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ON AN ASYMPTOTICALLY LINEAR SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. We prove the existence of positive solutions for the singular boundary value problems

$$\begin{cases} -\Delta u = \dfrac{p(x)}{u^\beta} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, 0 < \beta < 1$, $\lambda > 0$ is a small parameter, $f: (0,\infty) \to \mathbb{R}$ is asymptotically linear at ∞ and is possibly singular at 0.

1. Introduction

Consider the boundary value problems:

$$\text{(I)} \qquad \begin{cases} -\Delta u = \dfrac{p(x)}{u^\beta} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, 0 < \beta < 1$, $p: \Omega \to \mathbb{R}$, and $f: (0,\infty) \to \mathbb{R}$ may be singular at 0.

Singular problems of the type (I) have been studied extensively in recent years (see [3], [4], [6]–[10], [12]–[16] and the references therein). When f is continuous and nonnegative on $[0,\infty)$, $\lim_{u\to\infty} f(u)/u = m \in (0,\infty)$ and f satisfies some

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additional conditions at 0, Z. Zhang [16] show that (I) has a positive solution for $\lambda \in (0, \lambda_1/m)$, provided that $p \geq 0$, $p \not\equiv 0$, $p\phi_1^{-\beta} \in L^q(\Omega)$, $n/2 < q$. Here λ_1 and ϕ_1 are the first eigenvalue and corresponding positive eigenfunction of $-\Delta$ with Dirichlet boundary conditions. Related results when $p \equiv 0$ and f is nonsingular can be found in [1]. In this paper, we are interested in the case when f is asymptotically linear at ∞ and is possibly singular at 0, and p may be negative. Our results extend corresponding results [16]. In particular, our results when applied to the model cases

$$\begin{cases} -\Delta u = \dfrac{a}{u^\beta} + \lambda\left(\dfrac{b}{u^\delta} + u\left(1 + \dfrac{1}{u+1}\right)\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta u = \dfrac{a}{u^\beta} + \lambda\left(\dfrac{b}{u^\delta} + ue^{1/(1+u)}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $a, b \in \mathbb{R}$, $\beta, \delta \in (0,1)$ give the existence of a positive solution to (1.1) provided that λ is close enough to λ_1 on the left, and the existence of a positive solution to (1.2) if and only if $\lambda < \lambda_1$. Our approach is based on the method of sub- and supersolutions.

2. Preliminary results

We shall denote the norms in $L^p(\Omega), C^1(\overline{\Omega})$, and $C^{1,\alpha}(\overline{\Omega})$ by $||\cdot||_p$, $|\cdot|_1$ and $|\cdot|_{1,\alpha}$, respectively. Throughout the paper we assume that $||\phi_1||_\infty = 1$.

Let $d(x)$ denote the distance from x to the boundary of Ω.

We first establish a regularity result, which plays a crucial role in the proofs of the existence results.

LEMMA 2.1. *Let $h \in L^1(\Omega)$ and suppose that there exist numbers $\gamma \in (0,1)$ and $C > 0$ such that*

$$|h(x)| \leq \frac{C}{\phi_1^\gamma(x)} \tag{2.1}$$

for almost every $x \in \Omega$. Let $u \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then there exist constants $\alpha \in (0,1)$ and $M > 0$ depending only on C, γ, Ω such that $u \in C^{1,\alpha}(\overline{\Omega})$ and $|u|_{1,\alpha} < M$.

PROOF. Note that Lemma 2.1 was proved in [8] under the additional assumptions that $h \geq 0$ and $u \leq \widetilde{C}d$ in Ω for some $\widetilde{C} > 0$.

