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NONEXPANSIVE MAPPINGS ON HILBERT'S METRIC SPACES

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Abstract. This paper deals with the iterative behavior of nonexpansive mappings on Hilbert's metric spaces (X, d_X). We show that if (X, d_X) is strictly convex and does not contain a hyperbolic plane, then for each nonexpansive mapping, with a fixed point in X, all orbits converge to periodic orbits. In addition, we prove that if X is an open 2-simplex, then the optimal upper bound for the periods of periodic points of nonexpansive mappings on (X, d_X) is 6. The results have applications in the analysis of nonlinear mappings on cones, and extend work by Nussbaum and others.

1. Introduction

In [11] Hilbert introduced the following metric spaces which generalize the Cayley–Klein model of the hyperbolic plane. Let $X \subseteq \mathbb{R}^n$ be a bounded open convex set. For $x \neq y$ in X define the distance between x and y to be the logarithm of the *cross-ratio*,

$$[a, x, y, b] = \frac{|ay|}{|ax|}\frac{|bx|}{|by|},$$

where a and b are the points of intersection of the straight-line through x and y and the (Euclidean) boundary, ∂X, of X such that x is between a and y, and y is between b and x, see Figure 1.

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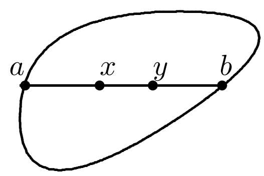

Figure 1. Hilbert's metric

So, for $x \neq y$ in X,

$$d_X(x,y) = \log[a,x,y,b].$$

The metric d_X is called *Hilbert's metric* on X. In case X is the interior of an ellipse, (X, d_X) is a model for the hyperbolic plane.

Hilbert's metric is a natural example of a projective metric, meaning that straight-line segments are geodesics, and plays a role in the solution of Hilbert's fourth problem [1]. In this paper we study the iterates of *nonexpansive mappings* $g: X \to X$ on Hilbert's metric spaces, so

$$d_X(g(x), g(y)) \leq d_X(x,y) \quad \text{for all } x, y \in X.$$

Motivating examples arise from nonlinear mappings on cones. More precisely, let $C \subseteq \mathbb{R}^{n+1}$ be a closed cone with non-empty interior C°, and let $C^* = \{\phi \in \mathbb{R}^{n+1}: \langle x, \phi\rangle \geq 0 \text{ for all } x \in C\}$ denote the *dual cone*. The cone C induces a partial ordering $\leq_C$ on $\mathbb{R}^{n+1}$ by $x \leq_C y$ if $y - x \in C$. Suppose that ϕ is in the interior of C^* and let $X_\phi = \{x \in C^\circ: \langle x, \phi\rangle = 1\}$, which is a bounded, convex, relatively open set. Now if $f: C^\circ \to C^\circ$ is a monotone mapping, i.e. f preserves $\leq_C$ on C°, and f is homogeneous of degree 1, then $g: X_\phi \to X_\phi$ given by,

$$g(x) = \frac{f(x)}{\langle f(x), \phi\rangle} \quad \text{for all } x \in X_\phi, \tag{1.1}$$

is nonexpansive under Hilbert's metric, see [5], [22]. The original idea to use Hilbert's metric in the analysis of cone mappings is due to Garrett Birkhoff [3] and H. Samelson [26], who used it to analyze eigenvalue problems for linear operators that leave a closed cone in a Banach space invariant.

Interesting nonlinear examples arise in optimal control and game theory [7], [25], matrix scaling problems [23], and the analysis of diffusions on fractals [18], [21]. In these applications it is often important to understand the iterative behavior of $g: X_\phi \to X_\phi$ in (1.1).

