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EXISTENCE OF MULTI-PEAK SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS IN $\mathbb{R}^N$

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Abstract. Using variational methods we establish existence of multi-peak solutions for the following class of quasilinear problems

$$-\varepsilon^p \Delta_p u + V(x)u^{p-1} = f(u), \quad u > 0, \text{ in } \mathbb{R}^N$$

where $\Delta_p u$ is the p-Laplacian operator, $2 \leq p < N$, $\varepsilon > 0$ and f is a continuous function with subcritical growth.

1. Introduction

Many recent studies have focused on the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad ih\frac{\partial \Psi}{\partial t} = -h^2\Delta\Psi + (V(z)+E)\Psi - f(\Psi) \quad \text{for all } z \in \Omega,$$

where $\varepsilon > 0$, Ω is a domain in $\mathbb{R}^N$, not necessarily bounded, with empty or smooth boundary. This class of equation is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics.

Knowledge of the solutions for the elliptic equation

$$\text{(S)}_\varepsilon \qquad \begin{cases} -\varepsilon^2\Delta u + V(z)u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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has a great importance in the study of standing-wave solutions of (NLS). In recent years, the existence and concentration of positive solutions for general semilinear elliptic equations $(\mathrm{S})_\varepsilon$ for the case $N \geq 3$ have been extensively studied, see for example, A. Floer and A. Weinstein [18], Y. J. Oh [26], P. H. Rabinowitz [27], X. Wang [30], S. Cingolani and M. Lazzo [14], A. Ambrosetti, M. Badiale and S. Cingolani [6], C. Gui [21], M. del Pino and P. L. Felmer [15], [16] and their references.

In [27], by a mountain pass argument, Rabinowitz proves the existence of positive solutions of $(\mathrm{S})_\varepsilon$, for $\varepsilon > 0$ small, whenever

$$\liminf_{|z|\to\infty} V(z) > \inf_{z\in\mathbb{R}^N} V(z) = \gamma > 0.$$

Later X. Wang [30] showed that these solutions concentrate at global minimum points of V as ε tends to 0.

In [15], M. del Pino and P. L. Felmer have found solutions which concentrate around local minimum of V by introducing a penalization method. More precisely, they assume that an open and bounded set Λ compactly contained in Ω satisfies

$$(\mathrm{V}_1) \qquad 0 < \gamma \leq V_0 = \inf_{z\in\Lambda} V(z) < \min_{z\in\partial\Lambda} V(z).$$

Existence of nodal solutions for general semilinear elliptic equations for the case $N \geq 3$ and $\varepsilon = 1$ have been established in T. Bartsch and Z.-Q. Wang in [10], [11] and T. Bartsch, K.-C. Chang and Z.-Q. Wang in [7], T. Bartsch, A. Pankov and Z.-Q. Wang in [9]. For the case involving $\varepsilon > 0$ sufficiently small, some results involving concentration of nodal solutions can be found in the works of T. Bartsch, M. Clapp and T. Weth [8], T. Bartsch, K.-C. Chang and Z.-Q. Wang in [7], E. S. Noussair and J. Wei [24], [25] and C. O. Alves and S. H. M. Soares [4], [5].

The existence of multi-peak solution has been considered in some papers. In [21], C. Gui showed the existence of a k-peak solution u_ε for the problem $(\mathrm{S})_\varepsilon$, under the assumptions that $\inf_{x\in\mathbb{R}^N} V(x) > V_0$ and there are bounded domains Ω_i; mutually disjoint, such that

$$\inf_{x\in\Omega_i} V(x) < \inf_{x\in\partial\Omega_i} V(x).$$

A similar result was also obtained by del M. Pino and P. L. Felmer in [16] using a different approach. In [19], A. Giacomini and M. Squassina showed the existence of multi-peak for a class of quasilinear problems, and in [13], D. Cao and E. S. Noussair have studied the existence of multi-bump standing waves with a critical frequency, that is, when $\inf_{x\in\mathbb{R}^N} V(x) = 0$.

In all references cited in this paper, the existence of multi-peak solution for problems involving the p-Laplacian operation with $p > 2$ was not considered.

Motivated by this fact, in the present paper we are concerned with the existence of positive multi-peak solutions for the following class of quasilinear elliptic problems

$$(\mathrm{P})_\varepsilon \qquad \begin{cases} -\varepsilon^p\Delta_p u + V(x)u^{p-1} = f(u) & \text{in } \mathbb{R}^N,\\ u > 0 & \text{in } \mathbb{R}^N,\\ u \in W^{1,p}(\mathbb{R}^N) \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $2 \leq p < N$, f is a continuous function with subcritical growth, $\varepsilon > 0$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ is continuous functions with $V(x) \geq 0$ for all $x \in \mathbb{R}^N$. The general hypotheses considered in this work are the following:

- (V_1) There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$.
- (V_2) There exist k disjoint bounded regions $\Omega_1, \ldots, \Omega_k$ such that

$$M_i = \min_{\partial\Omega_i} V(x) > \alpha_i = \inf_{\Omega_i} V(x), \quad i = 1, \ldots, k.$$

- (f_1) There exists $p < q < p^* = Np/(N-p)$ such that

$$\frac{f(t)}{t^{q-1}} \to 0 \quad \text{as } t \to \infty.$$

- (f_2) $f(t) = o(|t|^{p-1})$ as $t \to 0$.
- (f_3) There exist $\theta \in (p, p^*)$ and $r > 0$ such that

$$\theta F(t) \leq t f(t) \quad \text{for all } t \geq r$$

  where $F(t) = \int_0^t f(\tau)\, d\tau$.
- (f_4) The function $f(t)/t^{p-1}$ is increasing for $t \in (0, \infty)$,

Motivated by papers [1], [2], [17] and [21], we show the existence of multi-peak solutions to $(\mathrm{P})_\varepsilon$ for general case $p \geq 2$. Our current framework is different of those used in [21], because the p-Laplacian is not linear, and in our opinion, some properties that occur for 2-Laplacian (Laplacian operator) are not standard that they hold for general case, $p \geq 2$, therefore, some modifications are necessary in the sets that appear in the minimax arguments explored in [21]. The arguments developed in this paper are variational, and our main result can be seen as a complement of the study made in [21], in the sense that, we are working with p-Laplacian operator and show the existence of multiple positive multi-peak solutions. Moreover, our main result also complete the study made in [28] which has considering the existence of multi-peak solution for a class of problem involving the p-Laplacian operator for the case $1 < p \leq 2$.

The our main result is the following

Theorem 1.1. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *occur. Then, for each* $\Gamma \subset \{1, \ldots, k\}$, *there exist* $\varepsilon^* > 0$ *such that, for* $0 < \varepsilon \leq \varepsilon^*$, $(\mathrm{P})_\varepsilon$ *has a family*

$\{u_\varepsilon\}$ *of positive solutions verifying the following property for* ε *small enough: There exists* $\delta > 0$ *such that*

$$\sup_{\mathbb{R}^N} u_\varepsilon(x) \geq \delta.$$

There exists $P_{\varepsilon,i} \in \Omega_i$ *for all* $i \in \Gamma$ *such that, for each* $\xi > 0$*, there exists* $r > 0$ *verifying*

$$\sup_{B_{\varepsilon r}(P_{\varepsilon,i})} u_\varepsilon(x) \geq \delta \quad \textit{for all } i \in \Gamma$$

and

$$\sup_{\mathbb{R}^N \setminus \bigcup_{j\in\Gamma} B_{\varepsilon r}(P_{\varepsilon,i})} u_\varepsilon(x) < \xi \quad \textit{for all } i \in \Gamma.$$

In the above theorem, if Γ has l elements, we say that u_ε is a l-peak solution.

2. Penalization and $(\mathrm{PS})_c$ condition

In this section, following the approach of del M. Pino and P. L. Felmer [15] and C. Gui [21], we define a suitable penalization of the functional energy associated to $(\mathrm{P})_\varepsilon$. To this end, we fix some notations and define some functionals that will be used in this work.

Hereafter, when h is a mensurable function, we denote by $\int_{\mathbb{R}^N} h$ the following integral $\int_{\mathbb{R}^N} h\,dx$. Moreover, we will use the symbols $\|u\|$, $|u|_r$ $(r > 1)$ and $\|u\|_\infty$ to denote the usual norms in the spaces $W^{1,p}(\mathbb{R}^N)$, $L^r(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, respectively. Moreover, since that we intend to find positive solutions, in all this paper let us assume that

$$f(t) = 0 \quad \text{for all } t \in (-\infty, 0].$$

Using the change variable $x = \varepsilon y$, it is immediate to check that problem the problem $(\mathrm{P})_\varepsilon$ is equivalent to the following problem

$$(\widetilde{\mathrm{P}})_\varepsilon \qquad \begin{cases} -\Delta_p u + V(\varepsilon x)u^{p-1} = f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases}$$

From now on, we will work with $(\widetilde{\mathrm{P}})_\varepsilon$ to get the multi-peak solution associated to $(\mathrm{P})_\varepsilon$. First of all, notice that nonnegative weak solutions of $(\widetilde{\mathrm{P}})_\varepsilon$ are the critical points of the functional $I\colon E_\varepsilon \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \int_{\mathbb{R}^N} F(u)$$

where E_ε is the space of functions defined by

$$E_\varepsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon = \bigg(\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p)\bigg)^{1/p}.$$

From (V_1), it is easy to see that $(E_\varepsilon, \|\cdot\|_\varepsilon)$ is a Banach space and $E_\varepsilon \subset W^{1,p}(\mathbb{R}^N)$ for all $\varepsilon > 0$.