It follows from [4] that the problem

$$\begin{cases} -\Delta v = \dfrac{1}{v^\gamma} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has a positive solution v which is Lipschitz continuous in $\overline{\Omega}$.Let C_1, $C_2 > 0$ be such that $v(x) \le C_1 d(x) \le C_2 \phi_1(x)$ in Ω. Then

$$-\Delta(CC_2^\gamma v) \ge \frac{C}{\phi_1^\gamma} \quad \text{in } \Omega.$$

Let $\widetilde{u}$ be the solution of

$$\begin{cases} -\Delta\widetilde{u} = |h| & \text{in } \Omega, \\ \widetilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $\overline{u} = u + \widetilde{u}$. Then

$$-\Delta\overline{u} = h + |h| \quad \text{in } \Omega.$$

By the maximum principle, $\widetilde{u}(x) \le CC_2^\gamma v(x) \le C_3 d(x)$ and $\overline{u}(x) \le 2C_3 d(x)$ for $x \in \Omega$. Using the regularity result in [8], we conclude that there exist $\alpha \in (0,1)$ and $M_0 > 0$ such that $\widetilde{u}, \overline{u} \in C^{1,\alpha}(\overline{\Omega})$ and $|\widetilde{u}|_{1,\alpha}, |\overline{u}|_{1,\alpha} < M_0$. Since $u = \overline{u} - \widetilde{u}$, Lemma 2.1 follows. □

Remark 2.2. Note that under the assumptions of Lemma 2.1, (2.2) has a unique solution $u \in H_0^1(\Omega)$. Indeed, for $u, \xi \in H_0^1(\Omega)$, define

$$a(u,\xi) = \int_\Omega \nabla u.\nabla\xi\,dx, \qquad \widehat{h}(\xi) = \int_\Omega h\xi\,dx.$$

Then $a(u,\xi)$ is bilinear, continuous, and coercive on $H_0^1(\Omega)\times H_0^1(\Omega)$. By Hardy's inequality (see e.g. [2, p. 194]) and the fact that d/ϕ_1 is bounded in Ω, we obtain

$$|\widehat{h}(\xi)| \le k_1 \int_\Omega \left|\frac{\xi}{d^\gamma}\right| dx \le k_1 ||d||_\infty^{1-\gamma} \int_\Omega \left|\frac{\xi}{d}\right| dx \le k_2 ||\nabla\xi||_2,$$

for all $\xi \in H_0^1(\Omega)$, where k_1, k_2 are constants independent on ξ. Thus $\widehat{h} \in H^{-1}(\Omega)$ (the dual of $H_0^1(\Omega)$), and the Lax–Milgram Theorem (see [2, Corollary V.8]) implies the existence of a unique $u \in H_0^1(\Omega)$ such that $a(u,\xi) = \widehat{h}(\xi)$ for all $\xi \in H_0^1(\Omega)$.

Lemma 2.3. *Let $h \in L^1(\Omega)$ satisfy* (2.1) *and let u be the solution of* (2.2). *Then $|u|_1 \to 0$ as $||h||_1 \to 0$.*

Proof. By Lemma 2.1, there exists $M > 0$ such that $|u|_{1,\alpha} < M$. Multiplying the equation in (2.2) by u and integrating gives

$$||\nabla u||_2^2 = \int_\Omega hu\,dx \le ||u||_\infty ||h||_1 \le M||h||_1,$$

which implies $u \to 0$ in $L^2(\Omega)$ as $||h||_1 \to 0$. Since $C^{1,\alpha}(\overline{\Omega})$ is compactly imbedded in $C^1(\overline{\Omega})$, it follows that $u \to 0$ in $C^1(\overline{\Omega})$ as $||h||_1 \to 0$. □

Now, consider the problem:

$$\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $h:\Omega \times (0,\infty) \to \mathbb{R}$ is continuous. Let $\phi, \psi \in C^1(\overline{\Omega})$ satisfy $\phi, \psi \geq l\phi_1$ in Ω for some $l > 0$ and suppose there exist$\gamma \in (0,1)$ and $C > 0$ such that

$$|h(x,w)| \leq \frac{C}{\phi_1^\gamma(x)} \tag{$*$}$$

for almost every $x \in \Omega$ and all $w \in C(\overline{\Omega})$ with $\phi \leq w \leq \psi$ in Ω. Suppose ϕ, ψ are sub- and supersolution of (2.3), respectively, i.e. for all $\xi \in H_0^1(\Omega)$ with $\xi \geq 0$,

$$\int_\Omega \nabla\phi.\nabla\xi\,dx \leq \int_\Omega h(x,\phi)\xi\,dx, \qquad \int_\Omega \nabla\psi.\nabla\xi\,dx \geq \int_\Omega h(x,\psi)\xi\,dx.$$

Note that the integrals on the right-hand side are defined by virtue of Hardy's inequality.

