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OPTIMAL REGULARITY OF STABLE MANIFOLDS OF NONUNIFORMLY HYPERBOLIC DYNAMICS

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Abstract. We establish the existence of smooth invariant stable manifolds for differential equations $u' = A(t)u + f(t,u)$ obtained from sufficiently small perturbations of a *nonuniform* exponential dichotomy for the linear equation $u' = A(t)u$. One of the main advantages of our work is that the results are optimal, in the sense that the invariant manifolds are of class C^k if the vector field is of class C^k. To the best of our knowledge, in the nonuniform setting this is the first general optimal result (for a large family of perturbations and not for some specific perturbations). Furthermore, in contrast to some former works, we do not require a strong nonuniform exponential behavior (we note that contrarily to what happens for autonomous equations, in the nonautonomous case a nonuniform exponential dichotomy need not be strong). The novelty of our proofs, in this setting, is the use of the fiber contraction principle to establish the smoothness of the invariant manifolds. In addition, we can also consider linear perturbations, and our results have thus immediate applications to the robustness of nonuniform exponential dichotomies.

1. Introduction

1.1. The stable manifold theorem. We consider a linear nonautonomous differential equation

$$u' = A(t)u, \tag{1.1}$$

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with $A(t)$ varying continuously with $t \geq 0$. We note that under this assumption all solutions of the equation are global. Assuming that equation (1.1) has a *nonuniform* exponential dichotomy (see Section 2 for the definition), we establish the existence of smooth invariant stable manifolds under sufficiently small nonlinear perturbations. We point out that it is easy to show that if an autonomous linear equation has a nonuniform exponential dichotomy, then the dichotomy must in fact be uniform. This is why in the context of *nonuniform* exponential behavior we are only interested in perturbations of *nonautonomous* linear differential equations. It turns out that the classical notion of (uniform) exponential dichotomy is very stringent for the dynamics and it is of interest to look for more general types of hyperbolic behavior. These generalizations can be much more typical than the notion of uniform exponential dichotomy. This is precisely what happens with the notion of nonuniform exponential dichotomy. In fact, essentially any linear equation as in (1.1) with nonzero Lyapunov exponents has such a dichotomy. We refer to [1], [6] for a precise formulation of the results, and for detailed discussions. On the other hand, there also exist large classes of linear differential equations with uniform exponential dichotomies, and the corresponding theory and its applications are widely developed. We refer to the books [8], [10], [11], [20] for details and references related to uniform exponential dichotomies.

In order to formulate our stable manifold theorem and some of its consequences, we consider the nonlinear equation

$$u' = A(t)u + f(t, u) \tag{1.2}$$

in $X = \mathbb{R}^p$, where A and f are C^1 functions such that $f(t, 0) = f(t, u) = 0$ for every $t \geq 0$ and $u \in X$ with $\|u\| \geq c$, for some constant $c > 0$. Clearly, $u(t) = 0$ is still a solution of equation (1.2). The following is our stable manifold theorem.

Theorem 1.1. *If equation* (1.1) *has a nonuniform exponential dichotomy, and*

$$\left\| \frac{\partial f}{\partial u}(t, u) \right\| \leq c e^{-t/c} \tag{1.3}$$

for every $t \geq 0$ *and* $u \in X$*, for some sufficiently small constant* $c > 0$*, then the zero solution of equation* (1.2) *has a* C^1 *invariant stable manifold.*

This is a simple consequence of Theorems 4.1 and 4.7 (that also establish the uniqueness of the stable manifold as a graph in a certain family of Lipschitz functions). We emphasize that Theorem 1.1 is an optimal result, in the sense that the stable manifold is as regular as the vector field. Reversing time we can also obtain invariant unstable manifolds with optimal regularity. It turns out that the exponential decay required in (1.3) compensates in an optimal manner for the nonuniform exponential behavior of the linear equation (1.2). As usual, the stable manifold is obtained as a graph over the stable subspace, or more precisely, since we are in the nonautonomous case, over the family of stable subspaces indexed by time. Nevertheless, the extra small exponentials in the notion of nonuniform exponential dichotomy substantially complicate this procedure. More precisely, the existence of a nonuniform exponential dichotomy for equation (1.1) ensures the existence of stable and unstable subspaces $E(t)$ and $F(t)$ for each $t \geq 0$ (see Section 2 for details) such that $X = E(t) \oplus F(t)$. We look for the stable manifold as the graph

$$V_\phi = \{(t, u, \phi(t, u)) : t \geq 0 \text{ and } u \in E(t)\}$$

of a function $\phi \colon \mathbb{R}_0^+ \times X \to X$ such that

$$\phi(t, 0) = 0 \quad \text{and} \quad \phi(t, E(t)) \subset F(t) \tag{1.4}$$

for each $t \geq 0$. The first property ensures that $\mathbb{R}_0^+ \times \{0\} \subset V_\phi$, and the second says that V_ϕ is a collection of graphs over the stable subspaces. The invariance of the stable manifold means that the set V_ϕ is invariant under the flow defined by the autonomous equation

$$t' = 1, \quad u' = A(t)u + f(t, u).$$

We emphasize that since equation (1.2) may not be autonomous, in general it does not define a flow, and thus in general the problem of finding an invariant stable manifold for this equation (without adding the component $t' = 1$) is simply meaningless. For the C^1 smoothness of the set V_ϕ we could try to show that the function ϕ is of class C^1. However, in general it makes no sense to discuss the regularity in the variable t, the reason being that the spaces $E(t)$ may change with t. Instead, we first show that for each $t \geq 0$ the map $E(t) \ni u \mapsto \phi(t, u)$ is of class C^1, and then we use this property to construct a C^1 parametrization of V_ϕ. The proof of the C^1 regularity uses the fiber contraction principle together with a modification of an argument sketched in [7] to establish the continuity of the fiber contraction, now in the nonautonomous setting.

1.2. Relation to former work. We discuss in this section the hypotheses of Theorem 1.1, as well as its relations to former work.

Our results are a contribution to the theory of nonuniform hyperbolicity. We refer to [1] for a detailed exposition of the theory, which goes back to the landmark works of V. Oseledets [13] and particularly Ya. Pesin [14]–[16]. We note that the notion of nonuniform hyperbolicity (here reformulated in terms of nonuniform exponential dichotomies) is related to the Lyapunov exponents. For example, almost all trajectories of a dynamical system preserving a finite invariant measure with nonzero Lyapunov exponents are nonuniformly hyperbolic. We emphasize that uniform hyperbolicity, although being robust, occurs much less than nonuniform hyperbolicity. Among the most important properties due to the nonuniform hyperbolicity is precisely the existence of invariant stable and unstable manifolds, established by Ya. Pesin in [14] with an elaboration of the classical work of Perron. In [18] D. Ruelle obtained a proof of the stable manifold theorem based on the study of perturbations of products of matrices in Oseledets' multiplicative ergodic theorem [13]. Another proof was given by C. Pugh and M. Shub in [17] with an elaboration of the classical work of Hadamard using graph transform techniques. In [9] A. Fathi, M. Herman and J.-C. Yoccoz provided a detailed exposition of the stable manifold theorem essentially following the approaches of Ya. Pesin and D. Ruelle.

There are also versions of the stable manifold theorem for some classes of dynamical systems in infinite-dimensional spaces. In [19] D. Ruelle established a version of the theorem in Hilbert spaces, following his approach in [18]. In [12] R. Mañé considered transformations in Banach spaces under certain compactness and invertibility assumptions, including the case of differentiable maps with compact derivative at each point.

We note that in all these works the vector field is assumed to be of class $C^{k+\varepsilon}$ for some $\varepsilon > 0$. C. Pugh and M. Shub obtained in [17] the optimal regularity of the stable manifolds for diffeomorphisms in finite-dimensional manifolds. Namely, they showed that the stable manifolds are of class $C^{k+\varepsilon}$ if the dynamics is of class $C^{k+\varepsilon}$. We point out that some parts of the nonuniform hyperbolicity theory may fail for an arbitrary C^1 dynamics, such as the absolute continuity of the families of stable and unstable manifolds, although some statements may hold for some classes of C^1 dynamics. In particular, see [21] for an entropy formula for a C^1 generic surface diffeomorphism preserving a measure absolutely continuous with respect to the volume, and see [2] for the existence of C^1 invariant stable manifolds for a certain class of dynamics that are only of class C^1.

Now we would like to point out some drawbacks of our approach in [2] that are circumvented in the present work:

Strong exponential dichotomies. In [2] the exponential dichotomy needs to be strong, in the sense that not only we assume exponential bounds for the evolution into the future in the stable direction and into the past in the unstable direction, but also for the evolution into the past in the stable direction and into the future in the unstable direction. This is due to some estimates that we were unable to obtain otherwise. We note that for an autonomous equation a nonuniform exponential dichotomy is necessarily strong, but in general this property fails for nonautonomous equations.

Asymptotic behavior of the perturbations. In [2] the perturbation is assumed to satisfy

$$\|f(t,u)-f(t,v)\| \le c\|u-v\|(\|u\|^q+\|v\|^q) \tag{1.5}$$

for some constants $c>0$ and $q>1$. This prevents for example the consideration of any perturbation that behaves as a multiple of u^2 near the origin, and thus it creates difficulties when we are trying to obtain (1.5) by using some cutoff function. We also would like to explain the relation of condition (1.5) to condition (1.3). When we obtain invariant stable manifolds in neighbourhoods with exponentially decaying size (as in [2]), say of size $\delta e^{-\varepsilon t/q}$ at time t (where the exponential rate is assumed to be small when compared to the Lyapunov exponents), it follows readily from (1.5) that in this neighbourhood we have

$$\|f(t,u)-f(t,v)\| \le 2c\delta e^{-\varepsilon t}\|u-v\|, \tag{1.6}$$

that is, (1.6) holds whenever $\|u\|,\|v\| \le \delta e^{-\varepsilon t/q}$. Provided that $f=0$ outside those neighbourhoods, which of course does not affect the behavior of the invariant manifolds inside the neighbourhoods, this yields condition (1.3). So in fact we can consider (1.3) as a generalization of condition (1.5). The assumption that the perturbation f vanishes outside the neighbourhoods with exponentially decaying size was considered in [3], although this condition may be difficult to obtain. We refer to [4] for a related approach, which allows one to obtain C^1 invariant manifolds, although at the expense of assuming much stronger hypotheses on the (nonlinear) part of the vector field: in addition to (1.3) it is also assumed in that paper that $\|f(t,u)\| \le ce^{-t/c}$ and

$$\left\|\frac{\partial f}{\partial u}(t,u)-\frac{\partial f}{\partial u}(t,v)\right\| \le ce^{-t/c}\|u-v\|$$

for every $t \ge 0$ and $u,v \in X$.

Linear perturbations. In [2], due to condition (1.5), the perturbation was not allowed to have a nonzero linear part (in fact, as explained above, it is also not possible for example to consider a perturbation behaving as u^2 near the origin). Although this is a natural assumption when studying the existence of invariant stable manifolds, it is nevertheless a drawback for some applications that the same proof does not allow sufficiently small linear perturbations. On the contrary, our present assumption (1.3) allows a certain class of linear perturbations, and thus our results have applications to the robustness problem of nonuniform exponential dichotomies.