In what follows, let $A > 1$ and $m > 0$ verifying $f(m)/m^{p-1} = V_0/A$ and $\widetilde{f}, \widetilde{F}: \mathbb{R} \to \mathbb{R}$ the following functions

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{if } s \le m, \\ \dfrac{V_0}{A} s^{p-1} & \text{if } s > m, \end{cases} \qquad \text{and} \qquad \widetilde{F}(s) = \int_0^s \widetilde{f}(\tau)\, d\tau.$$

Moreover, we fix $\Gamma \subset \{1, \dots, k\}$, $\Omega = \bigcup_{j\in\Gamma} \Omega_j$ and the functions

$$\zeta(x) = \begin{cases} 1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega, \end{cases}$$

$$(\mathrm{g}_1) \qquad g(x,s) = \zeta(x) f(s) + (1-\zeta(x))\widetilde{f}(s)$$

and

$$(\mathrm{g}_2) \qquad G(x,s) = \int_0^s g(x,t)\, dt = \zeta(x)F(s) + (1-\zeta(x))\widetilde{F}(s).$$

From (g_1) and (g_2),

$$(\mathrm{g}_3) \qquad \theta G(x,s) \le s g(x,s), \quad x \in \Omega,\ s \ge 0$$

and

$$(\mathrm{g}_4) \qquad p G(x,s) \le s g(x,s) \le \frac{V_0}{A}|s|^p, \quad x \in \mathbb{R}^N \setminus \Omega,\ s \ge 0.$$

In what follows, for each $\varepsilon > 0$, we denote by $g_\varepsilon(x,s)$ and $G_\varepsilon(x,s)$ the functions given by

$$g_\varepsilon(x,s) = g(\varepsilon x, s) \quad \text{and} \quad G_\varepsilon(\varepsilon x, s) = \int_0^s g(\varepsilon x, \tau)\, d\tau$$

and we set the functional $\Phi_\varepsilon: E_\varepsilon \to \mathbb{R}$ given by

$$\Phi_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \int_{\mathbb{R}^N} G_\varepsilon(x,u).$$

Under the conditions (V_1), (g_1)–(g_4), $\Phi_\varepsilon \in C^1(E_\varepsilon, \mathbb{R})$ and its critical points are nonnegative weak solutions of the quasilinear problem

$$(\mathrm{A})_\varepsilon \qquad -\Delta_p u + V(\varepsilon x)|u|^{p-2}u = g_\varepsilon(x,u) \quad \text{in } \mathbb{R}^N.$$

Notice that positive solutions of the last equation are associated with the positive solutions of $(\widetilde{\mathrm{P}})_\varepsilon$, because if v_ε is a positive solution of $(\mathrm{A})_\varepsilon$ verifying $v_\varepsilon(x) \le m$ in $\mathbb{R}^N \setminus \Omega_\varepsilon$ with $\Omega_\varepsilon = \Omega/\varepsilon$, it follows that it is a positive solution to $(\widetilde{\mathrm{P}})_\varepsilon$.

2.1. The Palais–Smale condition and its consequences. We start this subsection studying the boundedness of Palais–Smale sequence associated to Φ_ε, that is, of a sequence $\{u_n\} \subset E_\varepsilon$ verifying

$$\text{(PS)} \qquad \Phi_\varepsilon(u_n) \to c \quad \text{and} \quad \Phi'_\varepsilon(u_n) \to 0$$

for some $c \in \mathbb{R}$ (shortly $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence).

Lemma 2.2. *Suppose $\{u_n\} \subset E_\varepsilon$ is a $(\mathrm{PS})_c$ sequence. Then, there exists a positive constant K, which is independent of $\varepsilon > 0$, that satisfies*

$$\|u_n\|_\varepsilon^p \le K \quad \textit{for all } n \in \mathbb{N}.$$

Proof. From definition of Palais-sequence, we derive easily

$$\Phi_\varepsilon(u_n) - \frac{1}{\theta}\Phi'_\varepsilon(u_n)u_n = c + o_n(1) + o_n(1)\|u_n\|_\varepsilon,$$

where $o_n(1) \to 0$. This equality combined with (g_3) and (g_4) yields

$$\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_\varepsilon^p - \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} (\widetilde{F}(u_n) - \frac{1}{\theta}\widetilde{f}(u_n)u_n) \le c + o_n(1) + o_n(1)\|u_n\|_\varepsilon. \tag{2.1}$$

Since $\widetilde{f}$ and $\widetilde{F}$ verify

$$\widetilde{F}(s) - \frac{1}{\theta}\widetilde{f}(s)s \le \left(\frac{1}{p} - \frac{1}{\theta}\right)\frac{V_0}{A}|s|^p \quad \text{for all } s \in \mathbb{R},$$

we obtain,

$$\left(\frac{1}{p} - \frac{1}{\theta}\right)\left(\|u_n\|_\varepsilon^p - \frac{V_0}{A}|u_n|_p^p\right) \le c + o_n(1) + o_n(1)\|u_n\|_\varepsilon.$$

From this,

$$\left(\frac{1}{p} - \frac{1}{\theta}\right)\left(1 - \frac{1}{A}\right)\|u_n\|_\varepsilon^p \le c + o_n(1) + o_n(1)\|u_n\|_\varepsilon$$

which implies that $\{u_n\}$ is bounded. Thereby,

$$\limsup_{n\to\infty} \|u_n\|_\varepsilon^p \le \left(\frac{1}{p} - \frac{1}{\theta}\right)^{-1}\left(1 - \frac{1}{A}\right)^{-1} c,$$

and, therefore, there exists $K > 0$, which is independent of ε, such that

$$\|u_n\|_\varepsilon^p \le K \quad \text{for all } n \in \mathbb{N}. \qquad \square$$

PROPOSITION 2.3. *For each $\varepsilon > 0$, Φ_ε satisfies $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$, that is, any $(\mathrm{PS})_c$ sequence $\{u_n\} \subset E_\varepsilon$ has a strongly convergent subsequence in E_ε.*

PROOF. Let $\{u_n\} \subset E_\varepsilon$ be a Palais–Smale sequence. By Lemma 2.2, $\{u_n\}$ is bounded in E_ε and for some subsequence, still denoted by $\{u_n\}$, there exists $u \in E_\varepsilon$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } E_\varepsilon \text{ and } W^{1,p}(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^{q+1}_{\mathrm{loc}}(\mathbb{R}^N) \text{ and } L^p_{\mathrm{loc}}(\mathbb{R}^N). \end{aligned}$$

Furthermore, setting $\varphi_n(x) = \eta_\varepsilon(x)u_n(x)$, we have $\Phi'_\varepsilon(u_n)\varphi_n \to 0$ where $\eta_\varepsilon \in C^\infty(\mathbb{R}^N)$ is given by

$$\begin{aligned} \eta_\varepsilon(x) &= 1 && \text{for all } x \in B^c_R(0), \\ \eta_\varepsilon(x) &= 0 && \text{for all } x \in B_{R/2}(0), \\ \eta_\varepsilon(x) &\in [0,1] && \text{with } \Omega_\varepsilon \subset B_R(0). \end{aligned}$$

This combined with [3, Lemma 1.1] implies that for each $\gamma > 0$ fixed, there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\{x\in\mathbb{R}^N:|x|\geq R\}} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) \leq \gamma \quad \text{for } n \in \mathbb{N}.$$

The last inequality together with the subcritical growth of g_ε implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (P^1_n + V(\varepsilon x)P^2_n) = 0$$

where

$$P^1_n = \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle$$

and

$$P^2_n = (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u).$$

Using the same type of arguments found in Jianfu [22, Lemma 4.2] (see also P. Tolksdorff [29]), it follows that $u_n \to u$ in E_ε. $\square$

Next, we study the behavior of a $(\mathrm{PS})^*_c$ sequence, that is, a sequence $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ satisfying:

$$u_n \in E_{\varepsilon_n} \quad \text{and} \quad \varepsilon_n \to 0, \qquad \Phi_{\varepsilon_n}(u_n) \to c, \qquad \|\Phi'_{\varepsilon_n}(u_n)\|^*_{\varepsilon_n} \to 0.$$

The proposition below was proved in [21] for the case $p = 2$.

Proposition 2.4. *Let $\{u_n\}$ be a $(\mathrm{PS})_c^*$ sequence. Then there exists a subsequence of $\{u_n\}$, still denoted by itself, a nonnegative integer s, sequences of points $\{y_{n,j}\} \subset \mathbb{R}^N$ with $j = 1, \dots, s$, such that*

$$\varepsilon_n y_{n,j} \to x_j \in \overline{\Omega} \quad \textit{and} \quad |y_{n,j} - y_{n,i}| \to \infty, \quad \textit{as } n \to \infty$$

and

$$\bigg\| u_n(\,\cdot\,) - \sum_{j=1}^{s} u_{0,j}(\,\cdot\, - y_{n,j})\chi_{\varepsilon_n}(\,\cdot\, - y_{n,j}) \bigg\|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty$$

where $\chi_\varepsilon(x) = \chi(x/(-ln\varepsilon))$ for $0 < \varepsilon < 1$, and χ is a cut-off function which is 1 for $|x| \le 1$, it is 0 for $|x| \ge 2$ and $|\nabla\chi| \le 2$. The function $u_{0,j} \ne 0$ is a nonnegative solution for

$$-\Delta_p u + V_j u^{p-1} = g_{0,j}(x,u), \quad x \in \mathbb{R}^N \tag{2.2}$$

where $V_j = V(x_j) \ge V_0 > 0$ and $g_{0,j}(x,u) = \lim_{n\to\infty} g_\varepsilon(\varepsilon_n x + \varepsilon_n y_{n,j}, u)$. Moreover, we have $c \ge 0$ and

$$c = \sum_{j=1}^{s} J_{0,j}(u_{0,j})$$

where $J_{0,j}\colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ denotes the functional given by

$$J_{0,j}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V_j|u|^p) - \int_{\mathbb{R}^N} G_{0,j}(x,u)$$

with $G_{0,j}(x,t) = \int_0^t g_{0,j}(x,\tau)\,d\tau$.