Lemma 2.4. *Under the above assumptions, there exists $\alpha \in (0,1)$ such that (2.3) has a solution $u \in C^{1,\alpha}(\overline{\Omega})$.*

Proof. For each $v \in C(\overline{\Omega})$, define $\widetilde{h}(x,v) = h(x, \min(\max(v,\phi),\psi))$. Then, in view of ($*$), we have

$$|\widetilde{h}(x,v)| \leq \frac{C}{\phi_1^\gamma(x)}$$

for almost every $x \in \Omega$, where C is a positive constant independent on v. Hence, it follows from Remark 2.2 and Lemma 2.1 that for each $v \in C(\overline{\Omega})$, the problem

$$\begin{cases} -\Delta u = \widetilde{h}(x,v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in C^{1,\alpha}(\overline{\Omega})$ with $|u|_{1,\alpha} < C$, where $\alpha \in (0,1)$ and $C > 0$ are constants independent on v. Define $Tv = u$. Then T is a bounded, compact, and continuous operator on $C(\overline{\Omega})$. Note that the continuity of T follows from Lemma 2.3, the fact that $1/\phi_1^\gamma \in L^1(\Omega)$, and the Lebesgue dominated convergence. Hence T has a fixed point u by Schauder fixed point theorem. Using standard arguments (see e.g. [5], [11]), we obtain $\phi \leq u \leq \psi$ in Ω, and Lemma 2.4 follows. □

3. Main results

We make the following assumptions:

(A.1) $p \in L^\infty(\Omega)$.

(A.2) $f:(0,\infty) \to \mathbb{R}$ is continuous and there exists $\delta \in (0,1)$ such that

$$\limsup_{u\to 0^+} u^\delta |f(u)| < \infty.$$

(A.3) There exist positive numbers m, k, A such that

$$\lim_{u\to\infty}\frac{f(u)}{u}=m \quad \text{and} \quad f(u)\geq mu+k \quad \text{for } u\geq A.$$

Let $\lambda_\infty=\lambda_1/m$. Then we have:

Theorem 3.1. *Let* (A.1)–(A.3) *hold. Then there exists a positive number* ε *such that for* $\lambda\in(\lambda_\infty-\varepsilon,\lambda_\infty)$*, problem* (I) *has a positive solution* $u_\lambda\in C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha\in(0,1)$*. Furthermore,*

$$u_\lambda\geq\frac{k\lambda_\infty}{4(\lambda_\infty-\lambda)}\phi_1 \quad \text{in } \Omega.$$

Theorem 3.2. *Let* (A.2) *hold,* $f\geq 0$ *and suppose*

(A.3') $\limsup_{u\to\infty}f(u)/u=m$ *for some* $m\in(0,\infty)$.

In addition, assume $p\geq 0$, $p\not\equiv 0$ *in* Ω *and either* (A.1) *or*

(A.1') $p\phi_1^{-\beta}\in L^q(\Omega)$ *for some* $q>n$

holds. Then, for $\lambda\in(0,\lambda_\infty)$, (I) *has a positive solution* $u_\lambda\in C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha\in(0,1)$*. If, in addition,* $f(u)\geq mu$ *for all* $u>0$*, then* (I) *has no positive solutions for* $\lambda\geq\lambda_\infty$*.*

Remark 3.3. When $p\equiv 0$ and f is nonsingular, the existence result in Theorem 3.1 was obtained in [1] using bifurcation theory. Theorem 3.2 improves Theorem 1 in [16], where f is required to be continuous on $[0,\infty)$, $f(0)=0$, and $\lim_{u\to 0^+}f(u)/u=m_1$.