Stable and unstable subspaces. In [2] we always assume that the exponential dichotomy has stable and unstable subspaces independent of time. While for uniform exponential dichotomies the changes required to consider arbitrary stable and unstable subspaces are straightforward, since the "angle" between them is uniformly bounded away from zero, in the case of nonuniform exponential dichotomies this "angle" may tend to zero, even exponentially (although often with small exponential speed when compared to the Lyapunov exponents), and thus some additional technical work is required to consider this general situation.

1.3. Some consequences of our work. We list in this section several nontrivial corollaries of our results. The first statement concerns the higher regularity of the stable manifold (see Theorem 5.1).

Theorem 1.2. *Let A and f be of class C^k for some $k \geq 2$. If equation* (1.1) *has a nonuniform exponential dichotomy, and*

$$\left\| \frac{\partial f}{\partial u}(t,u) \right\| \leq ce^{-t/c} \quad \text{and} \quad \left\| \frac{\partial^2 f}{\partial u^2}(t,u) \right\| \leq ce^{-t/c}$$

for every $t \geq 0$ and $u \in X$, for some sufficiently small constant $c > 0$ (depending on k), then the invariant stable manifold of the zero solution of equation (1.2) *is of class C^k.*

We emphasize that this is again an optimal result, since the stable manifold is as regular as the vector field.

The second consequence of Theorem 1.1 concerns the problem of robustness of nonuniform exponential dichotomies in linear equations. Namely, we consider the linear equation

$$u' = [A(t) + B(t)]u, \tag{1.7}$$

where $t \mapsto A(t)$ and $t \mapsto B(t)$ are C^1 functions. The robustness problem asks under what assumptions the exponential behavior of a nonuniform exponential dichotomy for equation (1.1) persists under such a linear perturbation. The following is an immediate consequence of Theorem 1.1.

Theorem 1.3. *If equation* (1.1) *has a nonuniform exponential dichotomy, and $\|B(t)\| \leq \kappa e^{-t/\kappa}$ for every $t \geq 0$ and $u \in X$, for some sufficiently small constant $\kappa > 0$, then equation* (1.7) *has invariant stable and unstable subspaces.*

This means that for each $t \geq 0$ there exist subspaces $\overline{E}(t)$ and $\overline{F}(t)$ with

$$X = \overline{E}(t) \oplus \overline{F}(t), \tag{1.8}$$

such that for every $t, s \geq 0$ we have

$$\overline{T}(t,s)\overline{E}(s) = \overline{E}(t) \quad \text{and} \quad \overline{T}(t,s)\overline{F}(s) = \overline{F}(t), \tag{1.9}$$

where $\overline{T}(t,s)$ is the linear evolution operator associated to equation (1.7), and that there exist $a < 0 \le b$ and $\varepsilon, D > 0$ such that for each $t \ge s \ge 0$ we have

$$\|\overline{T}(t,s)|\overline{E}(s)\| \le De^{a(t-s)+\varepsilon s}, \quad \|\overline{T}(t,s)^{-1}|\overline{F}(t)\| \le De^{-b(t-s)+\varepsilon t}. \tag{1.10}$$

Theorem 1.3 is also a consequence of Theorem 3 in [5]. In this (linear) setting the stable manifold of (the zero solution of) equation (1.7) is the graph

$$V_\phi = \{(t,u,\phi(t,u)) : t \ge 0 \text{ and } u \in E(t)\}$$

of a function $\phi\colon \mathbb{R}_0^+ \times X \to X$ satisfying (1.4), with the additional property that $u \mapsto \phi(t,u)$ is linear for each t. This implies that there exist subspaces $\overline{E}(t)$ for each t such that

$$V_\phi = \{(t,v) : t \ge 0 \text{ and } v \in \overline{E}(t)\}.$$

Moreover, the exponential behavior of the solutions in Theorem 4.1 ensures that $\overline{E}(t)$ are stable subspaces for equation (1.7), that is, they satisfy the first inequality in (1.10). In this case the invariance property in Theorem 1.1 means that the set V_ϕ is invariant under the flow defined by the autonomous equation

$$t' = 1, \quad u' = [A(t) + B(t)]u,$$

and this yields the first identity in (1.9). Reversing time we obtain invariant unstable subspaces $\overline{F}(t)$ for equation (1.7) satisfying (1.8) for each t.

2. Standing assumptions

We present in this section the standing assumptions in the paper. Let X be a Banach space, and let $A\colon \mathbb{R}_0^+ \to B(X)$ be a continuous function, where $B(X)$ is the set of bounded linear operators in X. We consider the initial value problem

$$u' = A(t)u, \quad u(s) = u_s, \tag{2.1}$$

for each $s \ge 0$ and $u_s \in X$. One can easily verify that its unique solution is defined for every $t > 0$, and we write it in the form $u(t) = T(t,s)u(s)$, where $T(t,s)$ is the associated linear evolution operator. We say that equation (2.1) admits a *nonuniform exponential dichotomy* if there exist constants

$$a < 0 \le b, \quad \varepsilon, D > 0,$$

and a continuous function $P\colon \mathbb{R}_0^+ \to B(X)$ such that $P(t)$ is a projection for $t \ge 0$, and for each $t \ge s \ge 0$ we have

$$P(t)T(t,s) = T(t,s)P(s), \tag{2.2}$$

and

$$\|T(t,s)P(s)\| \le De^{a(t-s)+\varepsilon s}, \quad \|T(t,s)^{-1}Q(t)\| \le De^{-b(t-s)+\varepsilon t}, \tag{2.3}$$

where $Q(t) = \mathrm{id} - P(t)$ is the complementary projection. We then define stable and unstable subspaces for each $s \ge 0$ by

$$E(s) = P(s)X \quad \text{and} \quad F(s) = Q(s)X.$$

It follows readily from (2.2) that for every $t, s \ge 0$ we have the invariance property

$$T(t,s)E(s) = E(t) \quad \text{and} \quad T(t,s)F(s) = F(t).$$

We also consider a continuous function $f: \mathbb{R}_0^+ \times X \to X$ with $f(t,0) = 0$ for every $t \ge 0$, and we assume that there exists a constant $\delta > 0$ such that

$$\|f(t,u) - f(t,v)\| \le \delta e^{-3\varepsilon t}\|u - v\| \tag{2.4}$$

for every $t \ge 0$ and $u, v \in X$. The following is a criterion for condition (2.4).

Proposition 2.1. *If $f(t,0) = 0$ for every $t \ge 0$, $f|Y = 0$, where*

$$Y = \{(t,u) \in \mathbb{R}_0^+ \times X : \|u\| \ge e^{-3\varepsilon t/q}\}$$

for some $q > 0$, and

$$\|f(t,u) - f(t,v)\| \le \frac{\delta}{2}\|u - v\|(\|u\|^q + \|v\|^q) \tag{2.5}$$

for every $t \ge 0$ and $u, v \in X$ with $\|u\|, \|v\| \le e^{-3\varepsilon t/q}$, then (2.4) holds.

Proof. We first assume that $\|u\|, \|v\| \le e^{-3\varepsilon t/q}$. In this case it follows from (2.5) that

$$\|f(t,u) - f(t,v)\| \le \frac{\delta}{2}\|u - v\|\big(e^{-3\varepsilon t} + e^{-3\varepsilon t}\big) = \delta e^{-3\varepsilon t}\|u - v\|.$$

When $\|u\|, \|v\| \ge e^{-3\varepsilon t/q}$ we have $f(t,u) = f(t,v) = 0$, and hence (2.4) also holds.

Now we assume that $\|u\| \le e^{-3\varepsilon t/q}$ and $\|v\| \ge e^{-3\varepsilon t/q}$. Let $w \in X$ be the unique point in the line segment between u and v with norm $\|w\| = e^{-3\varepsilon t/q}$. Then $f(t,v) = f(t,w) = 0$, and by the first case we have

$$\|f(t,u) - f(t,v)\| = \|f(t,u) - f(t,w)\| \le \delta e^{-3\varepsilon t}\|u - w\| \le \delta e^{-3\varepsilon t}\|u - v\|. \quad \square$$

Given $s \ge 0$ and $u_s = (\xi, \eta) \in E(s) \times F(s)$, we denote by

$$(x(t), y(t)) = (x(t,s,u_s), y(t,s,u_s)) \in E(t) \times F(t)$$

the unique solution of the initial value problem

$$u' = A(t)u + f(t,u), \quad u(s) = u_s, \tag{2.6}$$

or equivalently of the problem

$$\begin{aligned} x(t) &= T(t,s)\xi + \int_s^t P(t)T(t,s)f(\tau, x(\tau), y(\tau))\,d\tau, \\ y(t) &= T(t,s)\eta + \int_s^t Q(t)T(t,s)f(\tau, x(\tau), y(\tau))\,d\tau. \end{aligned}$$

We note that by (2.4), each solution is defined for every $t > 0$. Moreover, $u(t) = 0$ is a solution of equation (2.6). For each $\tau \geq 0$ we write

$$\Psi_\tau(s, u_s) = (s+\tau, x(s+\tau, s, u_s), y(s+\tau, s, u_s)). \tag{2.7}$$

This is the semiflow defined by the autonomous equation

$$t' = 1, \quad u' = A(t)u + f(t,u).$$

3. Lipschitz stable manifolds

We establish in this section the existence of a Lipschitz stable manifold for the equation $u' = A(t)u + f(t,u)$. It is obtained as a graph of a Lipschitz function. More precisely, let $\mathcal{X}$ be the space of continuous functions

$$\phi\colon \{(s,\xi) \in \mathbb{R}_0^+ \times X : \xi \in E(s)\} \to X$$

such that for each $s \geq 0$ and $\xi, \bar{\xi} \in E(s)$:

$$\begin{gathered} \phi(s,0) = 0 \quad \text{and} \quad \phi(s, E(s)) \subset F(s); \\ \|\phi(s,\xi) - \phi(s,\bar{\xi})\| \leq \|\xi - \bar{\xi}\|. \end{gathered} \tag{3.1}$$

Given a function $\phi \in \mathcal{X}$ we consider its graph

$$V_\phi = \{(s, \xi, \phi(s,\xi)) : (s,\xi) \in \mathbb{R}_0^+ \times E(s)\}. \tag{3.2}$$

The following is our Lipschitz stable manifold theorem. See Sections 4 and 5 for the smoothness of the stable manifold.