Proof. As in the proof of Lemma 2.2, it is easy to check that there exists $K > 0$ such that

$$\|u_n\|_{\varepsilon_n}^p \le K \quad \text{for all } n \in \mathbb{N}.$$

Thus $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and we can assume that for some $u \in W^{1,p}(\mathbb{R}^N)$

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^{q+1}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } q \in [1, p^*-1) \end{aligned}$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Using the properties (g_3)–(g_4) together with the definition of the functions g_ε and G_ε, we derive that $c \ge 0$. Moreover, it is immediate to check that $\|u_n\|_{\varepsilon_n} \to 0$ if $c = 0$. This way, we will consider only the case $c > 0$.

We claim that there exist positive constants R, a, a subsequence of $\{u_n\}$, still denoted by itself, and a sequence $\{y_{n,1}\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_{n,1})} |u_n(x)|^p > a. \tag{2.3}$$

Otherwise, by Lions' Lemma (see [23]), $u_n \to 0$ in $L^{q+1}(\mathbb{R}^N)$. Using, the assumptions on f, for $\gamma \in (0, V_0)$, there exists $C > 0$ such that

$$\Phi'_{\varepsilon_n}(u_n)u_n \geq \|u_n\|_{\varepsilon_n} - \int_{\mathbb{R}^N} (\gamma|u_n|^p + C|u_n|^{q+1}) \geq \left(1 - \frac{\gamma}{V_0}\right)\|u_n\|^p_{\varepsilon_n} - C|u_n|^{q+1}_{q+1}$$

from where it follows that

$$0 \leq \left(1 - \frac{\gamma}{V_0}\right)\|u_n\|^p_{\varepsilon_n} \leq \Phi'_{\varepsilon_n}(u_n)u_n + C|u_n|^{q+1}_{q+1} = o_n(1).$$

This leads to $c = 0$, which is a contradiction. Therefore (2.3) holds. Now, letting $w_n(x) = u_n(x + y_{n,1})$, we have that $\{w_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and therefore for a subsequence, still denoted by itself, it converges weakly to $u_{0,1}$ in $W^{1,p}(\mathbb{R}^N)$. From (2.3), it follows that $u_{0,1} \neq 0$.

CLAIM 1. *The sequence $\{\varepsilon_n y_{n,1}\}$ is bounded. Moreover, there exists $x_1 \in \overline{\Omega}$ and a subsequence of $\{\varepsilon_n y_{n,1}\}$, still denoted by itself, such that $\varepsilon_n y_{n,1} \to x_1$.*

In fact, assuming by contradiction that $\{\varepsilon_n y_{n,1}\}$ is a unbounded sequence, we can assume without of loss generality, that $|\varepsilon_n y_{n,1}| \to \infty$. Since $\varepsilon \ln \varepsilon \to 0$ as $\varepsilon \to 0$ and Ω is a bounded domain, we have $\varepsilon_n y_{n,1} + \varepsilon_n x \in \mathbb{R}^N \setminus \Omega$ for all $|x| \leq 2|\ln \varepsilon_n|$ and n large enough. This together with (g_4) leads to

$$\begin{aligned}
&\Phi'_{\varepsilon_n}(u_n)(u_n(x)\chi_{\varepsilon_n}(x - y_{n,1})) \\
&\qquad \geq \int_{\mathbb{R}^N} \{|\nabla u_n(x)|^p + V(\varepsilon_n x)|u_n(x)|^p\}\chi_{\varepsilon_n}(x - y_{n,1}) \\
&\qquad\quad - \frac{2}{|\ln \varepsilon_n|} \int_{\mathbb{R}^N} |\nabla u_n(x)|^{p-1}|u_n(x)| \\
&\qquad\quad - \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n(x))u_n(x)\chi_{\varepsilon_n}(x - y_{n,1}) \\
&\qquad \geq \int_{\mathbb{R}^N} \{|\nabla w_n(x)|^p + V(\varepsilon_n x + \varepsilon_n y_{n,1})|w_n|^p\}\chi_{\varepsilon_n}(x) \\
&\qquad\quad - \int_{\mathbb{R}^N} \frac{V_0}{A}|w_n(x)|^p\chi_{\varepsilon_n}(x) + o_n(1) \\
&\qquad \geq \int_{\mathbb{R}^N} \{|\nabla w_n(x)|^p + V_0(1 - 1/A)|w_n(x)|^p\}\chi_{\varepsilon_n}(x) + o_n(1) \\
&\qquad \geq \int_{\mathbb{R}^N} |\nabla u_{0,1}|^p + V_0(1 - 1/A)|u_{0,1}|^p + o_n(1).
\end{aligned}$$

Since $\{u_n\}$ is $(\mathrm{PS})^*_c$ and $\{\|u_n\chi_{\varepsilon_n}(\,\cdot\, - y_{n,1})\|_{\varepsilon_n}\}$ is bounded in $\mathbb{R}$, we know that $\Phi'_{\varepsilon_n}(u_n)(u_n(x)\chi_{\varepsilon_n}(x - y_{n,1})) = o_n(1)$. Hence, the last inequality leads to

$$\int_{\mathbb{R}^N} (|\nabla u_{0,1}|^p + V_0(1 - 1/A)|u_{0,1}|^p) = 0$$

which is a contradiction, because $u_{0,1} \neq 0$. Therefore $\{\varepsilon_n y_{n,1}\}$ is bounded. From this, we can assume that for some subsequence, still denote by $\{w_n\}$, there exists

$x_1 \in \mathbb{R}^N$ such that $\varepsilon_n y_{n,1} \to x_1$. Analogous arguments as above can be used to prove that $x_1 \in \overline{\Omega}$, and the proof of the Claim 1 is over.

Next we outline the proof that $u_{0,1}$ is a solution of (2.2). To this end, we need to prove the below limits involving the sequence $\{w_n\}$ and $u_{0,1}$

Claim 2. *There exists a subsequence of $\{w_n\}$, still denoted by itself, such that*

$$w_n(x) \to u_{0,1}(x) \quad \textit{a.e. in } \mathbb{R}^N$$

and

$$\nabla w_n(x) \to \nabla u_{0,1}(x) \quad \textit{a.e. in } \mathbb{R}^N.$$

In fact, for each $R > 0$, let us consider a function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\begin{aligned} \phi_R(x) &= 1 \quad \text{for all } x \in B_R(0), \\ \phi_R(x) &= 0 \quad \text{for all } x \in B_{2R}^c(0), \\ 0 \le \phi_R(x) &\le 1 \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

Since the sequences $\{\|w_n\phi_R\|_{\varepsilon_n}\}$ and $\{\|u_{0,1}\phi_R\|_{\varepsilon_n}\}$ are bounded in $\mathbb{R}$, we reach that

$$\Phi'_{\varepsilon_n}(u_n)((w_n\phi_R)(x-y_{n,1})) = o_n(1) \quad \text{and} \quad \Phi'_{\varepsilon_n}(u_n)((u_{0,1}\phi_R)(x-y_{n,1})) = o_n(1).$$

Thereby

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_n|^{p-2}\nabla w_n \nabla(w_n\phi_R) + V(\varepsilon_n y_{n,1} + \varepsilon_n x)|w_n|^p\phi_R \\ - \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n)(w_n\phi_R) = o_n(1) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_n|^{p-2}\nabla w_n \nabla(u_{0,1}\phi_R) + V(\varepsilon_n y_{n,1} + \varepsilon_n x)|w_n|^{p-2}w_n u_{0,1}\phi_R \\ - \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n)(u_{0,1}\phi_R) = o_n(1). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5)

$$\int_{B_R(0)} \langle |\nabla w_n|^{p-2}\nabla w_n - |\nabla u_{0,1}|^{p-2}\nabla u_{0,1}, \nabla w_n - \nabla u_{0,1}\rangle \, dx = o_n(1)$$

and thus, using well known arguments

$$\int_{B_R(0)} |\nabla w_n - \nabla u_{0,1}|^p \to 0$$

so that, for some subsequence, still denoted by $\{w_n\}$,

$$\nabla w_n(x) \to \nabla u_{0,1}(x) \quad \text{a.e. in } B_R(0).$$

Since $R > 0$ is arbitrary, the proof of the Claim 2 is complete.

For any $\phi \in C_0^\infty(\mathbb{R}^N)$, a direct calculus shows that $\{\|\phi(x - y_{n,1})\|_{\varepsilon_n}\}$ is bounded in $\mathbb{R}$ and

$$\Phi'_{\varepsilon_n}(u_n)(\phi(x - y_{n,1})) = o_n(1).$$

This yields

$$\int_{\mathbb{R}^N} |\nabla w_n|^{p-2}\nabla w_n \nabla\phi + V(\varepsilon_n y_{n,1} + \varepsilon_n x)|w_n|^{p-2} w_n\phi = \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n)\phi + o_n(1). \tag{2.6}$$

Now, Claim 2 together with the definition of $g_{0,1}$ leads to

$$\int_{\mathbb{R}^N} |\nabla u_{0,1}|^{p-2}\nabla u_{0,1}\nabla\phi + V_1|u_{0,1}|^{p-2}u_{0,1}\phi = \int_{\mathbb{R}^N} g_{0,1}(x, u_{0,1})\phi$$

showing that $u_{0,1}$ is a nontrivial solution of (2.2). Moreover, from the definition of $g_{0,1}$, it follows that $u_{0,1}$ is nonnegative.