Remark 3.4. It should be noted that Theorem 3.1 may not be true if $k=0$ in (A.3). Indeed, consider the problem

$$(**) \qquad \begin{cases} -\Delta u=-\dfrac{1}{u^\beta}+\lambda u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Then, by multiplying the equation in $(**)$ by ϕ_1 and integrating, we see that $(**)$ does not have any positive solutions for $\lambda<\lambda_1=\lambda_\infty$.

We are ready to give the proofs of the main results. Without loss of generality, we assume $m=1$.

Proof of Theorem 3.1. Let $\lambda_1/2<\lambda<\lambda_1$ and $c=k\lambda_1/(4(\lambda_1-\lambda))$. Let ϕ_0,z_0 satisfy

$$-\Delta\phi_0=\begin{cases}\lambda(c+k)\phi_1 & \text{if } \phi_1\geq A/c,\\ 0 & \text{if } \phi_1<A/c,\end{cases} \qquad \phi_0=0 \quad \text{on } \partial\Omega,$$

and

$$-\Delta z_0 = \begin{cases} \lambda(c+k)\phi_1 & \text{in } \Omega, \\ z_0 = 0 & \text{on } \partial\Omega, \end{cases} \qquad z_0 = 0 \quad \text{on } \partial\Omega.$$

Note that $z_0 = (\lambda(c+k)/\lambda_1)\phi_1$. Then

$$-\Delta(z_0 - \phi_0) = h \equiv \begin{cases} 0 & \text{if } \phi_1 \geq A/c, \\ \lambda(c+k)\phi_1 & \text{if } \phi_1 < A/c, \end{cases}$$

Note that

$$|\lambda(c+k)\phi_1| \leq \lambda_1(A+k)$$

if $\phi_1 < A/c$, and so $||h||_1 \to 0$ as $\lambda \to \lambda_1^-$. Hence Lemma 2.3 implies

$$|z_0 - \phi_0|_1 \to 0 \quad \text{as } \lambda \to \lambda_1^-.$$

Let $c_0 > 0$ be such that $d \leq c_0\phi_1$ in Ω. Then there exists $\varepsilon > 0$ such that, for $\lambda_1 - \lambda < \varepsilon$, we have

$$|\phi_0 - z_0|_1 < \frac{k}{8c_0}.$$

Hence, for such λ,

$$\phi_0 \geq z_0 - \frac{k}{8c_0}d \geq z_0 - \frac{k}{8}\phi_1 = \left(\frac{\lambda(c+k)}{\lambda_1} - \frac{k}{8}\right)\phi_1$$

in Ω. Since $\lambda > \lambda_1/2$, this implies

$$\phi_0 \geq \left(\frac{\lambda c}{\lambda_1} + \frac{3k}{8}\right)\phi_1 = \left(\frac{k\lambda_1}{4(\lambda_1 - \lambda)} + \frac{k}{8}\right)\phi_1 = \left(c + \frac{k}{8}\right)\phi_1$$

in Ω. Let z be the solution of

$$\begin{cases} -\Delta z = \dfrac{1}{\phi_1^\gamma} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\gamma = \max(\beta, \delta)$, and let $c_1 > 0$ be such that $z \leq c_1\phi_1$ in Ω. Then

$$\phi_0 \geq c\phi_1 + k_1 z,$$

in Ω, where $k_1 = k/8c_1$. By decreasing ε further if necessary, we can assume that

$$\frac{\lambda_1 K}{c^\delta} + \frac{||p||_\infty}{c^\beta} < k_1,$$

where$K > 0$ is such that

$$|f(u)| \leq \frac{K}{u^\delta} \tag{3.2}$$

for $u \in (0, A)$. Note that the existence of K follows from (A.2).

