* (H′) *hold, then problem* ((1.1) *has at least one nontrivial smooth solution* $\widehat{u} \in C^1(\overline{\Omega})$.

Proof. As in Perera [15] (see Lemma 4.1), we can find $R > 0$ and a functional $\widetilde{\varphi} \in C^1(H^1_n(\Omega))$ such that

$$\widetilde{\varphi}(u) = \begin{cases} \psi(u) & \text{if } ||u|| \leq R, \\ \varphi(u) & \text{if } ||u|| \geq \sqrt{2}R, \end{cases}$$

and $K_{\widetilde{\varphi}} \subseteq K_\varphi$. We have

$$C_k(\widetilde{\varphi}, 0) = C_k(\psi, 0) \quad \text{for all } k \geq 0. \tag{4.1}$$

Hypothesis (H′)(c) and Lemma 2.3 imply that $u = 0$ is a nondegenerate critical point of ψ with Morse index $d_\ell = \dim \overline{H}_\ell \geq 2$ (recall $\psi \in C^1(H^1_n(\Omega))$, $\ell \geq 1$). Hence

$$\begin{aligned} &C_k(\psi, 0) = \delta_{k,d_\ell}\mathbb{Z} \quad \text{for all } k \geq 0 \quad \text{(see [10, p. 188])}, \\ \Rightarrow\ &C_k(\widetilde{\varphi}, 0) = \delta_{k,d_\ell}\mathbb{Z} \quad \text{for all } k \geq 0 \quad \text{(see (4.1))}. \end{aligned} \tag{4.2}$$

On the other hand, we have

$$C_k(\widetilde{\varphi}, \infty) = C_k(\varphi, \infty) \quad \text{for all } k \geq 0. \tag{4.3}$$

Hypothesis (H′)(c) implies that

$$\varphi|_{E(\widehat{\lambda}_0)=\mathbb{R}} \text{ is anticoercive and } \varphi|_{\widehat{H}_1=\mathbb{R}^\perp} \text{ is coercive.}$$

So, by virtue of Proposition 3.8 of T. Bartsch and S. Li [2], we have

$$C_1(\varphi, \infty) \neq 0, \ \Rightarrow\ C_1(\widetilde{\varphi}, \infty) \neq 0 \quad \text{(see (4.3))}. \tag{4.4}$$

From (4.4) it follows that we can find $\widehat{u} \in K_{\widetilde{\varphi}}$ such that

$$C_1(\widetilde{\varphi}, \widehat{u}) \neq 0. \tag{4.5}$$

Comparing (4.2) and (4.5) we see that $\widehat{u} \neq 0$ (recall $d_\ell \geq 2$). Since $K_{\widetilde{\varphi}} \subseteq K_\varphi$ we have $\widehat{u} \in K_\varphi$ and so $\widehat{u} \in C^1_n(\overline{\Omega})$ (regularity theory) solves problem (1.1). □

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