Hereafter, we consider $u_n^1(x) = u_n(x) - (u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})$. Using the definition of the function χ_{ε_n}, a straightforward computation shows that $\{\|u_n^1\|_{\varepsilon_n}\}$ is bounded in $\mathbb{R}$ and the below limits hold

$$\int_{\mathbb{R}^N} |u_n^1(x + y_{n,1}) - (w_n(x) - u_{0,1}(x))|^p = o_n(1) \tag{2.7}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n^1(x + y_{n,1}) - (\nabla w_n(x) - \nabla u_{0,1}(x))|^p = o_n(1). \tag{2.8}$$

Our goal is to prove that

$$\|\Phi'_{\varepsilon_n}(u_n^1)\|^*_{\varepsilon_n} \to 0 \quad \text{and} \quad \Phi_{\varepsilon_n}(u_n^1) \to c - J_{0,1}(u_{0,1}).$$

To this end, we observe that the limits (2.7) and (2.8) imply in the following limits

CLAIM 3.

$$\int_{\mathbb{R}^N} [G(\varepsilon_n x, u_n(x)) - G(\varepsilon_n x, u_n^1(x)) - G(\varepsilon_n x, (u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1}))] = o_n(1), \tag{2.9}$$

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)[|u_n^1(x)|^p - |u_n(x)|^p - |(u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})|^p] = o_n(1), \tag{2.10}$$

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)|B(u_n^1(x)) - B(u_n(x)) - B((u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1}))|^{p/(p-1)} = o_n(1), \tag{2.11}$$

$$\int_{\mathbb{R}^N} [\nabla u_n^1(x)|^p - |\nabla u_n(x)|^p - |\nabla(u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})|^p] = o_n(1), \tag{2.12}$$

$$\int_{\mathbb{R}^N} |A(\nabla u_n^1(x)) - A(\nabla u_n(x)) - A(\nabla(u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1}))|^{p/(p-1)} = o_n(1), \tag{2.13}$$

where

$$A(y) = |y|^{p-2}y \quad \textit{for all } y \in \mathbb{R}^N \quad \textit{and} \quad B(t) = |t|^{p-2}t \quad \textit{for all } t \in \mathbb{R}.$$

In what follows, we will prove only (2.9), (2.10) and (2,12), because the same kinds of arguments can be used to prove (2.11) and (2.13).

Proof of (2.9). By a change variable, (2.9) is equivalent to

$$\int_{\mathbb{R}^N} [G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n) - G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n - u_{0,1}\chi_{\varepsilon_n}) \\ - G(\varepsilon_n x + \varepsilon_n y_{n,1}, u_{0,1}\chi_{\varepsilon_n})] = o_n(1).$$

Since

$$u_{0,1}\chi_{\varepsilon_n} \to u_{0,1} \quad \text{in } W^{1,p}(\mathbb{R}^N),$$

from (f_1)–(f_2) follow that given $\gamma > 0$ there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\left| \int_{|x| \geq R} G(\varepsilon_n x + \varepsilon_n y_{n,1}, u_{0,1}\chi_{\varepsilon_n}) \right| \leq \frac{\gamma}{6} \quad \text{for all } n \geq n_0$$

and

$$\left| \int_{|x| \geq R} [G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n) - G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n - u_{0,1}\chi_{\varepsilon_n})] \right| \leq \frac{\gamma}{6} \tag{2.14}$$

for all $n \geq n_0$. On the other hand, using the compact Sobolev embeddings, it follows that there exists $n_1 \in \mathbb{N}$ such that

$$\int_{|x| \leq R} |G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n) - G(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n - u_{0,1}\chi_{\varepsilon_n}) \\ - G(\varepsilon_n x + \varepsilon_n y_{n,1}, u_{0,1}\chi_{\varepsilon_n})]| \leq \frac{\gamma}{6} \quad \text{for all } n \geq n_1. \tag{2.15}$$

From (2.14) and (2.15), the proof of (2.9) is complete.

Proof of (2.10). Next, A_n denotes the following integral

$$A_n = \int_{\mathbb{R}^N} V(\varepsilon_n x)[|u_n^1(x)|^p - |u_n(x)|^p - |(u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})|^p].$$

Notice that

$$A_n = \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n y_{n,1})[|u_n^1(x + y_{n,1})|^p - |u_n(x + y_{n,1})|^p - |u_{0,1}\chi_{\varepsilon_n}|^p]$$

or equivalently

$$A_n = \int_{B_{2|\ln \varepsilon_n|}(0)} V(\varepsilon_n x + \varepsilon_n y_{n,1})[|u_n^1(x + y_{n,1})|^p - |w_n(x)|^p - |u_{0,1}\chi_{\varepsilon_n}|^p].$$

Thereby,

$$A_n = V(x_1) \int_{B_{2|\ln \varepsilon_n|}(0)} [|u_n^1(x + y_{n,1})|^p - |w_n(x)|^p - |(u_{0,1}\chi_{\varepsilon_n})(x)|^p] + o_n(1)$$

which implies that

$$A_n = V(x_1) \int_{B_{2|\ln \varepsilon_n|}(0)} [|w_n(x) - u_{0,1}(x)|^p - |w_n(x)|^p - |u_{0,1}(x)|^p] + o_n(1).$$

From a result due to Brezis and Lieb [12], we know that

$$\int_{\mathbb{R}^N} [|w_n(x) - u_{0,1}(x)|^p - |w_n(x)|^p - |u_{0,1}(x)|^p] = o_n(1).$$

This combined with the last inequality proves (2.10).

Proof of (2.12). From now on, B_n denotes the following integral

$$B_n = \int_{\mathbb{R}^N} [|\nabla u_n^1|^p - |\nabla u_n|^p - |\nabla (u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})|^p].$$

Notice that

$$B_n = \int_{B_{2|\ln \varepsilon_n|}(0)} [|\nabla u_n^1(x + y_{n,1})|^p - |\nabla w_n|^p - |\nabla (u_{0,1}\chi_{\varepsilon_n})|^p]$$

which gives

$$B_n = \int_{B_{2|\ln \varepsilon_n|}(0)} [|\nabla w_n - \nabla u_{0,1}|^p - |\nabla w_n|^p - |\nabla u_{0,1}|^p] + o_n(1).$$

Since

$$\nabla w_n(x) \to \nabla u_{0,1}(x) \quad \text{a.e. in } \mathbb{R}^N,$$

by using again Brezis and Lieb [12], we have that

$$\int_{\mathbb{R}^N} [|\nabla w_n - \nabla u_{0,1}|^p - |\nabla w_n|^p - |\nabla u_{0,1}|^p] \to 0$$

and this finishes the proof of (2.12).

The same type of arguments explored in the proof of (2.9) can be applied to show that for any $\phi_n \in E_{\varepsilon_n}$ with $\|\phi_n\|_{\varepsilon_n} \le 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} [g(\varepsilon_n x + \varepsilon_n y_{n,1}, w_n(x)) - g(\varepsilon_n x + \varepsilon_n y_{n,1}, u_n^1(x + \varepsilon_n y_{n,1}))\\ - g(\varepsilon_n x + \varepsilon_n y_{n,1}, (u_{0,1}\chi_{\varepsilon_n})]\phi_n(x + \varepsilon_n y_{n,1}) = o_n(1)\end{aligned} \tag{2.16}$$

uniformly in $\|\phi_n\|_{E_{\varepsilon_n}} \le 1$ as $n \to +\infty$.

An immediate consequence from (2.9)–(2.13) and (2.16) are the following limits

$$\Phi_{\varepsilon_n}(u_n^1) = \Phi_{\varepsilon_n}(u_n) - \Phi_{\varepsilon_n}((u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})) + o_n(1)$$

and

$$\Phi'_{\varepsilon_n}(u_n^1) = \Phi'_{\varepsilon_n}(u_n) - \Phi'_{\varepsilon_n}((u_{0,1}\chi_{\varepsilon_n})(x - y_{n,1})) + o_n(1).$$

On the other hand, a direct calculus shows that

$$\Phi_{\varepsilon_n}((u_{0,1}\chi_{\varepsilon_n})(\,\cdot - y_{n,1})) \to J_{0,1}(u_{0,1})$$

and

$$\|\Phi'(u_{0,1}\chi_{\varepsilon_n})(\,\cdot\, - y_{n,1})\|^*_{E_{\varepsilon_n}} \to 0.$$

Therefore,

$$\Phi_{\varepsilon_n}(u_n^1) \to c - J_{0,1}(u_{0,1}) \quad \text{and} \quad \|\Phi'_{\varepsilon_n}(u_n^1)\|^*_{E_{\varepsilon_n}} \to 0$$

showing that $\{u_n^1\}$ is a $(\mathrm{PS})^*_{c-J_{0,1}(u_{0,1})}$ sequence. Now, the proof follows as in [21, Proposition 2.2]. □

3. The existence of multi-peak positive solutions

In this section, for each $i \in \Gamma$, let us denote by $I_i: W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ and $I_{\varepsilon,i}: W^{1,p}(\Omega_{\varepsilon,i}) \to \mathbb{R}$ the following functionals

$$I_i(v) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + \alpha_i |v|^p) - \int_{\mathbb{R}^N} F(v)$$

and

$$I_{\varepsilon,i}(v) = \frac{1}{p}\int_{\Omega_{\varepsilon,i}} (|\nabla v|^p + V(\varepsilon x)|v|^p) - \int_{\Omega_{\varepsilon,i}} F(v).$$