Theorem 3.1. *Let A and f be continuous functions. If the equation $u' = A(t)u$ admits a nonuniform exponential dichotomy with*

$$a - b + \varepsilon < 0, \tag{3.3}$$

$f(t,0) = 0$ for every $t \geq 0$, and (2.4) holds with δ sufficiently small, then there exists a unique function $\phi \in \mathcal{X}$ such that

$$\Psi_\tau(V_\phi) = V_\phi \quad \textit{for every } \tau \geq 0. \tag{3.4}$$

Moreover, for every $s \geq 0$, $\xi, \bar{\xi} \in E(s)$, and $\tau \geq 0$ we have

$$\|\Psi_\tau(s,\xi,\phi(s,\xi)) - \Psi_\tau(s,\bar{\xi},\phi(s,\bar{\xi}))\| \leq 2De^{(a+2\delta D)\tau + \varepsilon s}\|\xi - \bar{\xi}\|.$$

Proof. In order that (3.4) holds we must have

$$
\begin{aligned}
x(t) &= T(t,s)\xi + \int_s^t P(t)T(t,s)f(\tau,x(\tau),\phi(\tau,x(\tau)))\,d\tau,\\
\phi(t,x(t)) &= T(t,s)\phi(s,\xi) + \int_s^t Q(t)T(t,s)f(\tau,x(\tau),\phi(\tau,x(\tau)))\,d\tau,
\end{aligned}
\tag{3.5}
$$

for every $t \geq s$. Given $s \geq 0$ we set

$$\rho(t) = a(t-s) + \varepsilon s,$$

and we consider the space $\mathcal{B} = \mathcal{B}_s$ of continuous functions

$$x\colon \{(t,\xi) : t \geq s \text{ and } \xi \in E(s)\} \to X$$

such that:

$$
\begin{gathered}
x(t,\xi) \in E(t) \quad \text{and} \quad x(s,\xi) = \xi \quad \text{for every } t \geq s \text{ and } \xi \in E(s);\\
\alpha(x) := \sup\left\{ \frac{\|x(t,\xi)\|}{\|\xi\|} e^{-\rho(t)} : t \geq s,\ \xi \in E(s)\setminus\{0\}\right\} \leq 2D.
\end{gathered}
\tag{3.6}
$$

We can easily verify that $\mathcal{B}$ is a complete metric space with the distance induced by the norm α. We notice that $x(t,0) = 0$ for every $t \geq s$, as a consequence of (3.6) and the continuity of x.

Lemma 3.2. *For every $\delta > 0$ sufficiently small, given $\phi \in \mathcal{X}$ and $s \geq 0$ there exists a unique function $x = x_\phi \in \mathcal{B}$ satisfying the first identity in* (3.5) *for every $t \geq s$ and $\xi \in E(s)$.*

Proof. Given $\phi \in \mathcal{X}$, we define an operator J in $\mathcal{B}$ by

$$(Jx)(t,\xi) = T(t,s)\xi + \int_s^t P(t)T(t,\tau)f(\tau,x(\tau,\xi),\phi(\tau,x(\tau,\xi)))\,d\tau$$

for each $t \geq 0$ and $\xi \in E(s)$. Clearly, Jx is a continuous function. Furthermore, $(Jx)(s,\xi) = \xi$ since $T(s,s) = \mathrm{id}$. For each $\tau \geq s$ we have

$$\|(x(\tau,\xi),\phi(\tau,x(\tau,\xi))) - (y(\tau,\xi),\phi(\tau,y(\tau,\xi)))\| \leq 2\|x(\tau,\xi) - y(\tau,\xi)\|.$$

Therefore, using (3.6),

$$
\begin{aligned}
K(\tau) &:= \|f(\tau,x(\tau,\xi),\phi(\tau,x(\tau,\xi))) - f(\tau,y(\tau,\xi),\phi(\tau,y(\tau,\xi)))\|\\
&\leq 2\delta e^{-3\varepsilon\tau}\|x(\tau,\xi) - y(\tau,\xi)\| \leq 4\delta D e^{\rho(\tau)} e^{-3\varepsilon\tau}\|\xi\|\alpha(x-y).
\end{aligned}
\tag{3.7}
$$

By the first inequality in (2.3) we obtain

$$\begin{aligned}\|(Jx)(t,\xi)-(Jy)(t,\xi)\| &\le \int_s^t \|P(t)T(t,\tau)\|K(\tau)\,d\tau \\ &\le 4\delta D^2\|\xi\|\alpha(x-y)\int_s^t e^{a(t-\tau)+\varepsilon\tau}e^{a(\tau-s)+\varepsilon s}e^{-3\varepsilon\tau}\,d\tau \\ &\le 4\delta D^2\|\xi\|\alpha(x-y)e^{\rho(t)}\int_s^\infty e^{-2\varepsilon\tau}\,d\tau \\ &\le \frac{2\delta D^2}{\varepsilon}\|\xi\|\alpha(x-y)e^{\rho(t)}.\end{aligned}$$

Therefore,

$$\alpha(Jx-Jy)\le\theta\alpha(x-y),\quad \theta=\frac{2\delta D^2}{\varepsilon}.$$

Taking δ sufficiently small so that $\theta<1/2$ the operator J becomes a contraction. Moreover, by the first inequality in (2.3) we have $\alpha(J0)\le D$, and hence,

$$\alpha(Jx)\le\alpha(J0)+\alpha(Jx-J0)\le D+\theta\alpha(x)\le D+2D/2=2D.$$

Therefore, there exists a unique function $x=x_\phi\in\mathcal{B}$ such that $Jx=x$. □

Now we establish a few properties of the function x_ϕ. We set $\nu=2\delta D$.

LEMMA 3.3. *For every $\delta>0$ sufficiently small, $\phi\in\mathcal{X}$ and $\xi,\overline{\xi}\in E(s)$ we have*

$$\|x_\phi(t,\xi)-x_\phi(t,\overline{\xi})\|\le De^{(a+\nu)(t-s)+\varepsilon s}\|\xi-\overline{\xi}\|,\quad t\ge s.$$

PROOF. Proceeding as in (3.7), for every $\tau\ge s$ we have

$$\begin{aligned}&\|f(\tau,x_\phi(\tau,\xi),\phi(\tau,x_\phi(\tau,\xi)))-f(\tau,\overline{x}_\phi(\tau,\overline{\xi}),\phi(\tau,\overline{x}_\phi(\tau,\overline{\xi})))\| \\ &\qquad\le 2\delta e^{-3\varepsilon\tau}\|x_\phi(\tau,\xi)-\overline{x}_\phi(\tau,\overline{\xi})\|.\end{aligned}$$

Setting $\Gamma(t)=\|x_\phi(t,\xi)-x_\phi(t,\overline{\xi})\|$, and using the first inequality in (2.3) we obtain

$$\begin{aligned}\Gamma(t)&\le\|P(t)T(t,s)\|\cdot\|\xi-\overline{\xi}\|+\int_s^t\|P(t)T(t,\tau)\|2\delta e^{-3\varepsilon\tau}\Gamma(\tau)\,d\tau \\ &\le De^{a(t-s)+\varepsilon s}\|\xi-\overline{\xi}\|+\nu\int_s^t e^{a(t-\tau)-2\varepsilon\tau}\Gamma(\tau)\,d\tau \\ &\le e^{a(t-s)}\biggl(De^{\varepsilon s}\|\xi-\overline{\xi}\|+\nu\int_s^t e^{-a(\tau-s)}\Gamma(\tau)\,d\tau\biggr).\end{aligned}$$

Applying Gronwall's lemma to the function $e^{-a(t-s)}\Gamma(t)$ yields

$$\Gamma(t)\le De^{(a+\nu)(t-s)+\varepsilon s}\|\xi-\overline{\xi}\|.$$

This completes the proof of the lemma. □

Now we equip the space $\mathcal{X}$ with the distance

$$d(\phi,\psi)=\sup\{\|\phi(t,x)-\psi(t,x)\|/\|x\| : t\ge 0 \text{ and } x\in E(t)\setminus\{0\}\}.$$

We can easily verify that $\mathcal{X}$ is a complete metric space with this distance.

Lemma 3.4. *For every $\delta>0$ sufficiently small, $\phi,\psi\in\mathcal{X}$ and $\xi\in E(s)$ we have*

$$\|x_\phi(t,\xi)-x_\psi(t,\xi)\|\le\frac{\nu D}{\varepsilon}e^{(a+2\nu)(t-s)}\|\xi\|d(\phi,\psi),\quad t\ge s.$$

Proof. Proceeding in a similar manner to that in (3.7) we obtain

$$\begin{aligned}&\|f(\tau,x_\phi(\tau,\xi),\phi(\tau,x_\phi(\tau,\xi)))-f(\tau,x_\psi(\tau,\xi),\psi(\tau,x_\psi(\tau,\xi)))\|\\&\qquad\le 2\delta e^{-3\varepsilon\tau}\|(x_\phi(\tau,\xi)-x_\psi(\tau,\xi),\phi(\tau,x_\phi(\tau,\xi))-\psi(\tau,x_\psi(\tau,\xi)))\|.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|\phi(\tau,x_\phi(\tau,\xi))-\psi(\tau,x_\psi(\tau,\xi))\|&\le\|\phi(\tau,x_\phi(\tau,\xi))-\psi(\tau,x_\phi(\tau,\xi))\|\\&\quad+\|\phi(\tau,x_\phi(\tau,\xi))-\psi(\tau,x_\psi(\tau,\xi))\|\\&\le\|x_\phi(\tau,\xi)\|d(\phi,\psi)+\|x_\phi(\tau,\xi)-x_\psi(\tau,\xi)\|,\end{aligned}$$

and thus,

$$\begin{aligned}&\|f(\tau,x_\phi(\tau,\xi),\phi(\tau,x_\phi(\tau,\xi)))-f(\tau,x_\psi(\tau,\xi),\psi(\tau,x_\psi(\tau,\xi)))\|\\&\qquad\le 2\delta e^{-3\varepsilon\tau}(\|x_\phi(\tau,\xi)\|d(\phi,\psi)+2\|x_\phi(\tau,\xi)-x_\psi(\tau,\xi)\|).\end{aligned}\tag{3.8}$$

Now we set $\Gamma(t)=\|x_\phi(t,\xi)-x_\psi(t,\xi)\|$. We have

$$\begin{aligned}\Gamma(t)&\le 2\delta\int_s^t\|P(t)T(t,\tau)\|e^{-3\varepsilon\tau}\|x_\phi(\tau,\xi)\|d(\phi,\psi)\,d\tau\\&\quad+4\delta\int_s^t\|P(t)T(t,\tau)\|e^{-3\varepsilon\tau}\|x_\phi(\tau,\xi)-x_\psi(\tau,\xi)\|\,d\tau\\&\le 2\nu D\|\xi\|d(\phi,\psi)\int_s^t e^{a(t-\tau)-2\varepsilon\tau}e^{a(\tau-s)+\varepsilon s}\,d\tau\\&\quad+2\nu\int_s^t e^{a(t-\tau)-2\varepsilon\tau}\Gamma(\tau)\,d\tau,\end{aligned}$$

and thus,

$$\begin{aligned}e^{-a(t-s)}\Gamma(t)&\le 2\nu D\|\xi\|d(\phi,\psi)\int_s^\infty e^{-2\varepsilon(\tau-s)}\,d\tau+\nu\int_s^t e^{-a(\tau-s)}\Gamma(\tau)\,d\tau\\&\le\frac{\nu D}{\varepsilon}\|\xi\|d(\phi,\psi)+2\nu\int_s^t e^{-a(\tau-s)}\Gamma(\tau)\,d\tau.\end{aligned}$$

Applying Gronwall's lemma to the function $e^{-a(t-s)}\Gamma(t)$ we obtain

$$e^{-a(t-s)}\Gamma(t)\le\frac{\nu D}{\varepsilon}\|\xi\|d(\phi,\psi)e^{2\nu(t-s)}.\qquad\square$$

Now we obtain an equivalent problem to the second equation in (3.5).

