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ON NONCOERCIVE PERIODIC SYSTEMS WITH VECTOR p-LAPLACIAN

Petru Jebelean — Nikolaos S. Papageorgiou

Abstract. We consider nonlinear periodic systems driven by the vector p-Laplacian. An existence and a multiplicity theorem are proved. In the existence theorem the potential function is p-superlinear, but in general does not satisfy the AR-condition. In the multiplicity theorem the problem is strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$. In both of the cases the Euler–Lagrange functional is noncoercive and the method is variational.

1. Introduction

In this paper we consider the following nonlinear periodic system driven by the p-Laplacian operator:

$$\begin{cases} -(|x'(t)|^{p-2}x'(t))' = \nabla F(t, x(t)) & \text{a.e. on } T = [0, b], \\ x(0) = x(b), \quad x'(0) = x'(b) & 1 < p < \infty. \end{cases} \tag{1.1}$$

Here $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^N$ and $F: T \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory mapping such that $F(t, \cdot\,)$ is of class C^1 for almost every $t \in T$.

Periodic systems were studied primarily within the context of semilinear equations (i.e. $p = 2$) and most of the works prove existence but not multiplicity results. In this direction we mention the works of M. S. Berger and M. Schecter [3] and C. L. Tang and X. P. Wu [25], who impose an anticoercivity condition

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on the potential $F(t,\cdot)$ which makes their Euler–Lagrange functional coercive. I. Ekeland and N. Ghoussoub [5] and N. Ghoussoub [10] employ the well-known Ambrosetti–Rabinowitz condition (AR-condition for short), which implies that the potential $F(t,\cdot)$ is superquadratic. On the other hand, C. L. Tang [24] considers second order systems with a subquadratic potential and uses minimax techniques, in particular the saddle point theorem. Finally, J. Mawhin and M. Willem [19] with a bounded potential function and F. Zhao and X. P. Wu [27] use the least action principle, while J. Mawhin [17] assumes that $F(t,\cdot)$ is convex and employs the dual action principle.

In contrast, the study of periodic systems driven by the vector p-Laplacian is in some sense lagging behind. We mention the works of R. P. Agarwal, H. Lü and D. O'Regan [1], G. Dincă and P. Jebelean [4], L. Gasinski [7], P. Jebelean [11], P. Jebelean and G. Moroşanu [13], [14], R. Manasevich and J. Mawhin [16], J. Mawhin [18], E. Papageorgiou and N. S. Papageorgiou [20]–[22], F. Papalini [23] and K. M. Teng and X. P. Wu [26]. In R. P. Agarwal, H. Lü and D. O'Regan [1] the authors deal with certain eigenvalue problems and prove multiplicity results valid for certain values of the parameter. L. Gasinski [7] proves multiplicity of solutions for systems with a coercive Euler–Lagrange functional. P. Jebelean [11] and P. Jebelean and G. Moroşanu [14] deal with problems with nonlinear boundary conditions and prove existence results using Szulkin's critical point theory (see, for example [8]). Also, such type of problems, but with nonpotential right hand term, are studied in G. Dincă and P. Jebelean [4] by the a priori estimates method. R. Manasevich and J. Mawhin [16] and J. Mawhin [18] obtain existence results by degree theoretic methods. The approach in E. Papageorgiou and N. S. Papageorgiou [21] is based on the theory of nonlinear operators of monotone type and they deal with problems which may have unilateral constraints, while in [20], [22] they prove multiplicity results using minimax techniques. Part of the results from [8], [20] and [22] were extended in F. Papalini [23]. Finally, K. M. Teng and X. P. Wu [26] obtain existence and multiplicity of solutions for $p \geq 2$. In [13], [20], [22], [23] and [26] the potential function is locally Lipschitz and in general nonsmooth. So, the method of proof relies on the nonsmooth critical point theory (see L. Gasinski and N. S. Papageorgiou [8]).

Here we prove an existence result and a multiplicity result for problem (1.1). In the existence theorem we assume that the potential $F(t,\cdot)$ is p-superlinear, but need not to satisfy the usual in such cases AR-condition. The multiplicity theorem concerns systems which are strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$. Such problems exhibit a partial lack of compactness, in the sense that the PS-condition is not globally satisfied.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space which admits a direct sum decomposition $X = Y \oplus V$ and let $\varphi \in C^1(X)$. We say that φ *has a local linking at the origin* (with respect to the decomposition (Y,V)) if there exists an $r > 0$ such that

$$\begin{cases} \varphi(y) \le 0 & \text{for all } y \in Y \text{ with } \|y\| \le r, \\ \varphi(v) \ge 0 & \text{for all } v \in V \text{ with } \|v\| \le r. \end{cases}$$

It is easy to see that if φ has a local linking at the origin, then $x = 0$ is a critical point of φ.