From (f_1)–(f_4), it follows that I_i and $I_{\varepsilon,i}$ have a ground state solution, that is, there exist $w_i \in W^{1,p}(\mathbb{R}^N)$ and $w_{\varepsilon,i} \in E_\varepsilon$ satisfying

$$\begin{aligned} I_i(w_i) &= \mu_i, & I'_i(w_i) &= 0,\\ I_{\varepsilon,i}(w_{\varepsilon,i}) &= \mu_{\varepsilon,i}, & I'_{\varepsilon,i}(w_{\varepsilon,i}) &= 0, \end{aligned}$$

where

$$\begin{aligned} \mu_i &= \inf_{v \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \sup_{t \ge 0} I_i(tv) = \inf_{h \in \Gamma_i} \sup_{t \in [0,1]} I_i(h(t)),\\ \mu_{\varepsilon,i} &= \inf_{v \in W^{1,p}(\Omega_{\varepsilon,i})\setminus\{0\}} \sup_{t \ge 0} I_{\varepsilon,i}(tv) = \inf_{h \in \Gamma_{\varepsilon,i}} \sup_{t \in [0,1]} I_{\varepsilon,i}(h(t)),\\ \Gamma_i &= \{h \in C([0,1], W^{1,p}(\mathbb{R}^N)) : h(0) = 0,\ I_i(h(1)) < 0\},\\ \Gamma_{\varepsilon,i} &= \{h \in C([0,1], W^{1,p}(\Omega_{\varepsilon,i})) : h(0) = 0,\ I_{\varepsilon,i}(h(1)) < 0\}. \end{aligned}$$

Lemma 3.1. *For for each $i \in \Gamma$, the following limits hold*

$$\mu_{\varepsilon,i} \to \mu_i \quad \textit{as } \varepsilon \to 0.$$

Proof. The same type of argument developed in [15] and [1] leads to

$$\mu_{\varepsilon,i} \le \mu_i + o(\varepsilon) \quad \text{for all } i \in \Gamma. \tag{3.1}$$

On the other hand, we can repeat, with some modifications, the arguments used in the proof of Proposition 2.4 to show that

$$\mu_i \le \mu_{\varepsilon,i} + o(\varepsilon) \quad \text{for all } i \in \Gamma. \tag{3.2}$$

Therefore, the proof is completed by invoking (3.1) and (3.2). □

Here and subsequently, without loss of generality, we will consider $\Gamma = \{1,\dots,l\}$ for some $1 \le l \le k$ and $R > 0$ verifying

$$I_i(R^{-1}w_i) < \frac{I_j(w_i)}{2} \quad \text{and} \quad I_i(Rw_i) < 0 \quad \text{for all } i \in \Gamma. \tag{3.3}$$

Furthermore, we fix

$$\widetilde{H}_\varepsilon(\overrightarrow{\theta})(z) = \sum_{i=1}^{l} \theta_i R(w_i \chi_\varepsilon)(z - x_i/\varepsilon) \tag{3.4}$$

for all $\overrightarrow{\theta} = (\theta_1,\dots,\theta_l) \in [1/R^2,1]^l$, where $x_i \in K_i = \{x \in \Omega_i : V(x_i) = \alpha_i\}$, the set

$$\begin{aligned}\Sigma_\varepsilon = \{H \in C([1/R^2,1]^l, E_\varepsilon);\ H = \widetilde{H}_\varepsilon \text{ on } \partial([1/R^2,1]^l),\\ H(\overrightarrow{\theta})|_{\Omega_{\varepsilon,i}} \neq 0 \text{ for all } i \in \Gamma \text{ and for all } \overrightarrow{\theta} \in [1/R^2,1]^l\}\end{aligned}$$

and the number

$$S_\varepsilon = \inf_{H \in \Sigma_\varepsilon} \max_{\overrightarrow{\theta} \in [1/R^2,1]^l} \Phi_\varepsilon(H(\overrightarrow{\theta})).$$

Since the assumption (V_1) implies that $d(x_i, \partial\Omega) > 0$ for all $i \in \Gamma$, we derive

$$\operatorname{supp}\left(u_i \chi_\varepsilon\left(z - \frac{x_i}{\varepsilon}\right)\right) \subset \Omega_{\varepsilon,i} = \Omega_i/\varepsilon, \quad \text{for all } i \in \Gamma,$$

for ε small enough. Consequently

$$\Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) = \sum_{i=1}^{l} I_{\varepsilon,i}(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \quad \text{for all } \overrightarrow{\theta} \in [1/R^2,1]^l.$$

Then, $\widetilde{H}_\varepsilon \in \Sigma_\varepsilon$, $\Sigma_\varepsilon \neq \emptyset$ and S_ε is well defined for ε small enough.

LEMMA 3.2. *For ε small enough, the following property holds: If H belongs to Σ_ε, then there exists $\overrightarrow{\theta_*} \in [1/R^2,1]^l$ such that*

$$I'_{\varepsilon,i}(H(\overrightarrow{\theta_*}))H(\overrightarrow{\theta_*}) = 0, \quad \textit{for all } i \in \Gamma.$$

In particular,

$$I_{\varepsilon,i}(H(\overrightarrow{\theta_*})) \ge \mu_{\varepsilon,i}, \quad i = 1,\dots,l.$$

PROOF. The proof follows by using similar arguments found in [1, Lemma 4.1] combined with the fact that for all $i \in \Gamma$ the following limits hold

$$I'_{\varepsilon,i}(\widetilde{H}_\varepsilon(\overrightarrow{\theta}))\widetilde{H}_\varepsilon(\overrightarrow{\theta}) \to I'(\theta_i R w_i)(\theta_i R w_i) \quad \text{as } \varepsilon \to 0.$$

uniformly in $\overrightarrow{\theta} \in [1/R^2,1]^l$. □

The next proposition is a key point in our arguments, because it establishes an important relation among S_ε and μ_i for $i = 1,\dots,l$.

Proposition 3.3. *The number S_ε fulfills the following limit*

$$\lim_{\varepsilon\to 0} S_\varepsilon = D_\Gamma \quad \textit{where } D_\Gamma = \sum_{i=1}^{l} \mu_i.$$

Proof. For each $H \in \Sigma_\varepsilon$, it follows from (g_1)–(g_2)

$$\Phi_\varepsilon(H(\overrightarrow{\theta})) \geq \sum_{i=1}^{l} I_{\varepsilon,i}(H(\overrightarrow{\theta})) \quad \text{for all } \overrightarrow{\theta} \in [1/R^2, 1]^l.$$

Hence, by Lemma 3.2

$$\max_{\overrightarrow{\theta}\in[1/R^2,1]^l} \Phi_\varepsilon(H(\overrightarrow{\theta})) \geq \sum_{i=1}^{l} \mu_{\varepsilon,i}.$$

On the other hand, from Lemma 3.2

$$\sum_{i=1}^{l} \mu_{\varepsilon,i} = \sum_{i=1}^{l} \mu_i + o_\varepsilon(1)$$

so that

$$\max_{\overrightarrow{\theta}\in[1/R^2,1]^l} \Phi_\varepsilon(H(\overrightarrow{\theta})) \geq D_\Gamma + o_\varepsilon(1)$$

and thus, given $\eta > 0$, there exists $\varepsilon_0 > 0$ such that

$$\Phi_\varepsilon(H(\overrightarrow{\theta})) \geq D_\Gamma - \eta \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

from where it follows that

$$S_\varepsilon \geq D_\Gamma - \eta \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{3.5}$$

To conclude the proof, it is sufficient to show the following inequality

$$\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \leq D_\Gamma + \eta \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

First of all, notice that

$$\Phi_\varepsilon((w_i\chi_\varepsilon)(\,\cdot\, - x_i/\varepsilon)) = I_i(w_i) + o_\varepsilon(1) = \mu_i + o_\varepsilon(1)$$

and

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(t(w_i\chi_\varepsilon)(\,\cdot\, - x_i/\varepsilon))) = I_i(tw_i) \quad \text{uniformly in } t \in [0, R] \text{ for } i = 1, \ldots, l.$$

This way, the function

$$\widetilde{H}_\varepsilon(\overrightarrow{\theta}) = \sum_{i=1}^{l} \theta_i R(w_i\chi_\varepsilon)(\,\cdot\, - x_i/\varepsilon)) \quad \text{for all } \overrightarrow{\theta} = (\theta_1, \ldots, \theta_l) \in [1/R^2, 1]^l$$

satisfies the following estimate

$$\limsup_{\varepsilon\to 0} \left[\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \right] \leq D_\Gamma$$

which leads to

$$\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \leq D_\Gamma + \eta \quad \text{for } \varepsilon \approx 0$$

so that

$$S_\varepsilon \leq D_\Gamma + \eta \quad \text{for } \varepsilon \approx 0. \tag{3.6}$$

Combining (3.5) with (3.6), we obtain the inequality

$$|S_\varepsilon - D_\Gamma| \leq \eta \quad \text{for } \varepsilon \approx 0,$$

which proves the lemma. □

Corollary 3.4. *For each $\alpha > 0$, there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that*

$$\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \leq D_\Gamma + \frac{\alpha}{2} \quad \textit{for all } \varepsilon \in (0,\varepsilon_0),$$

where $\widetilde{H}_\varepsilon(\overrightarrow{\theta})$ is the function given in (3.4).