Lemma 3.5. *For every $\delta > 0$ sufficiently small, given $\phi \in \mathfrak{X}$ the following properties are equivalent:*

(a) *for every $s \geq 0$, $\xi \in E(s)$ and $t \geq s$,*

$$\phi(t, x_\phi(t,\xi)) = T(t,s)\phi(s,\xi) + \int_s^t Q(t)T(t,\tau)f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\,d\tau; \tag{3.9}$$

(b) *for every $s \geq 0$ and $\xi \in E(s)$,*

$$\phi(s,\xi) = -\int_s^\infty Q(s)T(\tau,s)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\,d\tau. \tag{3.10}$$

Proof. For each $\tau \geq s$ we have

$$\begin{aligned}\|f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\| &\leq 2\delta e^{-3\varepsilon\tau}\|x_\phi(\tau,\xi)\| \\ &\leq 4\delta D e^{a(\tau-s)+\varepsilon s} e^{-3\varepsilon\tau}\|\xi\|.\end{aligned}$$

It follows from the second inequality in (2.3) that

$$\begin{aligned}\int_s^\infty &\|Q(s)T(\tau,s)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\|\,d\tau \\ &\leq 4\delta D^2\|\xi\| \int_s^\infty e^{(a-b-\varepsilon)(\tau-s)}\,d\tau < \infty,\end{aligned}$$

and the integral in (3.10) is well-defined. Now we assume that identity (3.9) holds, and we write it in the equivalent form

$$\begin{aligned}\phi(s,\xi) &= T(t,s)^{-1}\phi(t, x_\phi(t,\xi)) \\ &\quad - \int_s^t Q(s)T(\tau,s)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\,d\tau.\end{aligned} \tag{3.11}$$

By the second inequality in (2.3), for every $t \geq s$ we have

$$\begin{aligned}\|T(t,s)^{-1}\phi(t, x_\phi(t,\xi))\| &\leq De^{-b(t-s)+\varepsilon t}\|x_\phi(t,\xi)\| \\ &\leq 2D^2\|\xi\| e^{(a-b)(t-s)+\varepsilon s+\varepsilon t} \leq 2D^2\|\xi\| e^{2\varepsilon s} e^{(a-b+\varepsilon)(t-s)}.\end{aligned}$$

By (3.3), we have $a - b + \varepsilon < 0$, and letting $t \to \infty$ in (3.11) yields (3.10).

Now we assume that identity (3.10) holds. Since $T(t,s)T(\tau,s)^{-1} = T(t,\tau)$ we obtain

$$\begin{aligned}T(t,s)\phi(s,\xi) &+ \int_s^t Q(t)T(t,\tau)f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\,d\tau \\ &= -\int_t^\infty Q(t)T(\tau,t)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\,d\tau.\end{aligned} \tag{3.12}$$

We also define a flow F_τ in $\{(s,\xi) : s \ge 0 \text{ and } \xi \in E(s)\}$ by

$$F_\tau(s,\xi) = (s+\tau, x_\phi(s+\tau,\xi)).$$

In view of (3.10) we have

$$\phi(s,\xi) = -\int_s^\infty Q(s)T(\tau,s)^{-1} f(F_{\tau-s}(s,\xi), \phi(F_{\tau-s}(s,\xi)))\, d\tau. \tag{3.13}$$

Furthermore,

$$F_{\tau-t}(t, x_\phi(t,\xi)) = F_{\tau-t}(F_{t-s}(s,\xi)) = F_{\tau-s}(s,\xi) = (\tau, x_\phi(\tau,\xi)),$$

and by (3.13) with (s,ξ) replaced by $(t, x_\phi(t,\xi))$ we obtain

$$\begin{aligned}\phi(t, x_\phi(t)) &= -\int_t^\infty Q(t)T(\tau,t)^{-1} f(F_{\tau-t}(t,x_\phi(t,\xi)), \phi(F_{\tau-t}(t,x_\phi(t,\xi))))\, d\tau \\ &= -\int_t^\infty Q(t)T(\tau,t)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\, d\tau\end{aligned}$$

for every $t \ge s$. Together with (3.12) this yields identity (3.9). □

Lemma 3.6. *For every $\delta > 0$ sufficiently small, there exists a unique function $\phi \in \mathcal{X}$ satisfying* (3.10) *for every $s \ge 0$ and $\xi \in E(s)$.*

Proof. We consider the operator T in $\mathcal{X}$ defined for each $\phi \in \mathcal{X}$ by

$$(T\phi)(s,\xi) = -\int_s^\infty Q(s)T(\tau,s)^{-1} f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi)))\, d\tau \tag{3.14}$$

for $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$. Since $x_\phi(t,0) = 0$ for every $\phi \in \mathcal{X}$ and $t \ge s$, it follows from (3.14) that $(T\phi)(s,0) = 0$ for every $s \ge 0$. Moreover, by Lemma 3.3,

$$\begin{aligned}&\|f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi))) - f(\tau, x_\phi(\tau,\overline{\xi}), \phi(\tau, x_\phi(\tau,\overline{\xi})))\| \\ &\qquad \le \nu e^{-3\varepsilon\tau} e^{(a+\nu)(\tau-s)+\varepsilon s}\|\xi - \overline{\xi}\|.\end{aligned}$$

Using the last inequality in (2.3) we find that

$$\begin{aligned}\|(T\phi)(s,\xi) - (T\phi)(s,\overline{\xi})\| &\le D\nu\|\xi-\overline{\xi}\| \int_s^\infty e^{(a-b+\nu)(\tau-s)+\varepsilon\tau+\varepsilon s-3\varepsilon\tau}\, d\tau \\ &= D\nu\|\xi-\overline{\xi}\| \int_s^\infty e^{(a-b+\nu-\varepsilon)(\tau-s)}\, d\tau = \frac{D\nu}{|a-b+\nu-\varepsilon|}\|\xi-\overline{\xi}\|,\end{aligned}$$

taking δ sufficiently small so that

$$a-b+\nu-\varepsilon < 0 \quad \text{and} \quad D\nu/|a-b+\nu-\varepsilon| < 1 \tag{3.15}$$

(recall that $\nu = 2\delta D$). We obtain

$$\|(T\phi)(s,\xi) - (T\phi)(s,\overline{\xi})\| \le \|\xi-\overline{\xi}\|,$$

for every $s \ge 0$ and $\xi, \overline{\xi} \in \mathcal{X}$. Therefore, $T(\mathcal{X}) \subset \mathcal{X}$.

Now we show that T is a contraction. By (3.8) and Lemma 3.4 we have

$$\begin{aligned}
L(\tau) &:= \|f(\tau, x_\phi(\tau,\xi), \phi(\tau, x_\phi(\tau,\xi))) - f(\tau, x_\psi(\tau,\xi), \psi(\tau, x_\psi(\tau,\xi)))\| \\
&\le 2\delta e^{-3\varepsilon\tau}\|x_\phi(\tau,\xi)\| d(\phi,\psi) + \frac{4\nu D\delta}{\varepsilon} e^{-3\varepsilon\tau} e^{(a+2\nu)(\tau-s)}\|\xi\| d(\phi,\psi) \\
&\le 4\delta D e^{-3\varepsilon\tau} e^{a(\tau-s)+\varepsilon s}\|\xi\| d(\phi,\psi) + \frac{4\nu D\delta}{\varepsilon} e^{-3\varepsilon\tau} e^{(a+2\nu)(\tau-s)}\|\xi\| d(\phi,\psi) \\
&\le \nu_1 \delta e^{-3\varepsilon\tau} e^{(a+2\nu)(\tau-s)+\varepsilon s}\|\xi\| d(\phi,\psi)
\end{aligned}$$

for some constant $\nu_1 > 0$. Hence,

$$\begin{aligned}
\|(T\phi)(s,\xi) - (T\psi)(s,\xi)\| &\le \int_s^\infty \|Q(s)T(\tau,s)^{-1}\| L(\tau)\, d\tau \\
&\le D\nu_1\delta\|\xi\| d(\phi,\psi) \int_s^\infty e^{(a-b+2\nu)(\tau-s)+\varepsilon\tau+\varepsilon s-3\varepsilon\tau}\, d\tau \\
&= D\nu_1\delta\|\xi\| d(\phi,\psi) \int_s^\infty e^{(a-b+2\nu-\varepsilon)(\tau-s)}\, d\tau \\
&= \frac{D\nu_1\delta}{|a-b+2\nu-\varepsilon|}\|\xi\| d(\phi,\psi).
\end{aligned}$$

Taking δ sufficiently small, we have

$$a-b+2\nu-\varepsilon < 0 \quad \text{and} \quad D\nu_1/|a-b+2\nu-\varepsilon| < 1.$$

In particular, the operator T becomes a contraction. Hence, there exists a unique function $\phi \in \mathcal{X}$ satisfying $T\phi = \phi$. □

By Lemma 3.2, provided that δ is sufficiently small, for each $\phi \in \mathcal{X}$ there exists a unique function $x = x_\phi \in \mathcal{B}$ satisfying the first identity in (3.5). Moreover, by Lemmas 3.5 and 3.6 there exists a unique function $\phi \in \mathcal{X}$ satisfying the second identity in (3.5) with $x = x_\phi$.

It remains to prove the last property in the theorem. By Lemma 3.3 we have

$$\begin{aligned}
&\|\Psi_{t-s}(s,\xi,\phi(s,\xi)) - \Psi_{t-s}(s,\bar\xi,\phi(s,\bar\xi))\| \\
&\qquad = \|(t, x(t,\xi), \phi(t, x(t,\xi))) - (t, x(t,\bar\xi), \phi(t, x(t,\bar\xi)))\| \\
&\qquad \le 2\|x(t,\xi) - x(t,\bar\xi)\| \le 2De^{(a+\nu)(t-s)+\varepsilon s}\|\xi - \bar\xi\|.
\end{aligned}$$

This completes the proof of the Theorem 3.1. □

4. C^1 regularity of the stable manifolds

For $X = \mathbb{R}^p$, we establish in this section the C^1 regularity of the Lipschitz manifold V_ϕ in Theorem 3.1. The following is our main result.