Let $\phi = \phi_0 - k_1 z$. Then $\phi \geq c\phi_1$ in Ω. We shall verify that ϕ is a subsolution of (I). Let $\xi \in H_0^1(\Omega)$ with $\xi \geq 0$. Then

$$\begin{aligned}(3.3)\qquad \int_\Omega \nabla\phi.\nabla\xi\,dx &= \int_\Omega (-\Delta\phi)\xi\,dx = \int_\Omega (-\Delta\phi_0)\xi\,dx - k_1\int_\Omega \frac{\xi}{\phi_1^\gamma}\,dx \\ &= \lambda\int_{\phi_1>A/c}(c+k)\phi_1\xi\,dx - k_1\int_\Omega \frac{\xi}{\phi_1^\gamma}\,dx.\end{aligned}$$

If $\phi_1(x) > A/c$ then $\phi(x) \geq A$ and so

$$f(\phi(x)) \geq \phi(x) + k \geq (c+k)\phi_1(x),$$

which implies

$$(3.4)\qquad \lambda\int_{\phi_1>A/c} f(\phi)\xi\,dx \geq \lambda\int_{\phi_1>A/c}(c+k)\phi_1\xi\,dx.$$

On the other hand, using (3.2) and the fact that $f(u) > 0$ for $u > A$, we get

$$\begin{aligned}(3.5)\quad \lambda\int_{\phi_1<A/c} f(\phi)\xi\,dx &\geq \lambda\int_{(\phi_1<A/c)\cap(\phi<A)} f(\phi)\xi\,dx \geq -\int_{\phi<A}\frac{\lambda K\xi}{\phi^\delta}\,dx \\ &\geq -\frac{\lambda K}{c^\delta}\int_\Omega\frac{\xi}{\phi_1^\delta}\,dx \geq -\frac{\lambda_1 K}{c^\delta}\int_\Omega \frac{\xi}{\phi_1^\gamma}\,dx.\end{aligned}$$

Also

$$(3.6)\quad \int_\Omega \frac{p(x)}{\phi^\beta}\xi\,dx \geq -||p||_\infty\int_\Omega\frac{\xi}{\phi^\beta}\,dx \geq -\frac{||p||_\infty}{c^\beta}\int_\Omega\frac{\xi}{\phi_1^\beta}\,dx \geq -\frac{||p||_\infty}{c^\beta}\int_\Omega\frac{\xi}{\phi_1^\gamma}\,dx.$$

Combining (3.3)–(3.6), we obtain

$$\int_\Omega \nabla\phi.\nabla\xi\,dx \leq \int_\Omega\left(\frac{p(x)}{\phi^\beta} + \lambda f(\phi)\right)\xi dx,$$

i.e. ϕ is a subsolution of (I).

Next, we shall construct a supersolution ψ of (I) with $\psi \geq \phi$. Let λ, c be as in the above and let $a > 1$ be such that

$$\lambda a < \lambda_1.$$

By (A.2) and (A.3), there exist $B, L > 0$ such that

$$(3.7)\qquad f(u) \leq au$$

for $u > B$, and

$$(3.8)\qquad |f(u)| \leq \frac{L}{u^\delta}$$

for $u < B$. Let$M_0 = \lambda L + ||p||_\infty$ and $M > \max\{(\lambda a c_1 M_0)/(\lambda_1 - \lambda a), 1\}$, where $c_1 > 0$ is such that $z \leq c_1\phi_1$ in Ω and z is defined in (3.1).

Let $\psi = M\phi_1 + M_0 z$. We shall verify that ψ is a supersolution of (I). Let $\xi \in H_0^1(\Omega)$ with $\xi \geq 0$. Then

$$\int_\Omega \nabla\psi.\nabla\xi \, dx = \lambda_1 M \int_\Omega \xi\phi_1 \, dx + M_0 \int_\Omega \frac{\xi}{\phi_1^\gamma} \, dx. \tag{3.9}$$

We have

$$\lambda \int_\Omega f(\psi)\xi \, dx = \lambda \int_{\psi>B} f(\psi)\xi \, dx + \lambda \int_{\psi<B} f(\psi)\xi \, dx. \tag{3.10}$$