Proof. Repeating the same arguments used in the proof of Proposition 3.3, it follows that

$$\limsup_{\varepsilon\to 0}\Big[\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta}))\Big] \leq D_\Gamma.$$

From this, there exists $\varepsilon_0 > 0$ such that

$$\sup_{\overrightarrow{\theta}\in[0,1]^l} \Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta})) \leq D_\Gamma + \frac{\alpha}{2} \quad \text{for } \varepsilon \in (0,\varepsilon_0),$$

and the corollary is proved. □

Hereafter, we fix $\rho > 0$ verifying

$$\liminf_{\varepsilon\to 0} \|\widetilde{H}_\varepsilon(\overrightarrow{\theta})\|_{\varepsilon,i} \geq \rho \quad \text{uniformly in } \overrightarrow{\theta} \in [1/R^2,1]^l \text{ and } i \in \Gamma,$$

where

$$\|u\|_{\varepsilon,i} = \bigg(\int_{\Omega_{\varepsilon,i}} |\nabla u|^p + V(\varepsilon x)|u|^p\bigg)^{1/p}$$

and let us define the set

$$Z_{\varepsilon,i} = \{u \in W^{1,p}(\Omega_{\varepsilon,i}) : \|u\|_{\varepsilon,i} \leq \rho/2\}.$$

From definition of $\widetilde{H}_\varepsilon(\overrightarrow{\theta})$ and $Z_{\varepsilon,i}$, there exist positive numbers τ and ε^* such that

$$\operatorname{dist}_{\varepsilon,i}(\widetilde{H}_\varepsilon(\overrightarrow{\theta}), Z_{\varepsilon,i}) > \tau \quad \text{for all } \overrightarrow{\theta} \in [1/R^2,1]^l,\ i \in \Gamma \text{ and } \varepsilon \in (0,\varepsilon^*),$$

where $\operatorname{dist}_{\varepsilon,i}(K,F)$ denotes the distance between sets of $(W^{1,p}(\Omega_{\varepsilon,i}), \|\cdot\|_{\varepsilon,i})$. Taking the numbers τ and $\varepsilon \in (0,\varepsilon^*)$ as above, we define

$$\Theta = \{u \in E_\varepsilon : \operatorname{dist}_{\varepsilon,i}(u, Z_{\varepsilon,i}) \geq \tau \text{ for all } i \in \Gamma\}.$$

Moreover, for any $c, \mu > 0$ and $0 < \delta < \tau/2$, we consider the sets

$$\Phi_\varepsilon^c = \{u \in E_\varepsilon : \Phi_\varepsilon(u) \le c\}$$

and

$$B_{\varepsilon,\mu} = \{u \in \Theta_{2\delta} : |\Phi_\varepsilon(u) - S_\varepsilon| \le \mu\}$$

where Θ_r, for $r > 0$, denotes the set

$$\Theta_r = \{u \in E_\varepsilon : \operatorname{dist}(u, \Theta) \le r\}.$$

Notice that for each $\mu > 0$, there exists $\varepsilon_1 = \varepsilon_1(\mu) > 0$ such that $u^*(z) = \sum_{i=1}^l (w_i \chi_\varepsilon(z - x_i/\varepsilon)) \in \Theta$ for all $\varepsilon \in (0, \varepsilon_1)$. Moreover, since $\Phi_\varepsilon(u^*) = \sum_{i=1}^l \mu_i + o(\varepsilon)$ and $S_\varepsilon \to \sum_{i=1}^l \mu_i$ as $\varepsilon \to 0$, we conclude that $B_{\varepsilon,\mu} \ne \emptyset$ for ε sufficiently small.

In what follows, let us consider $\overline{B}_{M+1}(0) = \{u \in E_\varepsilon; \|u\|_\varepsilon \le M+1\}$ where M is a positive constant large enough, independent of ε, verifying

$$\|\widetilde{H}_\varepsilon(\overrightarrow{\theta})\|_\varepsilon \le \frac{M}{2} \quad \text{for all } \overrightarrow{\theta} \in [1/R^2, 1]^{2l}.$$

Moreover, we denote by $\mu^* > 0$ the following number

$$\mu^* = \min\left\{\frac{I_i(w_i)}{4}, \frac{M}{4}, \frac{\delta}{4};\ i \in \Gamma\right\}.$$

Proposition 3.5. *For each $\mu > 0$ fixed, there exist $\sigma_o = \sigma_o(\mu) > 0$ and $\varepsilon^* = \varepsilon(\mu) \approx 0$, such that*

$$\|\Phi_\varepsilon'(u)\|_\varepsilon^* \ge \sigma_o \quad \textit{for } \varepsilon \ge \varepsilon^* \textit{ and for all } u \in (B_{\varepsilon,2\mu} \setminus B_{\varepsilon,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_\varepsilon^{D_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\varepsilon_n \to 0$ and

$$v_n \in (B_{\varepsilon_n,2\mu} \setminus B_{\varepsilon_n,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_{\varepsilon_n}^{D_\Gamma}$$

such that $\|\Phi'_{\varepsilon_n}(v_n)\|_{\varepsilon_n}^* \to 0$. Since $v_n \in B_{\varepsilon_n,2\mu}$ and $\{\|v_n\|_{\varepsilon_n}\}$ is a bounded sequence, it follows that $\{\Phi_{\varepsilon_n}(v_n)\}$ is also bounded. Thus, we may assume

$$\Phi_{\varepsilon_n}(v_n) \to c \in (-\infty, D_\Gamma], \tag{3.7}$$

after extracting a subsequence if necessary. Applying Proposition 2.4, there exist a integer s, points $x_i \in \overline{\Omega}$, sequences $\{y_{n,i}\} \subset \mathbb{R}^N$ and $u_{0,i}$ functions for $i = 1, \dots, s$ such that

$$\left\| v_n - \sum_{i=1}^s u_{0,i}(\,\cdot\, - y_{n,i}) \chi_{\varepsilon_n}(\,\cdot\, - y_{n,i}) \right\|_{\varepsilon_n} \to 0$$

with

$$\varepsilon_n y_{n,i} \to x_i \quad \text{for } i = 1, \dots, s. \tag{3.8}$$

Since $v_n \in \Theta_{2\delta}$, it follows that $s \geq l$ and $x_i \in \overline{\Omega}_i$ for $i \in \{1, \dots, l\}$. On the other hand, the limit $S_{\varepsilon_n} \to D_\Gamma$ combined with (3.8) and (3.7) yields $s = l$ and $x_i \in K_i$ for all $i = 1, \dots, l$. This way,

$$\Phi_\varepsilon(v_n) \to D_\Gamma$$

showing that $v_n \in B_{\varepsilon_n,\mu}$ for n large enough, which is a contradiction. □

Proposition 3.6. *For each $\mu \in (0, \mu^*/2)$, there exists $\varepsilon^* = \varepsilon^*(\mu) > 0$ such that Φ_ε has a critical point in $B_{\varepsilon,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_\varepsilon^{D_\Gamma}$ for all $\varepsilon \in (0, \varepsilon^*)$.*

Proof. Arguing again by contradiction, we assume that there exists $\mu \in (0, \mu^*/2)$ and a sequence $\varepsilon_n \to 0$, such that Φ_{ε_n} has not critical points in $B_{\varepsilon_n,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_{\varepsilon_n}^{D_\Gamma}$. Since the Palais–Smale condition holds for Φ_{ε_n} (see Proposition 2.3), there exists a constant $d_{\varepsilon_n} > 0$ such that

$$\|\Phi'_{\varepsilon_n}(u)\|^*_{\varepsilon_n} \geq d_{\varepsilon_n} \quad \text{for all } u \in B_{\varepsilon_n,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_{\varepsilon_n}^{D_\Gamma}.$$

Moreover, from Proposition 3.5, we also have

$$\|\Phi'_{\varepsilon_n}(u)\|^*_{\varepsilon_n} \geq \sigma_o \quad \text{for all } u \in (B_{\varepsilon_n,2\mu} \setminus B_{\varepsilon_n,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_{\varepsilon_n}^{D_\Gamma}$$

where $\sigma_o > 0$ is independent of ε_n for n large enough. In what follows, $\Psi_n\colon \mathcal{H}_{\varepsilon_n} \to \mathbb{R}$ and $Q_n\colon \Phi_{\varepsilon_n}^{D_\Gamma} \to \mathbb{R}$ are continuous functions verifying

$$\begin{aligned} \Psi_n(u) &= 1 \quad \text{for } u \in B_{\varepsilon_n,3\mu/2} \cap \Theta_\delta \cap \overline{B}_M(0), \\ \Psi_n(u) &= 0 \quad \text{for } u \notin B_{\varepsilon_n,2\mu} \cap \overline{B}_{M+1}(0), \\ 0 \leq \Psi_n(u) &\leq 1 \quad \text{for } u \in \mathcal{H}_{\varepsilon_n} \end{aligned}$$

and

$$Q_n(u) = \begin{cases} -\Psi_n(u)\|Y_n(u)\|^{-1}\|Y_n(u)\| & \text{for } u \in B_{\varepsilon_n,2\mu} \cap \overline{B}_{M+1}(0), \\ 0 & \text{for } u \notin B_{\varepsilon_n,2\mu} \cap \overline{B}_{M+1}(0), \end{cases}$$

where Y_n is a pseudo-gradient vector field for Φ_{ε_n} on $\mathcal{M}_n = \{u \in \mathcal{H}_{\varepsilon_n} : \Phi'_{\varepsilon_n} \neq 0\}$. Next, we denote by m_0^n the real number given by

$$m_0^n = \sup\{\Phi_{\varepsilon_n}(u);\ u \in \widetilde{H}_{\varepsilon_n}([1/R^2, 1])^{2l} \setminus (B_{\varepsilon_n,\mu} \cap \overline{B}_M(0))\}$$

which verifies $\limsup_{n\to\infty} m_0^n < D_\Gamma$, and by $K > 0$ the real number verifying

$$|\Phi_{\varepsilon_n}(u) - \Phi_{\varepsilon_n}(v)| \leq K\|u - v\|_{\varepsilon_n} \quad \text{for all } u, v \in \overline{B}_{M+1}(0) \text{ and for all } j \in \Gamma.$$

From definition of Q_n,

$$\|Q_n(u)\| \leq 1 \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \Phi_{\varepsilon_n}^{D_\Gamma},$$

consequently, there is a deformation flow $\eta_n\colon [0,\infty) \times \Phi_{\varepsilon_n}^{D_\Gamma} \to \Phi_{\varepsilon_n}^{D_\Gamma}$ defined by

$$\frac{d\eta}{dt} = Q_n(\eta), \quad \eta_n(0,u) = u \in \Phi_{\varepsilon_n}^{D_\Gamma}.$$