Theorem 4.1. *Let A and f be C^1 functions. If the equation $u' = A(t)u$ admits a nonuniform exponential dichotomy satisfying* (3.3),

$$f(t,0) = f(t,u) = 0 \tag{4.1}$$

for every $t \geq 0$ and $u \in X$ with $\|u\| \geq c$, for some constant $c > 0$, and condition (2.4) *holds with δ sufficiently small, then the unique function ϕ in Theorem* 3.1 *is of class C^1 in ξ. If in addition $(\partial f/\partial u)(t,0) = 0$ for every $t \geq 0$, then $(\partial\phi/\partial\xi)(s,0) = 0$ for every $s \geq 0$.*

Proof. We first briefly recall the fiber contraction principle. Given metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, we define a distance in $X \times Y$ by

$$d((x,y),(\overline{x},\overline{y})) = d_X(x,\overline{x}) + d_Y(y,\overline{y}).$$

We consider transformations $S: X \times Y \to X \times Y$ of the form

$$S(x,y) = (T(x), A(x,y)),$$

for some functions $T: X \to X$ and $A: X \times Y \to Y$. We say that S is a *fiber contraction* if there exists $\lambda \in (0,1)$ such that

$$d_Y(A(x,y), A(x,\overline{y})) \leq \lambda d_Y(y,\overline{y})$$

for every $x \in X$ and $y, \overline{y} \in Y$. For each $x \in X$ we define a transformation $A_x: Y \to Y$ by $A_x(y) = A(x,y)$. We also say that a fixed point $x_0 \in X$ of T is *attracting* if $T^n(x) \to x_0$ when $n \to \infty$ for every $x \in X$.

Lemma 4.2 (Fiber contraction principle). *If S is a continuous fiber contraction, $x_0 \in X$ is an attracting fixed point of T, and $y_0 \in Y$ is a fixed point of A_{x_0}, then (x_0, y_0) is an attracting fixed point of S.*

Now we proceed with the proof of the theorem. We consider the space $\mathcal{F}$ of continuous functions

$$\Phi: \{(s,\xi) \in \mathbb{R}_0^+ \times X : \xi \in E(s)\} \to \coprod_{s \in \mathbb{R}_0^+} L(s),$$

where $L(s)$ is the family of linear transformations from $E(s)$ to $F(s)$, such that $\Phi(s,\xi) \in L(s)$ for every $s \geq 0$ and $\xi \in E(s)$, with

$$\|\Phi\| := \sup\{\|\Phi(s,\xi)\| : (s,\xi) \in \mathbb{R}_0^+ \times E(s)\} \leq 1. \tag{4.2}$$

We also consider the subset $\mathcal{F}_0 \subset \mathcal{F}$ composed of the functions $\Phi \in \mathcal{F}$ such that $\Phi(s,0) = 0$ for every $s \geq 0$. We can easily verify that $\mathcal{F}$ and $\mathcal{F}_0$ are complete metric spaces with the distance induced by this norm.

Given δ as in Theorem 3.1 and $\phi \in \mathfrak{X}$, we consider the (unique) function $x = x_\phi$ given by Lemma 3.2. We notice that it is the unique solution of the differential equation

$$x' = P(t)A(t)x + P(t)f(t, x, \phi(t, x)) \tag{4.3}$$

with initial condition $\xi \in E(s)$ at time s. In particular, due to the continuous dependence of the solutions of a differential equation on the initial conditions, this ensures that the function $(t, s, \xi) \mapsto x_\phi(t, \xi)$ is continuous. Thus, it follows from Lemma 3.4 that the function $(t, \phi, s, \xi) \mapsto x_\phi(t, \xi)$ is also continuous. For simplicity we write

$$y_\phi(t) = (t, x_\phi(t, \xi), \phi(t, x_\phi(t, \xi))) \quad \text{and} \quad z_\phi(t) = (t, x_\phi(t, \xi)) \tag{4.4}$$

in what follows.

We define a linear transformation $A(\phi, \Phi)$ for each $(\phi, \Phi) \in \mathfrak{X} \times \mathfrak{F}$ by

$$\begin{aligned} A(\phi, \Phi)(s, \xi) &= -\int_s^\infty Q(s)T(\tau, s)^{-1} \frac{\partial f}{\partial u}(y_\phi(\tau)) \begin{pmatrix} \mathrm{id}_{E(s)} \\ \Phi(z_\phi(\tau)) \end{pmatrix} W(\tau)\, d\tau \\ &= -\int_s^\infty Q(s)T(\tau, s)^{-1} \biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \biggr)\, d\tau, \end{aligned} \tag{4.5}$$

where $(x, y) \in E(s) \times F(s)$, and where the function $W = W_{\phi,\Phi,\xi}$ is uniquely determined by the identities

$$\begin{aligned} W(t) &= P(t)T(t, s) \\ &\quad + \int_s^t P(t)T(t, \tau) \biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \biggr)\, d\tau \end{aligned} \tag{4.6}$$

for $t \geq s$. We notice that each $W(t)$ is a linear transformation from $E(s)$ to $E(t)$, with $W(s) = \mathrm{id}_{E(s)}$. It follows from the continuity of the solutions of a differential equation with respect to parameters, together with the continuity of the functions $(t, \phi, s, \xi) \mapsto x_\phi(t, \xi)$, ϕ, and Φ, that the function $(t, \phi, s, \xi) \mapsto W_{\phi,\Phi,\xi}(t)$ is also continuous.

Lemma 4.3. *The operator A is well-defined, and, $A(\mathfrak{X} \times \mathfrak{F}) \subset \mathfrak{F}$.*

Proof. To show that A is well-defined, we set

$$B = \int_s^\infty \biggl\| Q(s)T(\tau, s)^{-1} \biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))W(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W(\tau) \biggr) \biggr\|\, d\tau.$$

By (2.4) we have

$$\biggl\| \frac{\partial f}{\partial u}(t, u) \biggr\| \leq \delta e^{-3\varepsilon t} \tag{4.7}$$

for every $t > 0$ and $u \in X$. It follows from (2.3) and (4.7) that

$$\begin{aligned} B &\le 2\delta D \int_s^\infty e^{-b(\tau-s)+\varepsilon\tau-3\varepsilon\tau} \|W(\tau)\| \, d\tau \\ &= 2\delta D \int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau} \|W(\tau)\| \, d\tau. \end{aligned} \tag{4.8}$$

On the other hand, by (4.6) and again (4.7) we have

$$\|W(t)\| \le De^{a(t-s)+\varepsilon s} + 2\delta D \int_s^t e^{a(t-\tau)+\varepsilon\tau-3\varepsilon\tau} \|W(\tau)\| \, d\tau. \tag{4.9}$$

Setting $\Gamma(t) = e^{-a(t-s)}\|W(t)\|$ we obtain

$$\Gamma(t) \le De^{\varepsilon s} + 2\delta D \int_s^t e^{-2\varepsilon\tau} \Gamma(\tau) \, d\tau \le De^{\varepsilon s} + 2\delta D \int_s^t \Gamma(\tau) \, d\tau.$$

It follows from Gronwall's lemma that $\Gamma(t) \le De^{\varepsilon s} e^{2\delta D(t-s)}$, and thus

$$\|W(t)\| \le De^{\varepsilon s} e^{(a+2\delta D)(t-s)}. \tag{4.10}$$

It follows from (3.15) and (4.8) that

$$B \le 2\delta D^2 \int_s^\infty e^{(-b+a-\varepsilon+2\delta D)(\tau-s)} \, d\tau = \frac{2\delta D^2}{|-b+a-\varepsilon+2\delta D|} < 1.$$

This shows that $A(\phi, \Phi)$ is well-defined. Since

$$\|A(\phi, \Phi)(s, \xi)\| \le B < 1$$

for every $s \ge 0$ and $\xi \in E(s)$, we obtain $\|A(\phi, \Phi)\| \le 1$. This shows that $A(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$. $\square$

Moreover, when $(\partial f/\partial u)(t, 0) = 0$ for every $t \ge 0$, since $x_\phi(t, 0) = 0$ for $\phi \in \mathcal{X}$ and $t \ge 0$, it follows from (4.5) that $A(\phi, \Phi)(s, 0) = 0$ for every $(\phi, \Phi) \in \mathcal{X} \times \mathcal{F}_0$ and $s \ge 0$. Therefore, in this case we have $A(\mathcal{X} \times \mathcal{F}_0) \subset \mathcal{F}_0$.

Now we consider the transformation $S: \mathcal{X} \times \mathcal{F} \to \mathcal{X} \times \mathcal{F}$ defined by

$$S(\phi, \Phi) = (T\phi, A(\phi, \Phi)),$$

where T is the operator in (3.14). Notice that when $(\partial f/\partial u)(t, 0) = 0$ for every $t \ge 0$, we have $S(\mathcal{X} \times \mathcal{F}_0) \subset \mathcal{X} \times \mathcal{F}_0$.

Lemma 4.4. *For every $\delta > 0$ sufficiently small, the operator S is a fiber contraction.*

Proof. Given $\xi \in E(s)$, $\phi \in \mathcal{X}$, and $\Phi, \Psi \in \mathcal{F}$, let

$$W_\Phi = W_{\phi,\Phi,\xi} \quad \text{and} \quad W_\Psi = W_{\phi,\Psi,\xi}.$$

We have

$$
\begin{aligned}
(4.11)\quad & \|A(\phi,\Phi)(s,\xi) - A(\phi,\Psi)(s,\xi)\| \\
&\le D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial x}W_\Phi + \frac{\partial f}{\partial y}\Phi W_\Phi - \frac{\partial f}{\partial x}W_\Psi - \frac{\partial f}{\partial y}\Psi W_\Psi\right\| d\tau \\
&\le \delta D\int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau}\big(\|W_\Phi - W_\Psi\| + \|\Phi W_\Phi - \Psi W_\Psi\|\big)\, d\tau \\
&\le \delta D\int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau}\big(\|W_\Phi - W_\Psi\| \\
&\quad + \|\Phi\|\cdot\|W_\Phi - W_\Psi\| + \|\Phi-\Psi\|\cdot\|W_\Psi\|\big)\, d\tau \\
&\le \delta D\int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau}(2\|W_\Phi - W_\Psi\| + \|\Phi-\Psi\|\cdot\|W_\Psi\|)\, d\tau,
\end{aligned}
$$

where for simplicity we have omitted the arguments inside the integrals. In an analogous manner to that in (4.9) and using (4.10) we obtain

$$
\begin{aligned}
\|W_\Phi(t) - W_\Psi(t)\| &\le 2\delta D\int_s^t e^{a(t-\tau)+\varepsilon\tau-3\varepsilon\tau}\|W_\Phi(\tau) - W_\Psi(\tau)\|\, d\tau \\
&\quad + \delta D\|\Phi-\Psi\|\int_s^t e^{a(t-\tau)+\varepsilon\tau-3\varepsilon\tau}\|W_\Psi(\tau)\|\, d\tau \\
&\le 2\delta D e^{a(t-s)}\int_s^t e^{-a(\tau-s)-2\varepsilon\tau}\|W_\Phi(\tau) - W_\Psi(\tau)\|\, d\tau \\
&\quad + \delta D^2 e^{a(t-s)}\|\Phi-\Psi\|\int_s^t e^{-(a+\varepsilon)(\tau-s)}e^{(a+2\delta D)(\tau-s)}\, d\tau \\
&= 2\delta D e^{a(t-s)}\int_s^t e^{-a(\tau-s)-2\varepsilon\tau}\|W_\Phi(\tau) - W_\Psi(\tau)\|\, d\tau \\
&\quad + \delta D^2 e^{a(t-s)}\|\Phi-\Psi\|\int_s^t e^{-(\varepsilon-2\delta D)(\tau-s)}\, d\tau.
\end{aligned}
$$