We say that φ *satisfies the Palais–Smale condition at the level* $c \in \mathbb{R}$ (the PS_c-condition for short) if every sequence $\{x_n\}_{n\ge1} \subset X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty,$$

has a strongly convergent subsequence. If φ satisfies the PS_c-condition at every level $c \in \mathbb{R}$, then we say that φ *satisfies the* PS*-condition*. Sometimes we need to use a weaker compactness-type condition on the functional φ. So, we say that φ *satisfies the Cerami condition at the level* $c \in \mathbb{R}$ (the C_c-condition for short) if every sequence $\{x_n\}_{n\ge1} \subset X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty,$$

has a strongly convergent subsequence. If φ satisfies the C_c-condition at every level $c \in \mathbb{R}$, then we say that φ *satisfies the* C*-condition*.

The next result is essentially du to S. J. Li and M. Willem [15]. In their formulation of the result they use a gradient version of the PS-condition. Noting that the deformation theorem remains true if the functional φ satisfies the C-condition instead of the PS-condition (see P. Bartolo, V. Benci and D. Fortunato [2]), we can state the following version of Theorem 2 in S. J. Li and M. Willem [15].

Theorem 2.1. *If X is a Banach space, $X = Y \oplus V$ with $\dim Y < \infty$ and $\varphi \in C^1(X)$ satisfies:*

(a) *φ has a local linking at the origin;*
(b) *φ satisfies the* C*-condition;*
(c) *φ maps bounded sets into bounded sets;*
(d) *for every $E \subset V$ finite dimensional subspace, $\varphi|_{Y\oplus E}$ is anticoercive (i.e. $\varphi(x) \to -\infty$ as $\|x\| \to \infty$, $x \in Y \oplus E$),*

then φ admits at least one nontrivial critical point.

In the proof of the multiplicity result we shall use the second deformation theorem (see L. Gasinski and N. S. Papageorgiou [9, p. 628]). Let K be the critical set of φ, i.e. $K = \{x \in X \mid \varphi'(x) = 0\}$. We introduce the following sets:

$$\varphi^c = \{x \in X \mid \varphi(x) \le c\} \quad \text{(the sublevel set of } \varphi \text{ at } c \in \mathbb{R})$$

and

$$K_c = \{x \in K \mid \varphi(x) = c\} \quad \text{(the critical set of } \varphi \text{ at the level } c).$$

In the next theorem we allow $c = +\infty$, in which case $\varphi^c \setminus K_c = X$.

Theorem 2.2. *If X is a Banach space, $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < c \le +\infty$, φ satisfies the* PS_r*-condition for every $r \in [a, c)$, $\varphi^{-1}(a, c) \cap K = \emptyset$ and $\varphi^{-1}(a) \cap K$ is finite, then there exists a homotopy $h: [0,1] \times (\varphi^c \setminus K_c) \to \varphi^c$ such that*

(a) *$h(1, \varphi^c \setminus K_c) \subset \varphi^a$;*
(b) *$h(t, x) = x$ for all $(t, x) \in [0,1] \times \varphi^a$;*
(c) *$\varphi(h(t,x)) \le \varphi(h(s,x))$ for all $t, s \in [0,1]$, $s \le t$ and all $x \in \varphi^c \setminus K_c$ (i.e. the homotopy h is φ-decreasing).*

According to Theorem 2.2 (the second deformation theorem), the set φ^a is a strong deformation retract of $\varphi^c \setminus K_c$.

Next, we present the functional framework and some basic results which are needed in the analysis of problem (1.1).

The Sobolev space

$$W^{1,p}_{\mathrm{per}}(T) := \{x \in W^{1,p}(T; \mathbb{R}^N) \mid x(0) = x(b)\}$$

is endowed with the norm

$$\|x\| = (\|x'\|_p^p + \|x\|_p^p)^{1/p},$$

where $\|\cdot\|_p$ stands for the usual norm on $L^p(T; \mathbb{R}^N)$. Note that since $W^{1,p}(T; \mathbb{R}^N)$ is embedded continuously (in fact compactly) in $C(T; \mathbb{R}^N)$, the evaluations at $t = 0$ and $t = b$ in the definition of $W^{1,p}_{\mathrm{per}}(T)$ make sense.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W^{1,p}_{\mathrm{per}}(T)^*, W^{1,p}_{\mathrm{per}}(T))$ and consider the nonlinear operator $A: W^{1,p}_{\mathrm{per}}(T) \to W^{1,p}_{\mathrm{per}}(T)^*$ defined by

$$\langle A(x), y \rangle = \int_0^b |x'(t)|^{p-2}(x'(t), y'(t))\,dt \quad \text{for all } x, y \in W^{1,p}_{\mathrm{per}}(T).$$

Here $(\cdot, \cdot)$ stands for the usual inner product in $\mathbb{R}^N$. It is a standard matter that A is monotone and continuous, hence it is maximal monotone. Also, the following result is known; however, for the sake of the completeness, we include a short proof.

Proposition 2.3. *The operator A is of type* $(\mathrm{S})_+$.