The goal of this paper is to analyze the iterates of mappings $g: X \to X$ which are nonexpansive with respect to Hilbert's metric, for strictly convex domains X, meaning that ∂X does not contain any straight-line segments, and for X an open n-simplex. It is important to distinguish two cases: g has a fixed point in X, and g does not have a fixed point in X. In the second case it follows from a result by A. Calka [6] that the limit points of each orbit of g are contained in the boundary of X. In fact, Nussbaum [24] and Karlsson have conjectured independently that, in that case, there exists $\Lambda \subseteq \partial X$ convex such that the limit points of all orbits of g are contained in Λ. If X is a strictly convex, the conjecture is known to be true [2] and Λ reduces to a single point. Partial results for general domains were obtained in [2], [12], [13], [17], [24].

In this paper we shall restrict ourselves to the first case and assume that g has a fixed point in X. We prove the following result.

THEOREM 1.1. *Suppose that (X, d_X) is a strictly convex Hilbert's metric space and there exists no 2-dimensional plane P such that $X \cap P$ is the interior of an ellipse. If $g\colon X \to X$ is nonexpansive under Hilbert's metric and g has a fixed point, then every orbit of g converges to a periodic orbit. In fact, there exists an integer $q \geq 1$ such that $(g^{qk}(x))_k$ converges for all $x \in X$.*

In other words, each orbit of a nonexpansive mapping converges to a periodic orbit if the domain does not contain a hyperbolic plane and the mapping has a fixed point.

If the domain X is an open polytope, i.e. the intersection of finitely many open half-spaces, then it is known [22], [27] that there is convergence to periodic orbits for nonexpansive mappings $g\colon X \to X$ with a fixed point in X. Moreover, there exists an a priori upper bound for the possible periods in terms of the number of facets of X. The current best estimate, obtained in [16], is

$$\max_{k=1,\dots,m} 2^k \binom{m}{k},$$

where $m = N(N-1)/2$ and N is the number of facets of X. This upper bound is believed to be far from optimal. Finding a sharp upper bound appears to be a hard combinatorial geometric problem, even when X is an open n-simplex. For the 2-simplex, however, we prove the following result.

THEOREM 1.2. *If X is an open 2-simplex, then the optimal upper bound for the periods of periodic points of nonexpansive mappings on (X, d_X) is 6.*

2. Preliminaries

Let (Y, d) be a complete metric space, and $g\colon Y \to Y$ be a continuous map. The *orbit* of y under g is denoted by $\mathcal{O}(y) = \{g^k(y)\colon k = 0, 1, \dots\}$. A point $y \in Y$ is called a *periodic point* if $g^p(y) = y$ for some integer $p \geq 1$, and the smallest such $p \geq 1$ is called the *period* of y. For $y \in Y$ the *ω-limit set* of y under g is given by,

$$\omega(y; g) = \Big\{x \in Y\colon \lim_{i\to\infty} g^{k_i}(y) = x \text{ for some subsequence } k_i \to \infty\Big\}.$$

Furthermore we write $\Omega_g = \bigcup_{y\in Y} \omega(y; g)$ to denote the *attractor* of g.

Clearly $\omega(y;g)$ is closed and $g(\omega(y;g)) \subseteq \omega(y;g)$. It is not hard to show that if $g\colon Y \to Y$ is continuous, $\mathcal{O}(y)$ is pre-compact, and $|\omega(y,g)| = q$, then $(g^{qk}(y))_k$ converges to a periodic point of g with period q. Moreover, if $g\colon Y \to Y$ is nonexpansive, g has a fixed point in Y, and the orbit of each point in Y is pre-compact, then each $\omega(y;g)$ is a non-empty compact set, and $g(\omega(y;g)) = \omega(y;g)$. Furthermore it was shown in [8] that $\omega(x;g) = \omega(y;g)$ for all $x \in \omega(y;g)$, and the restriction of g to $\omega(y;g)$ is an isometry, see [10].

A metric space (Y,d) is called *proper* if every closed ball is compact. Hilbert's metric spaces are separable and proper, since their topology coincides with the norm topology, see [22]. As the iterates of a nonexpansive mapping form an equicontinuous family, one can use an Arzelà–Ascoli type argument to prove the following assertion, see [4, p. 9] for details.