Setting $\Gamma(t) = e^{-a(t-s)}\|W_\Phi(t) - W_\Psi(t)\|$, we thus have

$$
\Gamma(t) \le \frac{\delta D^2}{|\varepsilon - 2\delta D|}\|\Phi-\Psi\| + 2\delta D\int_s^t \Gamma(\tau)\, d\tau,
$$

provided that δ is sufficiently small. It follows from Gronwall's lemma that

$$
(4.12)\qquad \|W_\Phi(t) - W_\Psi(t)\| \le \frac{\delta D^2}{|\varepsilon - 2\delta D|}\|\Phi-\Psi\| e^{(a+2\delta D)(t-s)}.
$$

Using inequalities (4.10) and (4.12), and in view of (3.15), it follows from (4.11) that

$$
\begin{aligned}
&\|A(\phi,\Phi)(s,\xi) - A(\phi,\Psi)(s,\xi)\| \\
&\le C_1\delta\|\Phi-\Psi\|\int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\tau-s)-\varepsilon\tau}\, d\tau + \delta D^2\|\Phi-\Psi\|\int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\tau-s)}\, d\tau \\
&\le K_1\delta\|\Phi-\Psi\|\int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\tau-s)}\, d\tau \le \frac{K_1\delta}{|a-b-\varepsilon+2\delta D|}\|\Phi-\Psi\|,
\end{aligned}
$$

for some constants $C_1, K_1 > 0$. We conclude that for δ sufficiently small the operator S is a fiber contraction. □

By Lemma 4.4, to apply Lemma 4.2 it remains to verify that S is continuous. This turns out to be a delicate part of the proof. We explore an argument sketched by C. Chicone in [7], now in the nonautonomous setting.

Lemma 4.5. *For every $\delta > 0$ sufficiently small, the operator S is continuous.*

Proof. Setting $W_\phi = W_{\phi,\Phi,\xi}$ and $W_\psi = W_{\psi,\Phi,\xi}$, we obtain

$$\begin{aligned}
&\|A(\phi,\Phi)(s,\xi) - A(\psi,\Phi)(s,\xi)\| \\
&\quad\le D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial x}(y_\phi(\tau))W_\phi(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Phi(z_\phi(\tau))W_\phi(\tau) \\
&\qquad - \frac{\partial f}{\partial x}(y_\psi(\tau))W_\psi(\tau) - \frac{\partial f}{\partial y}(y_\psi(\tau))\Phi(z_\psi(\tau))W_\psi(\tau)\bigg\|\,d\tau,
\end{aligned}$$

with $y_\phi(\tau)$ and $z_\phi(\tau)$ as in (4.4). It follows from (4.7) and (4.10) that

$$\begin{aligned}
&\|A(\phi,\Phi)(s,\xi) - A(\psi,\Phi)(s,\xi)\| \\
&\quad\le D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau))\bigg\|\cdot\|W_\phi(\tau)\|\,d\tau \\
&\qquad + D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial x}(y_\psi(\tau))\bigg\|\cdot\|W_\phi(\tau) - W_\psi(\tau)\|\,d\tau \\
&\qquad + D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau))\bigg\|\cdot\|\Phi(z_\phi(\tau))W_\phi(\tau)\|\,d\tau \\
&\qquad + D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial y}(y_\psi(\tau))\bigg\|\cdot\|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\|\cdot\|W_\phi(\tau)\|\,d\tau \\
&\qquad + D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\bigg\|\frac{\partial f}{\partial y}(y_\psi(\tau))\bigg\|\cdot\|\Phi(z_\psi(\tau))\|\cdot\|W_\phi(\tau) - W_\psi(\tau)\|\,d\tau,
\end{aligned}$$

and this yields

$$\begin{aligned}
&\|A(\phi,\Phi)(s,\xi) - A(\psi,\Phi)(s,\xi)\| \\
&\quad\le D^2e^{2\varepsilon s}\int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\tau-s)}\bigg\|\frac{\partial f}{\partial x}(y_\phi(\tau)) - \frac{\partial f}{\partial x}(y_\psi(\tau))\bigg\|\,d\tau \\
&\qquad + \delta D\int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau}\|W_\phi(\tau) - W_\psi(\tau)\|\,d\tau \\
&\qquad + D^2e^{2\varepsilon s}\int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\tau-s)}\bigg\|\frac{\partial f}{\partial y}(y_\phi(\tau)) - \frac{\partial f}{\partial y}(y_\psi(\tau))\bigg\|\,d\tau \\
&\qquad + \delta D^2\int_s^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)-\varepsilon\tau}\|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\|\,d\tau \\
&\qquad + \delta D\int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau}\|W_\phi(\tau) - W_\psi(\tau)\|\,d\tau
\end{aligned}\tag{4.13}$$

$$\leq 2D^2 e^{2\varepsilon s} \int_s^\infty e^{(a+2\delta D+\varepsilon-b)(\tau-s)} \left\| \frac{\partial f}{\partial u}(y_\phi(\tau)) - \frac{\partial f}{\partial u}(y_\psi(\tau)) \right\| d\tau$$
$$+ 2\delta D \int_s^\infty e^{-b(\tau-s)-2\varepsilon\tau} \|W_\phi(\tau) - W_\psi(\tau)\| \, d\tau$$
$$+ \delta D^2 \int_s^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)-\varepsilon\tau} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \, d\tau.$$

Again by (4.7) and (4.10), and in view of (3.15), given $\gamma > 0$ there exists $\sigma > 0$ (independent of s and ξ) such that

$$\text{(4.14)} \quad 2D^2 e^{2\varepsilon s} \int_{s+\sigma}^\infty e^{(a+2\delta D+\varepsilon-b)(\tau-s)} \left\| \frac{\partial f}{\partial u}(y_\phi(\tau)) - \frac{\partial f}{\partial u}(y_\psi(\tau)) \right\| d\tau$$
$$\leq 4\delta D^2 \int_{s+\sigma}^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)} \, d\tau = \frac{4\delta D^2 e^{(a+2\delta D-\varepsilon-b)\sigma}}{|a+2\delta D-\varepsilon-b|} < \gamma,$$

$$\text{(4.15)} \quad 2\delta D \int_{s+\sigma}^\infty e^{-b(\tau-s)-2\varepsilon\tau} \|W_\phi(\tau) - W_\psi(\tau)\| \, d\tau$$
$$\leq 4\delta D^2 \int_{s+\sigma}^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)} \, d\tau < \gamma,$$

and

$$\text{(4.16)} \quad \delta D^2 \int_{s+\sigma}^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)-\varepsilon\tau} \|\Phi(z_\phi(\tau)) - \Phi(z_\psi(\tau))\| \, d\tau$$
$$\leq 2\delta D^2 \int_{s+\sigma}^\infty e^{(a+2\delta D-\varepsilon-b)(\tau-s)} \, d\tau < \gamma.$$

Now we consider the integrals from s to $s+\sigma$. We must show that given $\gamma > 0$ there exists $\eta > 0$ (independent of s and ξ) such that each integral from s to $s+\sigma$ is bounded by γ whenever $d(\phi,\psi) < \eta$. For this we consider the functions

$$\begin{aligned} B(p,\phi)(s,\xi) &= 2D^2 e^{2\varepsilon s} e^{(a+2\delta D+\varepsilon-b)p} \frac{\partial f}{\partial u}(y_\phi(s+p)), \\ C(p,\phi)(s,\xi) &= 2\delta D e^{-bp-2\varepsilon(s+p)} W_\phi(s+p), \\ D(p,\phi)(s,\xi) &= \delta D^2 e^{(a+2\delta D-\varepsilon-b)p-\varepsilon(s+p)} \Phi(z_\phi(s+p)), \end{aligned}$$

for each $p \in [0,\sigma]$ and $\phi \in \mathfrak{X}$. We note that

$$2D^2 e^{2\varepsilon s} \int_s^{s+\sigma} e^{(a+2\delta D+\varepsilon-b)(\tau-s)} \frac{\partial f}{\partial u}(y_\phi(\tau)) \, d\tau$$
$$+ 2\delta D \int_s^{s+\sigma} e^{-b(\tau-s)-2\varepsilon\tau} W_\phi(\tau) \, d\tau$$
$$+ \delta D^2 \int_s^{s+\sigma} e^{(a+2\delta D-\varepsilon-b)(\tau-s)-\varepsilon\tau} \Phi(z_\phi(\tau)) \, d\tau$$
$$= \int_0^\sigma \big[B(p,\phi) + C(p,\phi) + D(p,\phi)\big](s,\xi) \, dp.$$

Therefore, by (4.13), it is sufficient to show that the integral

$$\phi \mapsto \int_0^\sigma \big[B(p,\phi) + C(p,\phi) + D(p,\phi)\big]\, dp \tag{4.17}$$

is continuous. Since the functions Φ,

$$(t,\phi,s,\xi) \mapsto x_\phi(t,\xi) \quad \text{and} \quad (t,\phi,s,\xi) \mapsto W_{\phi,\Phi,\xi}(t)$$

are continuous, the functions

$$(p,\phi,s,\xi) \mapsto B(p,\phi)(s,\xi),\ C(p,\phi)(s,\xi),\ D(p,\phi)(s,\xi) \tag{4.18}$$

are also continuous. Furthermore, by (4.7), (4.10), and (3.15), for each $p \in [0,\sigma]$ and $\phi \in \mathcal{X}$ we have

$$\begin{aligned}
\|B(p,\phi)\| &\le 2\delta D^2 e^{(a+2\delta D-\varepsilon-b)p-\varepsilon(s+p)} \le 2\delta D^2 e^{-\varepsilon s},\\
\|C(p,\phi)\| &\le 2\delta D^2 e^{(a+2\delta D-\varepsilon-b)p-\varepsilon(s+p)} \le 2\delta D^2 e^{-\varepsilon s},\\
\|D(p,\phi)\| &\le \delta D^2 e^{(a+2\delta D-\varepsilon-b)p-\varepsilon(s+p)} \le \delta D^2 e^{-\varepsilon s}.
\end{aligned}$$