Proof. Let $x_n \xrightarrow{w} x$ in $W^{1,p}_{\mathrm{per}}(T)$ and assume that

$$\limsup_{n \to \infty} \langle A(x_n), x_n - x \rangle \le 0.$$

We need to show that $x_n \to x$ in $W^{1,p}_{\rm per}(T)$. From

$$0 \le \langle A(x_n) - A(x), x_n - x\rangle = \langle A(x_n), x_n - x\rangle - \langle A(x), x_n - x\rangle \\ \le \sup_{k\ge n}\langle A(x_k), x_k - x\rangle - \langle A(x), x_n - x\rangle$$

it follows $\langle A(x_n) - A(x), x_n - x\rangle \to 0$ as $n \to \infty$. Then, the inequality

$$0 \le (\|x_n'\|_p^{p-1} - \|x'\|_p^{p-1})(\|x_n'\|_p - \|x'\|_p) \le \langle A(x_n) - A(x), x_n - x\rangle$$

yields $\|x_n'\|_p \to \|x'\|_p$ as $n \to \infty$.

We know that $x_n' \xrightarrow{w} x'$ in $L^p(T;\mathbb{R}^N)$. The space $L^p(T;\mathbb{R}^N)$ being uniformly convex, it has the Kadec–Klee property, which implies $x_n' \to x'$ in $L^p(T;\mathbb{R}^N)$. We also have $x_n \to x$ in $C(T;\mathbb{R}^N)$ (by the compactness of the embedding of $W^{1,p}_{\rm per}(T)$ into $C(T;\mathbb{R}^N)$). Therefore, we conclude that $x_n \to x$ in $W^{1,p}_{\rm per}(T)$. □

3. Existence of nontrivial solutions

In this section we prove an existence theorem for problem (1.1) under the hypothesis that the potential $F(t,\cdot)$ exhibiths p-superlinear growth near infinity, but need not to satisfy the AR-condition. The precise hypotheses on F are the following:

(H$_1$) $F\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

- (i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable;
- (ii) for almost all $t \in T$, $x \mapsto F(t,x)$ is C^1 and $F(t,0) = 0$;
- (iii) for almost all $t \in T$ and all $x \in \mathbb{R}^N$

$$|\nabla F(t,x)| \le a(t) + c|x|^{r-1}$$

with $a \in L^1(T)_+$, $c > 0$ and $p < r < \infty$;
- (iv) $\lim_{|x|\to\infty}(F(t,x)/|x|^p) = +\infty$ uniformly for almost all $t \in T$ and there exists $\mu > r - p$ such that

$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x), x) - pF(t,x)}{|x|^\mu} > 0 \quad \text{uniformly for a.a. } t \in T; \tag{3.1}$$

- (v) $\limsup_{x\to 0}(pF(t,x)/|x|^p) < 1/b^p$ uniformly for almost all $t \in T$ and there exists $\delta > 0$ such that $F(t,x) \ge 0$ for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $|x| \le \delta$.

Remark 3.1. Hypothesis (H$_1$) implies that $F(t,\cdot)$ is p-superlinear. However, we do not assume the AR-condition, very common in such cases. We recall that the AR-condition says that there exist $\beta > p$ and $M > 0$ such that

$$0 < \beta F(t,x) \le (\nabla F(t,x), x) \quad \text{for a.a. } t \in T \text{ and all } |x| \ge M. \tag{3.2}$$

Integrating (3.2) we get

$$c_1|x|^\beta \le F(t,x) \quad \text{for a.a. } t\in T, \text{ all } |x|\ge M, \text{ for some } c_1>0. \tag{3.3}$$

Clearly, (3.3) is stronger than the condition

$$\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^p} = +\infty \quad \text{uniformly for a.a. } t\in T.$$

Here, instead of (3.2) we use the weaker condition (3.1). Note that (3.1) was used earlier in the frame of semilinear (i.e. $p=2$) Hamiltonian systems by G. Fei [6]. The following example provides a function F which satisfies (3.1) but not (3.2).

Example 3.2. Consider the function $F\colon\mathbb{R}^N\to\mathbb{R}$ (for the sake of simplicity we drop the t-dependence), defined by

$$F(x) = \frac{1}{p}|x|^p\ln(1+|x|^\alpha)$$

with $\alpha>1$. Then F satisfies hypothesis (H_1) (with $r=p+\varepsilon$, $\varepsilon\in(0,p)$ and $\mu=p$), but it does not satisfy the AR-condition (see (3.2)).

The Euler–Lagrange functional for problem (1.1) is defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b F(t,x(t))\,dt \quad \text{for all } x\in W^{1,p}_{\mathrm{per}}(T).$$

It is known that $\varphi\in C^1(W^{1,p}_{\mathrm{per}}(T),\mathbb{R})$. Also, we shall consider the direct sum decomposition

$$W^{1,p}_{\mathrm{per}}(T) = \mathbb{R}^N\oplus V,$$

with $V=\{x\in W^{1,p}_{\mathrm{per}}(T)\mid \int_0^b x(t)\,dt=0\}$.

Proposition 3.3. *If hypotheses* (H_1) *hold, then* φ *satisfies the* C*-condition.*

Proof. Let $\{x_n\}_{n\ge1}\subset W^{1,p}_{\mathrm{per}}(T)$ be a sequence such that

$$|\varphi(x_n)|\le M_1 \quad \text{for some } M_1>0 \text{ and all } n\ge 1, \tag{3.4}$$

and

$$(1+\|x_n\|)\varphi'(x_n)\to 0 \quad \text{in } W^{1,p}_{\mathrm{per}}(T)^*, \text{ as } n\to\infty. \tag{3.5}$$

We know that

$$\varphi'(x_n) = A(x_n) - N(x_n) \tag{3.6}$$

with $N(u)(\cdot\,) = \nabla F(\cdot\,,u(\cdot\,))$ for all $u\in W^{1,p}_{\mathrm{per}}(T)$ (see, for example P. Jebelean [12]).