Lemma 2.1. *If (Y,d) is a separable proper metric space and $g\colon Y \to Y$ is a nonexpansive mapping with a fixed point in Y, then every subsequence of $(g^k)_k$ has a further subsequence which converges uniformly on compact subsets of Y.*

Lemma 2.1 can be used to prove the following proposition concerning the existence of a nonexpansive retraction on Ω_g. The argument is a straightforward adaptation of [15, Proposition 2.1].

Proposition 2.2. *If (Y,d) is separable proper metric space and $g\colon Y \to Y$ is a nonexpansive mapping with a fixed point on Y, then there exists a nonexpansive retraction $r\colon Y \to Y$ onto Ω_g and the restriction of g to Ω_g is an isometry.*

3. Strictly convex domains

Recall that a path $\gamma\colon [r,s] \to (X,d_X)$ is called a *geodesic* if

$$d_X(\gamma(t_1),\gamma(t_2)) = |t_1 - t_2| \quad \text{for all } t_1,t_2 \in [r,s].$$

A *geodesic segment* is the image of a geodesic in (X,d_X). It is a well-known fact that if X is strictly convex, then (X,d_X) is uniquely geodesic [4, p. 106]; so, the only geodesic segments are the straight-line segments. For $x,y \in (X,d_X)$ we write $[x,y]$ to denote the straight-line segment connecting x and y, and $\ell_{x,y}$ to denote the straight-line through x and y.

Lemma 3.1. *If (X,d_X) is a strictly convex Hilbert's metric space and $g\colon X \to X$ is a nonexpansive mapping with a fixed point in X, then Ω_g is convex.*

Proof. By Proposition 2.2 there exists a nonexpansive retraction $r\colon X \to X$ onto Ω_g. Thus, $r(r(x)) = r(x)$ for all $x \in X$.

Now let $x,y \in \Omega_g$ and put $s = d_X(x,y)$. Let $\gamma\colon [0,s] \to X$ be the unique geodesic path connecting $[x,y]$, so the image of γ is the straight-line segment between x and y. Now let $u = \gamma(t)$, so $d_X(x,u) = t$ and $d_X(u,y) = s - t$. As r is nonexpansive, we get that $d_X(x, r(u)) \leq t$ and $d_X(y, r(u)) \leq s - t$. Now using

$$d_X(x, r(u)) + d_X(r(u), y) \leq d_X(x,y) = s,$$

we find that $d_X(x, r(u)) = t$ and $d_X(y, r(u)) = s - t$. This implies that $r(u)$ lies on the unique geodesic connecting x and y, which is $[x,y]$. Thus, $r(u) = u$ and hence $u \in \Omega_g$. □

It is convenient to embed the domain X into the affine hyperplane $H = \{(x,1) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}$ by identifying X with $\{(x,1) \in \mathbb{R}^{n+1} : x \in X\}$. Let $P(\mathbb{R}^{n+1})$ denote the real n-dimensional projective space. So, $P(\mathbb{R}^{n+1})$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. Recall that $P(\mathbb{R}^{n+1})$ can be partitioned into the set of lines intersecting the hyperplane H and the set of lines parallel to H. In other words, X is contained in the open cell of $P(\mathbb{R}^{n+1})$. Let $V' = \mathrm{aff}(\Omega_g)$ denote the affine span of Ω_g inside H, and put $X' = V' \cap X$. Let V be the vector space above V' in $\mathbb{R}^{n+1}$. Thus, X' lies in the open cell of the real projective space $P(V) = V' \cup P(V')$. We write

$$\mathrm{Coll}(X') = \{h \in \mathrm{PGL}(V) : h(X') = X'\}$$

to denote the *collineation group* of X'.