Here we are using the norm $\|\cdot\|$ in (4.2). In particular, $B(p,\phi)$, $C(p,\phi)$, and $D(p,\phi)$ are in $\mathcal{F}$ provided that δ is sufficiently small. We proceed with the proof of the continuity of the map in (4.17). We first note that there exists $R > 0$ such that

$$\begin{aligned}
\|B(p,\phi)(s,\xi) - B(p,\psi)(s,\xi)\| &\le 4\delta D^2 e^{-\varepsilon s} < \gamma,\\
\|C(p,\phi)(s,\xi) - C(p,\psi)(s,\xi)\| &\le 4\delta D^2 e^{-\varepsilon s} < \gamma,\\
\|D(p,\phi)(s,\xi) - D(p,\psi)(s,\xi)\| &\le 4\delta D^2 e^{-\varepsilon s} < \gamma
\end{aligned}$$

for every $s > R$, $p \in [0,\sigma]$, and $\xi \in E(s)$. Now we consider the case when $s \le R$. Given $s \in \mathbb{R}_0^+$ and $(\phi,\xi) \in \mathcal{X} \times E(s)$, due to the continuity in (4.18) there exists $\delta > 0$ such that

$$\|B(p,\phi)(s,\xi) - B(q,\psi)(\overline{s},\overline{\xi})\| < \gamma$$

whenever $d(\phi,\psi) < \delta$ and $\|(p,s,\xi) - (q,\overline{s},\overline{\xi})\| < \delta$. Since $u \mapsto f(t,u)$ vanishes for $\|u\| \ge c$, given s it is sufficient to establish the desired continuity for ξ inside a certain ball in $E(s)$, possibly depending (continuously) on p and s, and thus for ξ in a certain compact set K. We can cover the compact set $[0,\sigma]\times[0,R]\times K$ with a finite number of balls B_i, $i = 1,\dots,r$ centered at points in this set, such that

$$\|B(p,\phi)(s,\xi) - B(\overline{p},\psi)(\overline{s},\overline{\xi})\| < \gamma$$

whenever $d(\phi,\psi) < \delta_i$ and $(p,s,\xi), (\overline{p},\overline{s},\overline{\xi}) \in B_i$, for $i = 1,\dots,r$ and some numbers $\delta_i > 0$. Therefore,

$$\|B(p,\phi)(s,\xi) - B(p,\psi)(s,\xi)\| < \gamma$$

whenever $d(\phi,\psi) < \delta = \min\{\delta_1,\dots,\delta_r\}$, for every $p \in [0,\sigma]$, $s \le R$, and $\xi \in K$. This shows that

$$\sup_{s\le R}\sup_{\xi\in K}\|B(p,\phi)(s,\xi) - B(p,\psi)(s,\xi)\| \le \gamma$$

whenever $d(\phi,\psi) < \delta$. Together with (4.14)–(4.16) this implies that $\phi \mapsto A(\phi,\Phi)$ is continuous, and thus the fiber contraction S is also continuous (we already know that the operator T in (3.14) is a contraction). □

To establish the C^1 regularity in ξ of the unique function ϕ in Theorem 3.1, we first obtain the following.

Lemma 4.6. *If ϕ is of class C^1, then $T\phi$ is also of class C^1, and*

$$\partial(T\phi)/\partial\xi = A(\phi, \partial\phi/\partial\xi). \tag{4.19}$$

Proof. If ϕ is of class C^1, then the function y defined by $y(t,\xi) = x_\phi(t,\xi)$ is also of class C^1 (when ϕ is of class C^1 the right-hand side of (4.3) is also of class C^1, and thus the solutions are C^1 in the initial conditions). Furthermore, for $\Phi = \partial\phi/\partial\xi$ the solution of equation (4.6) is given by $W(t) = \partial y/\partial\xi$ (this follows simply by comparison with the linear variational equation). Therefore, repeating arguments in the proof of Lemma 4.3 we can apply Leibnitz's rule to obtain

$$\begin{aligned} A\left(\frac{\phi,\partial\phi}{\partial\xi}\right)(s,\xi) &= -\int_s^\infty \frac{\partial}{\partial\xi}[Q(s)T(\tau,s)^{-1}f(\tau,x_\phi(\tau),\phi(\tau,x_\phi(\tau)))]\,d\tau \\ &= \left(\frac{\partial(T\phi)}{\partial\xi}\right)(s,\xi) \end{aligned}$$

for every $s \in \mathbb{R}_0^+$ and $\xi \in X$. □

End of the Proof of Theorem 4.1. To complete the proof, we consider the pair $(\phi_1,\Phi_1) = (0,0) \in \mathcal{X}\times\mathcal{F}$. Clearly, $\Phi_1 = \partial\phi_1/\partial\xi$. We define recursively a sequence $(\phi_n,\Phi_n) \in \mathcal{X}\times\mathcal{F}$ by

$$(\phi_{n+1},\Phi_{n+1}) = S(\phi_n,\Phi_n) = (T\phi_n, A(\phi_n,\Phi_n)). \tag{4.20}$$

Assuming that ϕ_n is of class C^1 with $\Phi_n = \partial\phi_n/\partial\xi$, it follows from Lemma 4.6 that $T\phi$ is of class C^1, and by (4.19) we have

$$\partial\phi_{n+1}/\partial\xi = \partial(T\phi_n)/\partial\xi = A(\phi_n,\Phi_n) = \Phi_{n+1}. \tag{4.21}$$

Now let ϕ_0 be the unique fixed point of T (that is, the unique function ϕ in Theorem 3.1), and let Φ_0 be the unique fixed point of $\Psi \mapsto A(\phi_0,\Psi)$. By Lemma 4.2 the sequences ϕ_n and Φ_n converge uniformly respectively to ϕ_0 and

Φ_0 on bounded subsets. More precisely, although the norm in $\mathcal{X}$ is not the supremum norm, for each $c > 0$ we have

$$\|\phi(t,x) - \psi(t,x)\| \leq \|x\| d(\phi,\psi) \leq c\, d(\phi,\psi)$$

whenever $t \geq 0$ and $x \in E(t)$ with norm $\|x\| \leq c$. This yields the desired uniform convergence on bounded subsets. It follows from (4.21) that ϕ_0 is of class C^1 in ξ, and that

$$\partial\phi_0/\partial\xi = \Phi_0 \tag{4.22}$$

(we recall that if a sequence f_n of C^1 functions converges uniformly, and the sequence f_n' of derivatives also converges uniformly, then the limit of f_n is of class C^1, and its derivative is the limit of f_n').

Finally, we assume that $(\partial f/\partial u)(t,0) = 0$ for every $t \geq 0$. Since the pair $(\phi_1, \Phi_1) = (0,0)$ is in $\mathcal{X} \times \mathcal{F}_0$, and $S(\mathcal{X} \times \mathcal{F}_0) \subset \mathcal{X} \times \mathcal{F}_0$, the sequence (ϕ_n, Φ_n) defined in (4.20) is also in $\mathcal{X} \times \mathcal{F}_0$. Therefore, $\Phi_0(s,0) = 0$ for every $s \geq 0$, and it follows from (4.22) that in this case $(\partial\phi_0/\partial\xi)(s,0) = 0$ for every $s \geq 0$. □

Now we show that the set V_ϕ is in fact a C^1 manifold. The reason why this is not an immediate consequence of Theorem 4.1 is that in general the unique function ϕ in Theorem 3.1 is not differentiable in s, simply because the spaces $E(s)$ may vary with s, and thus differentiability may make no sense. Due to this difficulty we need an additional argument.

Theorem 4.7. *Let A and f be C^1 functions. If the equation $u' = A(t)u$ admits a nonuniform exponential dichotomy satisfying* (3.3)*, and conditions* (2.4) *and* (4.1) *hold with δ sufficiently small, then for the unique function ϕ in Theorem* 3.1 *the set V_ϕ in* (3.2) *is a C^1 manifold, and*

$$T_{(s,0)}V_\phi = \mathbb{R} \times E(s), \quad s > 0.$$

Proof. By Theorem 4.1, the function $\xi \mapsto \phi(s,\xi)$ is of class C^1 for each fixed $s \in \mathbb{R}_0^+$. We consider the map

$$F = F_s \colon (-s,\infty) \times E(s) \to \mathbb{R}^+ \times X$$

defined by

$$F(t,\xi) = \Psi_t(s,\xi,\phi(s,\xi)). \tag{4.23}$$

Since A and f are of class C^1, the map

$$\mathbb{R}^+ \times \mathbb{R}^+ \times X \ni (t,s,v) \mapsto \Psi_t(s,v)$$

is also of class C^1, and the same happens with F. Moreover, we can easily verify that F is injective, and thus it is a parametrization of class C^1 of V_ϕ. This shows that V_ϕ is a C^1 manifold. □

It follows readily from the proof of Theorem 4.7 that to establish the C^1 regularity of V_ϕ it would be sufficient to know that $\xi \mapsto \phi(s, \xi)$ is of class C^1 for some $s > 0$.

5. Higher regularity of the stable manifolds

For $X = \mathbb{R}^p$, we show in this section that the stable manifold V_ϕ in Theorem 3.1 is a C^k manifold when A and f are of class C^k. We emphasize that this is an optimal result.

THEOREM 5.1. *Let A and f be of class C^k for some $k \geq 2$. If $u' = A(t)u$ admits a nonuniform exponential dichotomy satisfying* (3.3)*, condition* (4.1) *holds, and*

$$\left\|\frac{\partial f}{\partial u}(t,u)\right\| \leq \delta e^{-2\varepsilon t} \quad \text{and} \quad \left\|\frac{\partial^2 f}{\partial u^2}(t,u)\right\| \leq \delta e^{-2\varepsilon t} \tag{5.1}$$

for every $t \geq 0$ and $u \in X$, and some sufficiently small δ (depending on k), then for the unique function ϕ in Theorem 3.1 the set V_ϕ is a C^k manifold. Moreover, for every $s \geq 0$, $\xi, v, \overline{v} \in E(s)$, and $\tau \geq 0$ we have

$$\left\|\frac{\partial \Psi_\tau}{\partial u}(p_{s,\xi})\left(v, \frac{\partial \phi}{\partial \xi}(s,\xi)v\right) - \frac{\partial \Psi_\tau}{\partial u}(p_{s,\xi})\left(\overline{v}, \frac{\partial \phi}{\partial \xi}(s,\xi)\overline{v}\right)\right\| \leq 2De^{a\tau+\varepsilon s}\|v - \overline{v}\|.$$

PROOF. The proof is based on the study of the linear variational equations of the solutions of equation (2.6). The regularity of the stable manifold of the original equation is obtained integrating the "stable manifolds" of the linear variational equations, which coincide with the tangent spaces of the stable manifolds, thus gaining one additional derivative in the process. We note that when $k = 1$ the statement in Theorem 5.1 is contained in Theorem 3.1.