Claim. $\{x_n\}_{n\ge1}$ is bounded in $W^{1,p}_{\mathrm{per}}(T)$.

Suppose that the Claim is not true. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. From (3.5) and (3.6) we have

$$\left|\langle A(x_n), u\rangle - \int_0^b (\nabla F(t, x_n), u)\, dt\right| \le \frac{\varepsilon_n}{1 + \|x_n\|}\|u\| \quad \text{for all } u \in W^{1,p}_{\text{per}}(T), \tag{3.7}$$

with $\varepsilon_n \to 0+$. From (3.7), with $u = x_n$, it follows

$$-\|x_n'\|_p^p + \int_0^b (\nabla F(t, x_n), x_n)\, dt \le \varepsilon_n \quad \text{for all } n \ge 1. \tag{3.8}$$

Also, from (3.4) we have

$$\|x_n'\|_p^p - \int_0^b pF(t, x_n)\, dt \le pM_1 \quad \text{for all } n \ge 1. \tag{3.9}$$

Adding (3.8) and (3.9), we obtain

$$\int_0^b [(\nabla F(t, x_n), x_n) - pF(t, x_n)]\, dt \le M_2 \quad \text{for some } M_2 > 0, \text{ all } n \ge 1. \tag{3.10}$$

By virtue of hypothesis (H_1)(iv) we can find $\beta > 0$ and $M_3 = M_3(\beta)$ with

$$0 < \beta|x|^\mu \le (\nabla F(t, x), x) - pF(t, x) \quad \text{for a.a. } t \in T \text{ and all } |x| \ge M_3. \tag{3.11}$$

Since $F(t, 0) = 0$, hypothesis (H_1)(iii) implies that there is some $M_4 \in L^1(T)_+$ such that

$$|(\nabla F(t, x), x) - pF(t, x)| \le M_4(t) \quad \text{for a.a. } t \in T \text{ and all } |x| < M_3. \tag{3.12}$$

Combining (3.11) and (3.12), we infer that

$$\beta|x|^\mu - c_1(t) \le (\nabla F(t, x), x) - pF(t, x) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N, \tag{3.13}$$

where $c_1(t) = M_4(t) + \beta M_3^\mu$.

We return to (3.10) and use (3.13). Then

$$\begin{aligned} &\beta\|x_n\|_\mu^\mu \le M_5 \quad \text{for some } M_5 > 0, \text{ all } n \ge 1, \\ &\qquad \Rightarrow \{x_n\}_{n\ge1} \text{ is bounded in } L^\mu(T, \mathbb{R}^N). \end{aligned} \tag{3.14}$$

It is clear from (3.1) that we can always assume, without loss of generality, that $\mu < r$. Then

$$\begin{aligned} \int_0^b |x_n|^r\, dt &= \int_0^b |x_n|^{r-\mu}|x_n|^\mu\, dt \le \|x_n\|_\infty^{r-\mu} \int_0^b |x_n|^\mu\, dt \\ &\le c_2\|x_n\|^{r-\mu} \quad \text{for some } c_2 > 0, \text{ all } n \ge 1 \text{ (see (3.14))}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\Rightarrow \|x_n\|_p^r \le c_3\|x_n\|^{r-\mu} && \text{for some } c_3 > 0, \text{ all } n \ge 1 \text{ (since } p < r) \\ &\Rightarrow \|x_n\|_p^p \le c_4\|x_n\|^{(r-\mu)p/r} && \text{with } c_4 = c_3^{p/r}, \text{ all } n \ge 1, \\ &\Rightarrow \|x_n\|_p^p \le c_4(1 + \|x_n\|^{r-\mu}) && \text{for all } n \ge 1. \end{aligned} \tag{3.16}$$

From (3.4), hypothesis (H_1)(iii) and (3.15), we successively have

$$\begin{aligned}\frac{1}{p}\|x_n'\|_p^p &\le M_1 + \int_0^b F(t,x_n)\,dt \\ &\le M_1 + \int_0^b (a(t)|x_n(t)| + c|x_n(t)|^r)\,dt \\ &\le \|x_n\|_\infty \|a\|_1 + c\|x_n\|_r^r \le \widetilde{c}\|x_n\| + cc_2\|x_n\|^{r-\mu}\end{aligned}$$

for some $\widetilde{c} > 0$ and all $n \ge 1$. This together with (3.16) yield

$$(3.17)\qquad \|x_n\|^p \le c_4 + \widehat{c}\|x_n\| + c_5\|x_n\|^{r-\mu} \quad \text{for some } \widehat{c},\ c_5 > 0 \text{ and all } n \ge 1.$$

But recall that by hypothesis (H_1)(iv) we have $p > \max\{1, r-\mu\}$. Hence, from (3.17) it follows that $\{x_n\}_{n\ge1} \subset W^{1,p}_{\rm per}(T)$ is bounded. This proves the Claim.