Proposition 3.2. *If $g: X \to X$ is a nonexpansive mapping on a strictly convex Hilbert's metric space (X, d_X), and g has a fixed point, then there exists a collineation $h \in \mathrm{Coll}(X')$ such that h coincides with g on Ω_g.*

Proof. Let $k = \dim V' \geq 1$. (The case $k = 0$ is trivial.) By Lemma 3.1 Ω_g is convex, and hence there exist $x^0, \ldots, x^k$ in Ω_g such that $\mathrm{conv}(x^0, \ldots, x^k)$ is a k-simplex in Ω_g Let $y \in \Omega_g$ be in the relative interior of $\mathrm{conv}(x^0, \ldots, x^k)$. So, $y, x^0, \ldots, x^k$ forms a projective basis for $P(V)$.

As g is an isometry on Ω_g and g maps Ω_g onto itself, $g_{|\Omega_g}$ and $g^{-1}_{|\Omega_g}$ map straight-line segments to straight-line segments in Ω_g. We now use a simple induction argument to prove the following claim.

Claim 1. For $0 \leq k_0 < \ldots < k_s \leq k$ we have that $z \in \mathrm{conv}(x^{k_0}, \ldots, x^{k_s})$ if and only if $g(z) \in \mathrm{conv}(g(x^{k_0}), \ldots, g(x^{k_s}))$.

If $s = 1$, then the assertion is clear as g maps the straight-line segment $[x^{k_0}, x^{k_1}]$ onto the straight-line segment $[g(x^{k_0}), g(x^{k_1})]$. Now suppose the assertion is true for all integers r with $1 \leq r < s \leq k$. Let $z \in \mathrm{conv}(x^{k_1}, \ldots, x^{k_s})$. The assertion is clearly true for $z = x^{k_s}$. So, suppose that $z \neq x^{k_s}$ and let ℓ be the straight-line through z and x^{k_s}. Note that ℓ intersects $\mathrm{conv}(x^{k_1}, \ldots, x^{k_{s-1}})$ in a point u. By the induction hypothesis $g(u) \in \mathrm{conv}(g(x^{k_1}), \ldots, g(x^{k_{s-1}}))$. As g maps $[u, x^{k_s}]$ onto $[g(u), g(x^{k_s})]$, we conclude that $g(z)$ is in $\mathrm{conv}(g(x^{k_0}), \ldots,$

$g(x^{k_s})$). The opposite implication is obtained by applying the same argument to $g^{-1}_{|\Omega_g}$.

Claim 1 implies that $\mathrm{conv}(g(x^0), \dots, g(x^k))$ is a k-dimensional simplex in Ω_g and $g(y)$ is in its relative interior. Thus, $g(y), g(x^0), \dots, g(x^k)$ is a projective basis for $P(V)$. Let $h \in \mathrm{PGL}(V)$ be the unique collineation that coincides with g on $y, x^0, \dots, x^k$. We will show that $h \in \mathrm{Coll}(X')$ and h coincides with g on the whole of Ω_g. We need the following claim.

Claim 2. If g coincides with h on 3 distinct collinear points $x, w, z \in \Omega_g$, then g coincides with h on the straight-line segment $\ell_{x,z} \cap \Omega_g$.

To prove the claim let $a, b \in \partial X'$ be the points of intersection of the straight-line through x and z such that x is between a and z, and z is between b and x. Likewise let $a', b' \in \partial X'$ be the points of intersection of the straight-line through $g(x)$ and $g(z)$ such that $g(x)$ is between a' and $g(z)$, and $g(z)$ is between b' and $g(x)$. There exists a collineation f on the projective line containing $\ell_{x,z}$ that maps a to a', b to b', and x to $g(x)$. We show that f coincides with g on $\ell_{x,z} \cap \Omega_g$. For $u \in \ell_{x,z} \cap \Omega_g$, with u between x and b,

$$[a', g(x), g(u), b'] = [a, x, u, b] = [a', f(x), f(u), b'].$$

As $f(x) = g(x)$, this equality uniquely determines $g(u)$, so that $g(u) = f(u)$. The case where u is between z and a is similar. Now note that, as x, w and z form a projective basis for the projective line containing $\ell_{x,z}$, and h and f coincides on these 3 points, f and h are identical on $\ell_{x,z}$. This implies that g and h are identical on $\ell_{x,z} \cap \Omega_g$.