Let $\alpha\colon X \to [0, 1]$ be a C^k function with compact support, such that $\alpha(z) = 1$ when $\|z\| \leq 1$, and satisfying

$$\|\alpha(z)z\| \leq C \quad \text{and} \quad \left\|\frac{d}{dz}[\alpha(z)z]\right\| \leq C \tag{5.2}$$

for every $z \in X$ and some constant $C > 0$. We consider the vector field $\Gamma\colon \mathbb{R}_0^+ \times X \times X \to X \times X$ given by

$$\Gamma(t,u,z) = \left(A(t)u + f(t,u), A(t)z + \alpha(z)\frac{\partial f}{\partial u}(t,u)z\right),$$

and the corresponding nonautonomous differential equation

$$(u', z') = \Gamma(t,u,z). \tag{5.3}$$

The first component $u(t)$ of a solution of (5.3) satisfies

$$u' = A(t)u + f(t,u), \tag{5.4}$$

while the second component $z(t)$ satisfies

$$z' = A(t)z + \alpha(z)\frac{\partial f}{\partial u}(t, u(t))z.$$

We also consider the vector field $\overline{\Gamma}\colon \mathbb{R}_0^+ \times X \times X \to X \times X$ given by

$$\overline{\Gamma}(t,u,z) = \left(A(t)u + f(t,u), A(t)z + \frac{\partial f}{\partial u}(t,u)z\right).$$

We notice that $\Gamma(t,u,z) = \overline{\Gamma}(t,u,z)$ when $\|z\| \le 1$. If Ψ_τ is the semiflow in (2.7), then the autonomous equation

$$(t', u', z') = (1, \overline{\Gamma}(t,u,z)) \tag{5.5}$$

generates the semiflow $\overline{\Theta}_\tau$ in $\mathbb{R}_0^+ \times X \times X$ given by

$$\overline{\Theta}_\tau(s,u,z) = \left(\Psi_\tau(s,u), \frac{\partial \Psi_\tau}{\partial u}(s,u)z\right). \tag{5.6}$$

Lemma 5.2. *For the vector field*

$$F(t,u,z) = \left(f(t,u), \alpha(z)\frac{\partial f}{\partial u}(t,u)z\right),$$

the following properties hold:

(a) *F is of class C^{k-1} and $F(t,0,0) = F(t,u,z) = 0$ for every $t \ge 0$ and $(u,z) \in X \times X$ with $\|(u,z)\| \ge c'$, for some constant $c' > 0$;*
(b) *for each $t \ge 0$ and $u_1, u_2, z_1, z_2 \in X$ we have*

$$\|F(t,u_1,z_1) - F(t,u_2,z_2)\| \le d\delta e^{-3\varepsilon t}\|(u_1,z_1) - (u_2,z_2)\|, \tag{5.7}$$

for some constant $d > 0$.

Proof. The first property follows immediately from the definitions, and the second property follows from (5.1) and (5.2). □

We emphasize that the constant δ in (5.7) is the same as in (2.4).

Now let $\mathfrak{Y}$ be the space of continuous functions

$$\psi\colon \{(s,\xi,v) \in \mathbb{R}_0^+ \times X \times X : \xi, v \in E(s)\} \to X$$

such that:

- $E(s) \ni v \mapsto \psi(s,\xi,v)$ is linear for each $s \ge 0$ and $\xi \in E(s)$;
- for each $s \ge 0$ and $\xi, v \in E(s)$ we have

$$\|\psi(s,\xi,v)\| \le \|v\|. \tag{5.8}$$

We also write $p_{s,\xi} = (s,\xi,\phi(s,\xi))$ with ϕ as in Theorem 3.1.

Lemma 5.3. *Provided that δ is sufficiently small, there exists a unique function $\psi \in \mathfrak{Y}$ such that the set*

$$T_\psi = \{(p_{s,\xi}, v, \psi(s,\xi,v)) : (s,\xi,v) \in \mathbb{R}_0^+ \times E(s) \times E(s)\}$$

satisfies

$$\overline{\Theta}_\tau(T_\psi) = T_\psi \quad \textit{for every } \tau \geq 0. \tag{5.9}$$

Furthermore, for every $s \geq 0$, $\xi, v, \overline{v} \in E(s)$, and $\tau \geq 0$ we have

$$\|\overline{\Theta}_\tau(p_{s,\xi}, v, \psi(s,\xi,v)) - \overline{\Theta}_\tau(p_{s,\xi}, \overline{v}, \psi(s,\xi,\overline{v}))\| \leq 2De^{a\tau+\varepsilon s}\|v - \overline{v}\|. \tag{5.10}$$

Proof. By Lemma 5.2 and Theorem 3.1, for each $s \geq 0$ and $\xi \in E(s)$, provided that δ is sufficiently small there exists a unique function $\overline{\phi} = \overline{\phi}_\xi \in \mathfrak{X}$ such that its graph

$$V_{\overline{\phi}} = \{(s, v, \overline{\phi}(s,v)) : (s,v) \in \mathbb{R}_0^+ \times E(s)\}$$

is invariant under the semiflow generated by the autonomous equation

$$t' = 1, \quad z' = A(t)z + \alpha(z)\frac{\partial f}{\partial u}(t, u(t))z,$$

where $u(t)$ is the solution of equation (5.4) with $u(s) = (\xi, \phi(s,\xi))$. Since

$$z \mapsto A(t)z + \alpha(z)\frac{\partial f}{\partial u}(t, u(t))z$$

is linear in a neighbourhood of zero, the functions $v \mapsto \overline{\phi}(s,v)$ are also linear in a neighbourhood of zero (possibly depending on s). Writing $\overline{\phi}_\xi(s,v) = \psi(s,\xi,v)$ for any sufficiently small v, and extending $\psi(s,\xi,\cdot)$ linearly to the whole X, it follows that $\psi \in \mathfrak{Y}$ (we note that condition (5.8) follows from (3.1)). Moreover, the uniqueness of $\overline{\phi}$ (for each ξ) implies that ψ is the unique function in $\mathfrak{Y}$ satisfying (3.9). Finally, inequality (5.10) follows readily from Theorem 3.1. □

We proceed by induction on k. Namely, let us assume that the statement in Theorem 5.1 holds for $k = l$, and that A and f are of class C^{l+1}. Then, by Lemma 5.2, the unique function $\psi \in \mathfrak{Y}$ in Lemma 5.3 is of class C^l in ξ. Actually, by the induction hypothesis the statement in Theorem 3.1 implies that the map $(\xi, v) \mapsto \overline{\phi}_\xi(s,v)$, with $\overline{\phi}_\xi$ as in the proof of Lemma 5.3, is of class C^l. Then this property extends to ψ due to the linearity of $v \mapsto \overline{\phi}_\xi(s,v)$ in some neighbourhood of zero (possibly depending on s).

Let $z(t) = (v(t), w(t)) \in E(t) \times F(t)$ be the solution of equation (5.5) with

$$u(t) = (x(t), y(t)) = (x_\phi(t,\xi), \phi(t, x_\phi(t,\xi))).$$

We write

$$w(t) = \psi(t, x_\phi(t,\xi), v(t)) = \Psi(t, x_\phi(t,\xi))v(t) = \Psi(z_\phi(t))u(t),$$

with $z_\phi(t)$ as in (4.4). We notice that $\Psi \in \mathcal{F}$. Indeed, it follows from (5.8) that $\|\Psi(s,\xi)\| \le 1$ for each $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$, and thus $\Psi \in \mathcal{F}$. By the variation-of-constants formula, we have

$$
\begin{aligned}
(5.11)\quad v(t) &= T(t,s)v(s) + \int_s^t P(t)T(t,\tau)\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))w(\tau)\biggr)\,d\tau \\
&= T(t,s)v(s) \\
&\quad + \int_s^t P(t)T(t,\tau)\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr)\,d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
(5.12)\quad \Psi(z_\phi(t))v(t) &= T(t,s)\Psi(s,\xi)v(s) \\
&\quad + \int_s^t Q(t)T(t,\tau)\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr)\,d\tau
\end{aligned}
$$

for every $t \ge s$. Comparing (5.11) with (4.6) we conclude that $v(t) = W(t)v(s)$ with $W(t)$ as in (4.6).

On the other hand, proceeding in a similar manner to that in the proof of Lemma 2.5 we can show that (5.12) is equivalent to

$$
\begin{aligned}
(5.13)\quad &\Psi(s,\xi)v(s) \\
&= -\int_s^\infty Q(s)T(\tau,s)^{-1}\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr)\,d\tau.
\end{aligned}
$$

We first note that by (4.10), for each $\tau \ge s$ we have

$$
\begin{aligned}
&\biggl\|\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr\| \\
&\qquad \le 2\delta e^{-3\varepsilon\tau}\|v(\tau)\| \le 2\delta D e^{(a+2\delta D)(\tau-s)+\varepsilon s} e^{-3\varepsilon\tau}\|v(s)\|.
\end{aligned}
$$

It follows from the second inequality in (2.3) that

$$
\begin{aligned}
&\int_s^\infty \biggl\|Q(s)T(\tau,s)^{-1}\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr)\biggr\|\,d\tau \\
&\qquad \le 2\delta D^2\|v(s)\| \int_s^\infty e^{(a-b-\varepsilon+2\delta D)(\tau-s)}\,d\tau,
\end{aligned}
$$

and in view of (3.15) the integral in (5.13) is well defined.

To show that (5.12) and (5.13) are equivalent, we first assume that identity (5.12) holds, and we write it in the equivalent form

$$
\begin{aligned}
(5.14)\quad \Psi(s,\xi)v(s) &= T(t,s)^{-1}\Psi(z_\phi(t))v(t) \\
&\quad - \int_s^t Q(s)T(s,\tau)\biggl(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\biggr)\,d\tau.
\end{aligned}
$$

We have

$$\|T(t,s)^{-1}\Psi(z_\phi(t))v(t)\| = \|T(t,s)^{-1}Q(t)\Psi(z_\phi(t))v(t)\| \\ \le De^{-b(t-s)+\varepsilon t}\|v(t)\| \le D^2e^{2\varepsilon s}e^{(a-b+\varepsilon+2\delta D)(t-s)}\|v(s)\|.$$

In view of (3.15), letting $t\to\infty$ in (5.14) we obtain (5.13). Now we assume that identity (5.13) holds. We have

$$\begin{aligned} T(t,s)\Psi(s,\xi)v(s) \\ + \int_s^t Q(t)T(t,\tau)\left(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\right)d\tau \\ = -\int_t^\infty Q(t)T(t,\tau)\left(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\right)d\tau. \end{aligned} \tag{5.15}$$

In view of (5.13) with (s,ξ) replaced by $(t,x_\phi(t,\xi))$ we obtain

$$\Psi(z_\phi(t))v(t) \\ = -\int_t^\infty Q(t)T(\tau,t)^{-1}\left(\frac{\partial f}{\partial x}(y_\phi(\tau))v(\tau) + \frac{\partial f}{\partial y}(y_\phi(\tau))\Psi(z_\phi(\tau))v(\tau)\right)d\tau,$$

which together with (5.15) yields identity (5.12).

Since $W(s)=\mathrm{id}_{E(s)}$, comparing (5.13) with (4.5) we find that

$$\Psi(s,\xi) = A(\phi,\Psi)(s,\xi)$$

for every $(s,\xi)\in\mathbb{R}_0^+\times E(s)$. Since $\Psi\in\mathcal{F}$, by the uniqueness of the fixed point of $\Phi\mapsto A(\phi,\Phi)$, which by (4.22) coincides with $\partial\phi/\partial\xi$, we have $\Psi=\partial\phi/\partial\xi$, and hence,

$$\psi(s,\xi,v) = (\partial\phi/\partial\xi)(s,\xi)v. \tag{5.16}$$

Since ψ is of class C^l in ξ, it follows from (5.16) that ϕ is of class C^{l+1} in ξ. Therefore, the statement in Theorem 5.1 holds for $k=l+1$. This completes the induction argument.

Finally, proceeding as in the proof of Theorem 4.7 we can show that the parametrization F in (4.23) is now of class C^k, since the same happens with the map $(t,s,v)\mapsto\Psi_t(s,v)$. Therefore, V_ϕ is a C^k manifold. The last property follows readily from (5.6), (5.10), and (5.16). □

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