Thanks to the Claim we may assume that

$$(3.18)\qquad x_n \xrightarrow{w} x \quad \text{in } W^{1,p}_{\rm per}(T) \quad \text{and} \quad x_n \to x \quad \text{in } C(T;\mathbb{R}^N).$$

In (3.7) we choose $u = x_n - x$. Then

$$\left|\langle A(x_n), x_n - x\rangle - \int_0^b (\nabla F(t,x_n), x_n - x)\,dt\right| \le \frac{\varepsilon_n}{1+\|x_n\|}\|x_n - x\| \quad \text{for all } n \ge 1.$$

Evidently

$$\int_0^b (\nabla F(t,x_n), x_n - x)\,dt \to 0 \quad \text{as } n\to\infty$$

(see (3.18) and (H_1)(iii)). Hence

$$\begin{aligned}\lim_{n\to\infty}\langle A(x_n), x_n - x\rangle \to 0 \ &\Rightarrow x_n \to x \text{ in } W^{1,p}_{\rm per}(T) \quad \text{(see Proposition 2.3)} \\ &\Rightarrow \varphi \text{ satisfies the } C\text{-condition.}\end{aligned}$$

□

Proposition 3.4. *If hypotheses* (H_1) *hold, then* φ *has a local linking at the origin with respect to* $(\mathbb{R}^N, V)$.

Proof. By virtue of hypothesis (H_1)(v), it is clear that we can find $\delta_0 > 0$ such that

$$(3.19)\qquad \varphi(x) = -\int_0^b F(t,x)\,dt \le 0 \quad \text{for all } x \in \mathbb{R}^N \subset W^{1,p}_{\rm per}(T) \text{ with } |x| \le \delta_0.$$

On the other hand, again from hypothesis (H_1)(v), there are constants $\varepsilon \in (0, 1/b^p)$ and $\delta_1 > 0$, such that

$$(3.20)\qquad F(t,x) \le \frac{1}{p}\left(\frac{1}{b^p} - \varepsilon\right)|x|^p \quad \text{for a.a. } t \in T \text{ and all } |x| \le \delta_1.$$

Since V is embedded continuously (in fact, compactly) into $C(T;\mathbb{R}^N)$, we can find $\delta_2 > 0$ such that

$$x \in V \text{ and } \|x\| \le \delta_2 \ \Rightarrow\ \|x\|_\infty \le \delta_1.$$

On account of the inequality (see J. Mawhin and M. Willem [19, p. 8]):

$$\|x\|_p^p \le b^p \|x'\|_p^p \quad \text{for all } x \in V, \tag{3.21}$$

we can estimate $\varphi(x)$ for $x \in V$, with $\|x\| \le \delta_2$, as follows

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|x'\|_p^p - \int_0^b F(t, x(t))\, dt \\ &\ge \frac{1}{p}\|x'\|_p^p - \frac{1}{p}\left(\frac{1}{b^p} - \varepsilon\right) \int_0^b |x(t)|^p\, dt \quad \text{(see (3.20))} \\ &\ge \frac{\varepsilon}{p}\|x\|_p^p \ge 0. \end{aligned} \tag{3.22}$$

Letting $\delta = \min\{\delta_0, \delta_2\}$, from (3.19) and (3.22) we infer that φ has a local linking at the origin with respect to $(\mathbb{R}^N, V)$. □

PROPOSITION 3.5. *If hypotheses* (H_1) *hold and* $E \subset V$ *is a finite dimensional subspace, then* $\varphi|_{\mathbb{R}^N \oplus E}$ *is anticoercive* (*i.e.* $\varphi(x) \to -\infty$ *as* $\|x\| \to \infty$, *for* $x \in \mathbb{R}^N \oplus E$).

PROOF. By virtue of hypothesis (H_1)(iv), given $\gamma > 0$, we can find $M_6 = M_6(\gamma) > 0$ such that

$$F(t, x) \ge \gamma |x|^p \quad \text{for a.a. } t \in T, \text{ all } |x| \ge M_6. \tag{3.23}$$

On the other hand, by hypothesis (H_1)(iii) we can find $\xi_\gamma \in L^1(T)_+$ such that

$$|F(t, x)| \le \xi_\gamma(t) \quad \text{for a.a. } t \in T, \text{ all } |x| \le M_6. \tag{3.24}$$

Combining (3.23) and (3.24), we have

$$F(t, x) \ge \gamma |x|^p - \widehat{\xi}_\gamma(t) \quad \text{for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N, \tag{3.25}$$

where $\widehat{\xi}_\gamma = \xi_\gamma + \gamma M_6^p \in L^1(T)_+$. Now, let $u \in \mathbb{R}^N \oplus E$. Then

$$\varphi(u) = \frac{1}{p}\|u'\|_p^p - \int_0^b F(t, u(t))\, dt \le \frac{1}{p}\|u'\|_p^p - \gamma \|u\|_p^p + c_6 \tag{3.26}$$

with $c_6 = \|\widehat{\xi}_\gamma\|_1$ (see (3.25)). Because $\mathbb{R}^N \oplus E$ is finite dimensional, all norms are equivalent and so from (3.26) we infer that

$$\varphi(u) \le \frac{1}{p}\|u\|^p - \gamma \|u\|_p^p + c_6 \le \frac{1}{p}(1 - \gamma c_7)\|u\|^p + c_6 \quad \text{for all } u \in \mathbb{R}^N \oplus E, \tag{3.27}$$

with $c_7 > 0$ independent of γ. Therefore, we can chose $\gamma > 1/c_7$ and (3.27) shows that $\varphi|_{\mathbb{R}^N \oplus E}$ is anticoercive. □

Now we are ready for the existence theorem.