By (3.7),

$$\begin{aligned} \lambda \int_{\psi>B} f(\psi)\xi \, dx &\leq \lambda a \int_{\psi>B} \psi\xi \, dx \\ &\leq \lambda a M \int_{\psi>B} \phi_1\xi \, dx + \lambda a M_0 \int_{\psi>B} z\xi \, dx \\ &\leq \lambda a M \int_{\psi>B} \phi_1\xi \, dx + \lambda a c_1 M_0 \int_{\psi>B} \phi_1\xi \, dx \\ &\leq \lambda_1 M \int_\Omega \xi\phi_1 \, dx. \end{aligned} \tag{3.11}$$

Next, using (3.8), we obtain

$$\lambda \int_{\psi<B} f(\psi)\xi \, dx \leq \lambda L \int_{\psi<B} \frac{\xi}{\psi^\delta} \, dx \leq \lambda L \int_{\psi<B} \frac{\xi}{\phi_1^\delta} \, dx \leq \lambda L \int_\Omega \frac{\xi}{\phi_1^\gamma} \, dx \tag{3.12}$$

Finally,

$$\int_\Omega \frac{p(x)\xi}{\psi^\beta} \, dx \leq ||p||_\infty \int_\Omega \frac{\xi}{\phi_1^\beta} \, dx \leq ||p||_\infty \int_\Omega \frac{\xi}{\phi_1^\gamma} \, dx. \tag{3.13}$$

Combining (3.9)–(3.13), we obtain

$$\int_\Omega \nabla\psi.\nabla\xi \, dx \geq \int_\Omega \left(\frac{p(x)}{\psi^\beta} + \lambda f(\psi)\right)\xi \, dx,$$

i.e. ψ is a supersolution of (I). Lemma 2.4 now gives the existence of a $C^{1,\alpha}(\overline{\Omega})$ solution u of (I) with $u \geq c\phi_1$ in Ω. □

PROOF OF THEOREM 3.2. Under the assumptions on p, it follows from Lemma 2.1 or regularity results (see e.g. [2]) that the problem

$$\begin{cases} -\Delta w = \dfrac{p(x)}{\phi_1^\beta} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

has a solution $w \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Let $m_0, m_1 > 0$ be such that $m_0\phi_1 \leq w \leq m_1\phi_1$ in Ω. For $v \in C(\overline{\Omega})$, let $u = Tv$ be the solution of

$$\begin{cases} -\Delta\phi = \dfrac{p(x)}{\max^\beta(v, c\phi_1)} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $c > 0$ is a small number so that $c^{1-\beta^2} \le m_1^{-\beta} m_0$ and $c^{1+\beta} \le m_1$. Then T is a bounded compact mapping on $C(\overline{\Omega})$ by Lemmas 2.1 and 2.3. Hence T has a fixed point ϕ. We claim that $\phi \ge c\phi_1$ in Ω. Indeed, since

$$-\Delta\phi \le \frac{p(x)}{c^\beta \phi_1^\beta}$$

in Ω, it follows from the weak maximum principle that

$$\phi \le c^{-\beta} w \le c^{-\beta} m_1 \phi_1$$

in Ω. Hence

$$-\Delta\phi \ge \frac{p(x)}{\max^\beta(c^{-\beta} m_1, c)\phi_1^\beta} = \frac{c^{\beta^2} m_1^{-\beta} p(x)}{\phi_1^\beta}$$

in Ω, and so

$$u \ge c^{\beta^2} m_1^{-\beta} w \ge c^{\beta^2} m_1^{-\beta} m_0 \phi_1 \ge c\phi_1$$

in Ω. Thus ϕ is a solution of

$$\begin{cases} -\Delta\phi = \dfrac{p(x)}{\phi^\beta} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and since $f \ge 0$, it is easily seen that ϕ is a subsolution of (I). The existence of a supersolution ψ with $\psi \ge \phi$ is derived exactly as in the proof of Theorem 3.1. Finally, the nonexistence result under the additional assumption follows upon multiplying the equation by ϕ_1 and integrating. □

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