This flow satisfies the following basic properties:

$$\Phi_{\varepsilon_n}(\eta_n(t,u)) \leq \Phi_{\varepsilon_n}(u) \quad \text{for all } t \geq 0 \text{ and } u \in \mathcal{H}_{\varepsilon_n}$$

and

$$\eta_n(t,u) = u \quad \text{for all } t \geq 0 \text{ and } u \notin B_{\varepsilon_n,2\mu} \cap \overline{B}_{M+1}(0).$$

Claim 1. *There exist $T_n = T(\varepsilon_n) > 0$ and $\xi > 0$ independent of n, such that*

$$\limsup_{n\to\infty} \left[\max_{\overrightarrow{\theta} \in [1/R^2,1]^l} \Phi_{\varepsilon_n}(\eta_n(T_n, \widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))) \right] < D_\Gamma - \xi.$$

In fact, set $u = \widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}), \widetilde{d}_{\varepsilon_n} = \min\{d_{\varepsilon_n}, \sigma_0\}, T_n = \frac{\sigma_0\mu}{2\widetilde{d}_{\varepsilon_n}}$ and $\widetilde{\eta_n}(t) = \eta_n(t,u)$. If $u \notin B_{\varepsilon_n,\mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, from definition of m_0^n we get

$$\Phi_{\varepsilon_n}(\eta_n(t,u)) \leq \Phi_{\varepsilon_n}(u) \leq m_0^n \quad \text{for all } t \geq 0.$$

On the other hand, if $u \in B_{\varepsilon_n,\mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, we have to consider the following cases:

Case 1. $\widetilde{\eta_n}(t) \in B_{\varepsilon_n,3\mu/2} \cap \overline{B}_M(0) \cap \Theta_\delta$ for all $t \in [0, T_n]$.

Case 2. $\widetilde{\eta_n}(t_0) \notin B_{\varepsilon_n,3\mu/2} \cap \overline{B}_M(0) \cap \Theta_\delta$ for some $t_0 \in [0, T_n]$.

Following the same arguments found in Y. H. Ding and K. Tanaka [17], Case 1 implies that there exists $\xi > 0$ independent of n such that

$$\Phi_{\varepsilon_n}(\widetilde{\eta_n}(T_n)) \leq D_\Gamma - \xi.$$

Related to Case 2, we have the following situations:

(A) There exists $t_2 \in [0, T_n]$ such that $\widetilde{\eta_n}(t_2) \notin \Theta_\delta$, and thus for $t_1 = 0$ it follows that

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\|_{\varepsilon_n} \geq \delta > \mu$$

because $\widetilde{\eta}(t_1) = u \in \Theta$.

(B) There exists $t_2 \in [0, T_n]$ such that $\widetilde{\eta_n}(t_2) \notin \overline{B}_M(0)$, so that for $t_1 = 0$ we get

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\|_{\varepsilon_n} \geq \frac{M}{2} > \mu$$

because $\widetilde{\eta_n}(t_1) = u \in \overline{B}_{\frac{M}{2}}(0)$.

(C) $\widetilde{\eta_n}(t) \in \Theta_\delta \cap \overline{B}_M(0)$ for all $t \in [0, T_n]$, and there are $0 \leq t_1 \leq t_2 \leq T_n$ such that $\widetilde{\eta_n}(t) \in B_{\varepsilon_n,3\mu/2} \setminus B_{\varepsilon_n,\mu}$ for all $t \in [t_1, t_2]$ with

$$|\Phi_{\varepsilon_n}(\widetilde{\eta_n}(t_1)) - S_{\varepsilon_n,\Gamma}| = \mu \quad \text{and} \quad |\Phi_{\varepsilon_n}(\widetilde{\eta_n}(t_2)) - S_{\varepsilon_n,\Gamma}| = 3\mu/2.$$

Using the definition of K, we have that

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\|_{\varepsilon_n} \geq \frac{\mu}{2K}.$$

The estimates showed in (A)–(C) yield there exists $C > 0$ such that $t_2 - t_1 \geq C\mu$. This combined with some arguments found in [17] gives that there exists $\xi > 0$ independent of n such that

$$\limsup_{n\to+\infty}\left[\max_{\overrightarrow{\theta}\in[1/R^2,1]^l}\Phi_{\varepsilon_n}(\eta_n(T_n,\widetilde{H}_\varepsilon(\overrightarrow{\theta})))\right]\leq D_\Gamma-\xi$$

and the proof of Claim 1 is complete.

Claim 2. *The function $\overrightarrow{\theta}\mapsto\eta_n(T_n,\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))$ belongs to Σ_{ε_n}.*

In fact, since $\eta_n(T_n,\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))$ is a continuous functions in $[1/R^2,1]^l$, we have to show that

$$\eta_n(T_n,\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))=\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta})\quad\text{for all }\overrightarrow{\theta}\in\partial([1/R^2,1]^l).$$

From (3.3),

$$\left|\Phi_{\varepsilon_n}(\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))-\sum_{j=1}^{l}\mu_i\right|\geq 2\mu^*\quad\text{for all }\overrightarrow{\theta}\in\partial([1/R^2,1]^l)\text{ and }n\text{ large enough.}$$

Hence, using again the fact that $S_{\varepsilon_n}\to D_\Gamma$ as $n\to\infty$, there is n_0 such that

$$\left|\Phi_\varepsilon(\widetilde{H}_\varepsilon(\overrightarrow{\theta}))-S_\varepsilon\right|\geq\mu^*>2\mu\quad\text{for all }\overrightarrow{\theta}\in\partial([1/R^2,1]^l)\text{ and }n\geq n_0$$

which implies that $\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta})\notin B_{\varepsilon_n,2\mu}$ for all $\overrightarrow{\theta}\in\partial([1/R^2,1]^l)$ and $n\geq n_0$. From this,

$$\eta_n(T_n,\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta}))=\widetilde{H}_{\varepsilon_n}(\overrightarrow{\theta})\quad\text{for all }\overrightarrow{\theta}\in\partial([1/R^2,1]^l).$$

This concludes the proof of the claim.

Combining the definition of $S_{\varepsilon_n,\Gamma}$ with Claim 1, and the fact that, for n large enough, $\eta_\varepsilon(T_{\varepsilon_n},\widetilde{H}_\varepsilon(\overrightarrow{\theta}))$ belongs to Σ_{ε_n}, we get the inequality

$$\limsup_{n\to+\infty}S_{\varepsilon_n,\Gamma}\leq D_\Gamma-\xi$$

which contradicts the Proposition 3.3. □

Here and subsequently, for each $\varepsilon\in(0,\varepsilon_0)$, v_ε denotes the solution given by Proposition 3.6.

Lemma 3.7. *There exist $\widetilde{\varepsilon},\widetilde{\mu}>0$, such that v_ε satisfies*

$$\max_{z\in\partial\Omega_\varepsilon}v_\varepsilon(z)<m$$

for all $\mu\in(0,\widetilde{\mu})$ and $\varepsilon\in(0,\widetilde{\varepsilon})$.

Proof. Arguing by contradiction, we assume that there exist $\varepsilon_n,\mu_n\to 0$ such that

$$v_{\varepsilon_n}\in B_{\varepsilon_n,\mu_n}\quad\text{and}\quad\max_{z\in\partial\Omega_{\varepsilon_n}}v_{\varepsilon_n}(z)\geq m\quad\text{for all }n\in\mathbb{N}.$$

Hereafter, we set $v_n(z) = v_{\varepsilon_n}(z)$. By Proposition 3.6,

$$|\Phi_{\varepsilon_n}(v_n) - S_{\varepsilon_n}| \leq \mu_n, \quad \Phi'_{\varepsilon_n}(v_n) = 0 \quad \text{and} \quad \operatorname{dist}(v_n, \Theta) \leq 2\delta.$$

Using Proposition 2.4, there exist a integer s, points $x_i \in \overline{\Omega}$ and $u_{0,i}$ functions for $i = 1, \ldots, l$ such that

$$\Big\| v_n - \sum_{i=1}^{s} u_{0,i}(\,\cdot\, - y_{n,i})\chi_{\varepsilon_n}(\,\cdot\, - y_{n,i}) \Big\|_{\varepsilon_n} \to 0. \tag{3.9}$$

and

$$\varepsilon_n y_{n,i} \to x_i \quad \text{for } i = 1, \ldots, s. \tag{3.10}$$

Since $dist(v_n, \Theta) \leq 2\delta$, the limit (3.9) implies that $s \geq l$ and $x_i \in \overline{\Omega}_i$ for $i \in \{1, \ldots, l\}$. On the other hand, the limit $S_{\varepsilon_n} \to D_\Gamma$, (3.10) and

$$|\Phi_{\varepsilon_n}(v_n) - S_{\varepsilon_n}| \to 0$$

imply that $s = l$ and $x_i \in K_i$ for all $i = 1, \ldots, l$. In what follows, we set $z_n \in \partial\Omega_{\varepsilon_n}$ verify $v_n(z_n) = \max_{z \in \partial\Omega_{\varepsilon_n}} v_\varepsilon(z)$ and $w_n(x) = v_n(x + z_n)$. Hence,

$$\Big\| w_n - \sum_{j=1}^{s} u_{0,j}(\,\cdot\, + z_n - y_{n,j})\chi_{\varepsilon_n}(\,\cdot\, + z_n - y_{n,j}) \Big\| \to 0.$$