Theorem 3.6. *If hypotheses* (H_1) *hold, then problem* (1.1) *has a nontrivial solution* $x_0 \in C^1(T;\mathbb{R}^N)$.

Proof. It is clear that φ maps bounded sets into bounded sets. This together with Propositions 3.3–3.5 allow us to use Theorem 2.1, which gives the existence of some $x_0 \in W^{1,p}_{\mathrm{per}}(T)$, $x_0 \neq 0$ such that $\varphi'(x_0) = 0$, which means

$$A(x_0) = N(x_0). \tag{3.28}$$

From (3.28), a standard reasoning using integration by parts, shows that $x_0 \in C^1(T;\mathbb{R}^N)$ and solves (1.1) (see e.g. L. Gasinski and N. S. Papageorgiou [8]). □

4. Existence of multiple solutions

We prove a multiplicity theorem for problem (1.1). Our hypotheses on the potential function $F(t,x)$ incorporate systems which are strongly resonant with respect to $\lambda_0 = 0$, the principal eigenvalue of the negative vector p-Laplacian. The Euler–Lagrange functional φ will be bounded below but not coercive.

The precise hypotheses on the potential function $F(t,x)$ are the following:

(H_2) $F\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \mapsto F(t,x)$ is C^1 and $F(t,0) = 0$;

(iii) for almost all $t \in T$ and all $x \in \mathbb{R}^N$

$$|\nabla F(t,x)| \leq a_0(t)c_0(|x|)$$

with $a_0 \in L^1(T)_+$, $c_0 \in C(\mathbb{R}_+)$, $c_0 \geq 0$;

(iv) there exists a function $F_\infty \in L^1(T)$ such that $\int_0^b F_\infty(t)\,dt \leq 0$ and

$$F(t,x) \to F_\infty(t) \quad \text{for a.a. } t \in T, \text{ as } |x| \to \infty;$$

(v) there exists a function $\eta \in L^1(T)_+$, $\eta \neq 0$ such that

$$\liminf_{x\to 0} \frac{pF(t,x)}{|x|^p} \geq \eta(t) \quad \text{uniformly for a.a. } t \in T;$$

(vi) $F(t,x) \leq \frac{1}{pb^p}|x|^p$ for almost all $t \in T$ and all $x \in \mathbb{R}^N$.

Remark 4.1. Hypothesis (H_2)(iv) implies that al infinity we may have strong resonance with respect to the principal eigenvalue $\lambda_0 = 0$. As it is well known, strongly resonant problems exhibit a partial lack of compactness. In our case this is reflected in Proposition 4.3 below.

Example 4.2. The function

$$F(x) = \begin{cases} \dfrac{1}{pb^p}|x|^p & \text{if } |x| \leq 1, \\ \dfrac{1}{pb^p|x|}(1 + (p+1)\ln|x|) & \text{if } |x| > 1, \end{cases}$$

satisfies hypotheses (H_2) (again, for the sake of simplicity, we dropped the t-dependence).

Proposition 4.3. *If hypotheses* (H_2) *hold, then* φ *satisfies the* PS_c*-condition at every level* $c < -\int_0^b F_\infty(t)\,dt$.

Proof. Let $\{x_n\}_{n\geq 1} \subset W^{1,p}_{\rm per}(T)$ be a sequence such that

$$\varphi(x_n) \to c, \quad \text{with } c < -\int_0^b F_\infty(t)\,dt \tag{4.1}$$

and

$$\varphi'(x_n) \to 0 \quad \text{in } W^{1,p}_{\rm per}(T)^*, \text{ as } n \to \infty. \tag{4.2}$$

Claim. $\{x_n\}_{n\geq 1}$ is bounded in $W^{1,p}_{\rm per}(T)$.

We proceed by contradiction. So, suppose that $\|x_n\| \to \infty$ and set $y_n = x_n/\|x_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ and we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\rm per}(T) \text{ and } y_n \to y \text{ in } C(T;\mathbb{R}^N). \tag{4.3}$$

From (4.1) we have

$$|\varphi(x_n)| \leq M_7 \quad \text{for some } M_7 > 0, \text{ and all } n \geq 1, \tag{4.4}$$
$$\Rightarrow \frac{1}{p}\|y_n'\|_p^p \leq \frac{M_7}{\|x_n\|^p} + \int_0^b \frac{F(t,x_n)}{\|x_n\|^p}\,dt.$$

From (H_2)(iv), for almost all $t \in T$ there is some $M_t > 0$ such that $|F(t,x)| \leq M_t$, for all $x \in \mathbb{R}^N$. Then, by virtue of (H_2)(vi) and Fatou's lemma, it follows

$$\limsup_{n\to\infty} \int_0^b \frac{F(t,x_n(t))}{\|x_n\|^p}\,dt \leq \int_0^b \limsup_{n\to\infty} \frac{F(t,x_n(t))}{\|x_n\|^p}\,dt = 0. \tag{4.5}$$

So, if in (4.4) we pass to the limit as $n \to \infty$, we get $\|y'\|_p = 0$ (see (4.3) and (4.5)), which means that $y = \xi \in \mathbb{R}^N$.

If $\xi = 0$, then $y_n \to 0$ in $W^{1,p}_{\rm per}(T)$, a contradiction to the fact that $\|y_n\| = 1$, for all $n \geq 1$.

If $\xi \neq 0$, then $|x_n(t)| \to \infty$ for all $t \in T$, as $n \to \infty$. Then, by virtue of (H_2)(iv), we have

$$F(t,x_n(t)) \to F_\infty(t) \quad \text{for a.a. } t \in T, \text{ as } n \to \infty.$$

Because of (4.1), given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geq 1$ such that

$$|\varphi(x_n) - c| \leq \varepsilon \quad \text{for all } n \geq n_0,$$
$$\Rightarrow \frac{1}{p}\|x_n'\|_p^p - \int_0^b F(t,x_n(t))\,dt \leq c + \varepsilon \quad \text{for all } n \geq n_0,$$
$$\Rightarrow -\int_0^b F_\infty(t)\,dt \leq c + \varepsilon \quad \text{(by Fatou's lemma).}$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \to 0+$ and obtain

$$-\int_0^b F_\infty(t)\,dt \le c,$$

which contradicts the choice of $c \in \mathbb{R}$ (see (4.1)). This proves the Claim.

Due to the Claim, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W^{1,p}_{\rm per}(T) \quad \text{and} \quad x_n \to x \quad \text{in } C(T;\mathbb{R}^N).$$

Then using (4.2) and arguing as in the proof of Proposition 3.3, exploiting the fact that the operator A is of type $(\mathrm{S})_+$, we conclude that $x_n \to x$ in $W^{1,p}_{\rm per}(T)$. Therefore, φ satisfies the PS_c-condition at every level $c < -\int_0^b F_\infty(t)\,dt$. □

Now we are ready for the multiplicity theorem.

Theorem 4.4. *If hypotheses* (H_2) *hold, then problem* (1.1) *has at least two nontrivial solutions* $x_0, u_0 \in C^1(T,\mathbb{R}^N)$.

Proof. By virtue of hypothesis (H_2)(v), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{p}(\eta(t)-\varepsilon)|x|^p \le F(t,x) \quad \text{for a.a. } t \in T \text{ and } x \in \mathbb{R}^N \text{ with } |x| \le \delta. \tag{4.6}$$

Let $x = \xi \in \mathbb{R}^N$ with $|\xi| \le \delta$. Then

$$\varphi(\xi) = -\int_0^b F(t,\xi)\,dt \le -\frac{|\xi|^p}{p}\bigg[\int_0^b \eta(t)\,dt - \varepsilon b\bigg] \quad \text{(see (4.6)}. \tag{4.7}$$

If we chose $\varepsilon \in (0, \|\eta\|_1/b)$, then from (4.7) it follows that

$$\varphi(\xi) < 0. \tag{4.8}$$

We show that φ is bounded below. Indeed, if this is not the case, then we can find a sequence $\{x_n\}_{n\ge1} \subset W^{1,p}_{\rm per}(T)$ such that

$$\varphi(x_n) \to -\infty \quad \text{as } n \to \infty. \tag{4.9}$$

Since φ maps bounded sets into bounded sets, we may assume that $\|x_n\| \to \infty$, as $n \to \infty$. As before, let $y_n = x_n/\|x_n\|$ and assume, without any loss of generality, that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\rm per}(T) \quad \text{and} \quad y_n \to y \quad \text{in } C(T;\mathbb{R}^N). \tag{4.10}$$

We have

$$\frac{\varphi(x_n)}{\|x_n\|^p} = \frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{F(t,x_n)}{\|x_n\|^p}\,dt.$$

From (4.9) and (4.10) it follows

$$0 \ge \liminf_{n\to\infty} \frac{\varphi(x_n)}{\|x_n\|^p} \ge \frac{1}{p}\|y'\|_p^p - \limsup_{n\to\infty} \int_0^b \frac{F(t,x_n)}{\|x_n\|^p}\,dt \ge \frac{1}{p}\|y'\|_p^p$$

(by Fatou's lemma; see (4.5)), meaning that $y = \xi \in \mathbb{R}^N$.

As before, if $\xi = 0$, then $y_n \to 0$ in $W^{1,p}_{\mathrm{per}}(T)$, a contradiction to the fact that $\|y_n\| = 1$, for all $n \geq 1$. If $\xi \neq 0$, then $|x_n(t)| \to +\infty$ for all $t \in T$, as $n \to \infty$ and so, via (H_2)(iv), (4.9) and Fatou's lemma, we have

$$-\infty = \lim_{n\to\infty} \varphi(x_n) \geq -\int_0^b F_\infty(t)\,dt \geq 0,$$

a contradiction. This proves that φ is bounded below.