A direct calculus shows that for each $R > 0$,

$$\Big\| \sum_{j=1}^{s} u_{0,j}(\,\cdot\, + z_n - y_{n,j})\chi_{\varepsilon_n}(\,\cdot\, + z_n - y_{n,j}) \Big\|_{W^{1,p}(B_R(0))} \to 0$$

and, thus

$$\|w_n\|_{W^{1,p}(B_R(0))} \to 0.$$

Since v_n is a solution of $(\widetilde{P}_{\varepsilon_n})$, the function w_n is a solution of the following problem

$$(\mathrm{A})_{\varepsilon_n} \qquad -\Delta_p w_n + V(\varepsilon_n x + \varepsilon_n z_n)|w_n|^{p-2}u = g(\varepsilon_n x + \varepsilon_n z_n, w_n) \quad \text{in } \mathbb{R}^N.$$

Combining Moser iteration technique, the growth of g and the boundedness the $\{w_n\}$ in $L^{p^*}(\mathbb{R}^N)$, it follows that there exists $C > 0$ and $\alpha \in (0,1)$ such that

$$\|w_n\|_{C^{0,\alpha}(\overline{B}_1(0))} \leq C \quad \text{for all } n \in \mathbb{N}$$

(see [3]). This inequality implies that for some subsequence, still denote by $\{w_n\}$, we have that

$$w_n \to w \quad \text{in } C^0(\overline{B}_1(0)).$$

Once

$$\max_{z \in \partial\Omega_{\varepsilon_n}} v_{\varepsilon_n}(z) \geq m \quad \text{for all } n \in \mathbb{N}$$

it follows that $w_n(0) \geq m$ for all $n \in \mathbb{N}$ and, therefore $w(0) \geq m$. Thereby, there exists $R \in (0,1)$ such that $w(x) \geq m/2$ for all $x \in B_R(0)$. From this, we deduce that $w \neq 0$ in $W^{1,p}(B_R(0))$, which is a contradiction, because $w_n \to 0$ in $W^{1,p}(B_R(0))$. □

4. Proof of Theorem 1.1

From Lemma 3.7, there exist $\widetilde{\mu}, \varepsilon_0 > 0$, such that the solution $v_\varepsilon \in B_{\varepsilon,\mu}$ given by Proposition 3.6 satisfies

$$\max_{z \in \partial\Omega_\varepsilon} v_\varepsilon(z) < m \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and all } \varepsilon \in (0, \varepsilon_0).$$

Repeating the same arguments found in [15] and [3], we get

$$v_\varepsilon(x) \leq m \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega_\varepsilon,$$

and, therefore, v_ε is a solution for $(\widetilde{\mathrm{P}})_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. To finish the proof, we will show that the family $\{v_\varepsilon\}$ has the property mentioned in the statement of the theorem. To this end, we set $\varepsilon_n \to 0$ and $v_n = v_{\varepsilon_n}$. Then, $\{v_n\}$ is a $(PS)^*_{D_\Gamma}$ sequence verifying

$$dist(v_n, \Theta) \leq 2\delta \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Applying Proposition 2.4, there exists a subsequence of $\{v_n\}$, still denoted by itself, a nonnegative integer s , $\{y_{n,j}\}$, $j = 1, \ldots, s$ and functions $u_{0,j} \neq 0$ for $i = 1, \ldots, s$ such that

$$\varepsilon_n y_{n,j} \to x_j \in \overline{\Omega}, \quad |y_{n,j} - y_{n,i}| \to \infty \tag{4.2}$$

and

$$\bigg\| v_n - \sum_{j=1}^{s} (u_{0,j}\chi_{\varepsilon_n})(\,\cdot\, - y_{n,j}) \bigg\|_{\varepsilon_n} \to 0 \tag{4.3}$$

where $\chi_\varepsilon(x) = \chi(x/{-ln\varepsilon})$ for $0 < \varepsilon < 1$, and χ is a cut-off function which is 1 for $|x| \leq 1$, is 0 for $|x| \geq 2$ and $|\nabla\chi| \leq 2$. From Proposition 2.4, the function $u_{0,j}$ is a nonnegative solution for

$$-\Delta_p u + V_j u^{p-1} = g_{0,j}(x, u), \quad x \in \mathbb{R}^N$$

where $V_j = V(x_j) \geq V_0 > 0$ and $g_{0,j}(x, u) = \lim_{n\to\infty} g_\varepsilon(\varepsilon_n x + \varepsilon_n y_{n,j}, u)$. Moreover,

$$\sum_{j=1}^{l} \mu_j = \sum_{j=1}^{s} J_{0,j}(u_j) \tag{4.4}$$

where $J_{0,j}\colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ denotes the functional given by

$$J_{0,j}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V_j|u|^p) - \int_{\mathbb{R}^N} G_{0,j}(x, u).$$

Repeating the same arguments used in the proof Lemma 3.7, the informations obtained in (4.1)–(4.4) leads to $l = s$ and $x_i \in \overline{\Omega}_i$, and so

$$\sum_{j=1}^{l} \mu_j = \sum_{j=1}^{l} J_{0,j}(u_{0,j}).$$

This equality implies that $x_i \in K_i$ for all $i = 1, \dots, l$, because if for some $i_0 \in \{1, \dots, l\}$ we have $x_{i_0} \in \partial\Omega_i$, the assumption (V_2) leads to $V(x_{i_0}) > w_{i_0}$, and thus $J_{0,i_0}(u_{i_0}) > \mu_{i_0}$.

On the other hand, we know that $J_{0,i}(u_i) \geq \mu_i$ for all $i = 1, \dots, l$ then,

$$\sum_{i=1}^{l} \mu_i < \sum_{i=1}^{l} J_{0,i}(u_i),$$

which is a contradiction. Then $x_i \in K_i$ and $V(x_i) = \alpha_i$ for all $i = 1, \dots, l$, from where it follows that $u_{0,i}$ is a nontrivial solution of the problem

$$-\Delta_p u + \alpha_i u^{p-1} = f(u), \quad x \in \mathbb{R}^N.$$

Now, we will show that for each $\xi > 0$, there exist $r > 0$ and $\varepsilon^* > 0$ such that

$$|v_n|_{\infty, \mathbb{R}^N \setminus \bigcup_{i \in \Gamma} B_r(y_{\varepsilon_n, i})} < \xi \quad \text{for all } \varepsilon \in (0, \varepsilon^*).$$

Moreover, there exists $\delta > 0$ such that

$$|v_n|_{\infty, B_r(y_{\varepsilon_n, i})} \geq \delta \quad \text{for all } i \in \Gamma.$$

Considering the function $w_{n,i}(x) = v_n(x + y_{n,i})$, we have that it is a nonnegative solution and nontrivial of the problem

$$(\mathrm{A})_{\varepsilon_n} \quad -\Delta_p w_{n,i} + V(\varepsilon_n x + \varepsilon_n y_{n,i})|w_{n,i}|^{p-2}u = g(\varepsilon_n x + \varepsilon_n y_{n,i}, w_{n,i}) \quad \text{in } \mathbb{R}^N.$$

Using (f_2) and (g_1), there exists $\delta > 0$ satisfying the following inequality

$$g_\varepsilon(x, t) \leq \frac{V_0}{2} t \quad \text{for all } t \in (0, 2\delta) \text{ and } x \in \mathbb{R}^N.$$

This way,

$$|w_{n,i}|_{\infty, \mathbb{R}^N} \geq 2\delta. \tag{4.5}$$

Adapting some arguments found in [3] and [20], for each $r > 1$, there exists $C > 0$ independent of r, such that

$$|w_{n,i}|_{\infty, \mathbb{R}^N \setminus B_r(0)} \leq C |w_{n,i}|_{L^{p^*}(|x| \geq r/2)}.$$

By Proposition 2.4, $w_n \to u_{0,i}$ in $W^{1,p}(\mathbb{R}^N)$, then for each $\xi > 0$, there is $r > 1$ and $n_0 \in \mathbb{N}$ such that

$$|w_{n,i}|_{\infty, \mathbb{R}^N \setminus B_r(0)} < \frac{\xi}{l} \quad \text{for all } n \geq n_0.$$

Since

$$|v_n|_{\infty,\mathbb{R}^N\setminus\bigcup_{i\in\Gamma}B_r(y_{n,i})} \le \sum_{i=1}^{l}|v_n|_{\infty,\mathbb{R}^N\setminus B_r(y_{n,i})} = \sum_{i=1}^{l}|w_{n,i}|_{\infty,\mathbb{R}^N\setminus B_r(0)}$$

it follows that

$$|v_n|_{\infty,\mathbb{R}^N\setminus\bigcup_{i\in\Gamma}B_r(y_{n,i})} < \xi \quad \text{for all } n \ge n_0.$$

Thereby, for $\xi < \delta$, the last inequality combined with (4.5) yields

$$|w_{n,i}|_{\infty,B_r(0)} \ge \delta,$$

that is,

$$|v_n|_{\infty,B_r(y_{n,i})} \ge \delta \quad \text{for all } i \in \Gamma.$$

Defining $u_n(x) = v_n(x/\varepsilon_n)$ and $P_{n,i} = \varepsilon_n y_{n,i}$, we get u_n is a solution of $(\mathrm{P}_{\varepsilon_n})$ verifying

$$|u_n|_{\infty,B_{\varepsilon_n r}(P_{n,i})} \ge \delta \quad \text{for all } i \in \Gamma$$

and

$$|u_n|_{\infty,\mathbb{R}^N\setminus\bigcup_{i\in\Gamma}B_{\varepsilon_n r}(P_{n,i})} \le \sum_{i=1}^{l}|u_n|_{\infty,\mathbb{R}^N\setminus B_{\varepsilon_n r}(P_{n,i})} = \sum_{i=1}^{l}|w_{n,i}|_{\infty,\mathbb{R}^N\setminus B_r(0)} < \xi,$$

for all $n \ge n_0$, finishing the proof of theorem. □

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