From (4.8) we infer

$$-\infty < m := \inf \varphi < 0 = \varphi(0) \leq -\int_0^b F_\infty(t)\,dt.$$

According to Proposition 4.3, φ satisfies the PS_m-condition. Hence, we can find $x_0 \in W^{1,p}_{\mathrm{per}}(T)$ such that

$$m = \varphi(x_0) < 0 = \varphi(0) \tag{4.11}$$

(see, for example, L. Gasinski and N. S. Papageorgiou [9, p. 650]). From (4.11) we see that $x_0 \neq 0$ and

$$\varphi'(x_0) = 0. \tag{4.12}$$

By virtue of (4.8), for $\rho > 0$ small enough, we have

$$\mu := \sup\{\varphi(x) \mid x \in \partial B_\rho \cap \mathbb{R}^N\} < 0. \tag{4.13}$$

As before, we consider the direct sum decomposition

$$W^{1,p}_{\mathrm{per}}(T) = \mathbb{R}^N \oplus V, \quad \text{with } V = \left\{ x \in W^{1,p}_{\mathrm{per}}(T) \;\middle|\; \int_0^b x(t)\,dt = 0 \right\}.$$

From (H_2)(vi) and (3.21) , for $x \in V$, we have

$$\varphi(x) \geq \frac{1}{p}\|x'\|_p^p - \frac{1}{pb^p}\|x\|_p^p \geq 0 \;\Rightarrow\; \inf_V \varphi \geq 0. \tag{4.14}$$

Suppose that x_0 is the only nontrivial critical point of φ (see (4.12)). Let $a := m < 0 =: c$ and apply Theorem 2.2. Then we can find a homotopy $h\colon [0,1] \times (\varphi^c \setminus K_c) \to \varphi^c$, such that $h(t,x) = x$ for all $(t,x) \in [0,1] \times \varphi^a$ and

$$h(1, \varphi^c \setminus K_c) \subset \varphi^a, \tag{4.15}$$

$$\varphi(h(t,x)) \leq \varphi(h(s,x)) \quad \text{for all } t,s \in [0,1],\; s \leq t,\; \text{all } x \in \varphi^c \setminus K_c. \tag{4.16}$$

Now, we consider the map $\overline{\gamma}\colon \overline{B}_\rho \cap \mathbb{R}^N \to W^{1,p}_{\mathrm{per}}(T)$ defined by

$$\overline{\gamma}(x) = \begin{cases} x_0 & \text{if } \|x\| \leq \rho/2, \\ h(2(\rho - \|x\|)/\rho, \rho x/\|x\|) & \text{if } \|x\| \in (\rho/2, \rho]. \end{cases} \tag{4.17}$$

If $x \in \mathbb{R}^N$, $\|x\| = \rho/2$, then $2x \in \varphi^c \setminus K_c$ (see (4.13)) and so, by (4.15)

$$h\bigg(\frac{2(\rho - \|x\|)}{\rho}, \frac{\rho x}{\|x\|}\bigg) = h(1, 2x) \in \varphi^a = \{x_0\},$$

showing that $\overline{\gamma}$ is continuous (see (4.17)). If $x \in \partial B_\rho \cap \mathbb{R}^N$ then $\overline{\gamma}(x) = h(0, x) = x$, because h is a homotopy. Therefore

$$\overline{\gamma} \in \Gamma = \{\gamma \in C(\overline{B}_\rho \cap \mathbb{R}^N, W^{1,p}_{\rm per}(T)) \mid \gamma|_{\partial B_\rho \cap \mathbb{R}^N} = {\rm id}|_{\partial B_\rho \cap \mathbb{R}^N}\}.$$

From L. Gasinski and N. S. Papageorgiou [9, p. 642], we know that the pair $\{\partial B_\rho \cap \mathbb{R}^N, \overline{B}_\rho \cap \mathbb{R}^N\}$ is linking with V in $W^{1,p}_{\rm per}(T))$. It follows that

$$\overline{\gamma}(\overline{B}_\rho \cap \mathbb{R}^N) \cap V \neq \emptyset,$$

which ensures that

$$\sup\{\varphi(\overline{\gamma}(x)) \mid x \in \overline{B}_\rho \cap \mathbb{R}^N\} \geq 0 \quad \text{(see (4.14))}. \tag{4.18}$$

On the other hand, using (4.17), (4.11), (4.13) and (4.16), we deduce

$$\varphi(\overline{\gamma}(x)) \leq \mu < 0 \quad \text{for all } x \in \overline{B}_\rho \cap \mathbb{R}^N. \tag{4.19}$$

Comparing (4.18) and (4.19), we reach a contradiction. This proves that φ has one more nontrivial critical point $u_0 \neq x_0$. □

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Petru Jebelean
Department of Mathematics
West University of Timişoara
Blvd. V. Pârvan, no. 4
300223 Timişoara, ROMANIA

E-mail address: jebelean@math.uvt.ro

Nikolaos S. Papageorgiou
National Technical University
Department of Mathematics
Zografou Campus
Athens 15780, GREECE

E-mail address: npapg@math.ntua.gr