Note that for each $0 \leq l \leq k$ there exists y^l in the relative interior of $\mathrm{conv}(\{x^1, \dots, x^k\} \setminus \{x^l\})$ such that $h(y^l) = g(y^l)$. Simply let y^l be the point of intersection of $\ell_{x^l,y}$ and $\mathrm{conv}(\{x^1, \dots, x^k\} \setminus \{x^l\})$. It follows from Claim 1 that $g(y^l)$ is in the relative interior of $\mathrm{conv}(\{g(x^0), \dots, g(x^k)\} \setminus \{g(x^l)\})$. As g maps straight-line segments to straight-line segments. $g(y^l)$ is the unique point of intersection of $\ell_{g(x^l),g(y)}$ and $\mathrm{conv}(\{g(x^0), \dots, g(x^k)\} \setminus \{g(x^l)\})$. Thus, $h(y^l) = g(y^l)$. Repeating this argument shows that for each $0 \leq k_0 < k_1 < \dots < k_s \leq k$ there exists w in the relative interior of $\mathrm{conv}(x^{k_0}, \dots, x^{k_s})$ such that $h(w) = g(w)$.

We shall now show by induction on $s \geq 1$ that g and h coincide on $\mathrm{conv}(x^{k_0}, \dots, x^{k_s})$. The induction basis is true by Claim 2. Suppose the assertion is true for all $m < s$. Let v and w be points in the relative interior of $\mathrm{conv}(x^{k_0}, \dots, x^{k_s})$ with $v \neq w$ and $h(w) = g(w)$. Then the straight-line $\ell_{w,v}$ intersects the relative boundary of $\mathrm{conv}(x^{k_0}, \dots, x^{k_s})$ in two distinct points p and q. By the induction hypothesis g and h coincide on p and q. As $h(w) = g(w)$, we can apply Claim 2 to deduce that $g(v) = h(v)$.

To see that g coincides with h on the whole of Ω_g we remark that for $v \in \Omega_g$ the straight-line $\ell_{y,v}$ contains at least 3 points of $\mathrm{conv}(x^0, \dots, x^k)$, as y is in the relative interior of $\mathrm{conv}(x^0, \dots, x^k)$. Thus, by Claim 2 g coincides with h on Ω_g.

It remains to show that $h(X') = X'$. Let $z_1 \neq z_2$ in the relative interior of $\mathrm{conv}(x^0, \dots, x^k)$, and let $a, b \in \partial X' \cap \ell_{z_1,z_2}$ such that z_1 is between z_2 and a, and z_2 is between z_1 and b. Then $[a, z_1, z_2, b] = [a', g(z_1), g(z_2), b']$, where $a', b' \in \partial X' \cap \ell_{g(z_1),g(z_2)}$, and $g(z_1)$ is between a' and $g(z_2)$. There exists a collineation f that maps a to a', b to b', and z_1 to $g(z_1)$. As in the proof of Claim 2 g coincides with f on $[z_1, z_2]$. This implies that h coincides with f on $\ell_{z_1,z_2} \cap X'$, and hence $h(a) = a'$ and $h(b) = b'$, which completes the proof. □

Ideas similar to the ones behind Lemma 3.1 and Proposition 3.2 were used by Edelstein in [9] to analyze nonexpansive mappings on strictly convex Banach spaces.

A key ingredient in the proof of Theorem 1.1 is a result by Y. I. Lyubich and A. I. Veitsblit [20]. To state it we need to recall a few definitions. Let C be a closed cone with non-empty interior in a finite dimensional real vector space W, so C is convex, $\lambda C \subseteq C$ for all $\lambda > 0$, and $C \cap (-C) = \{0\}$. A linear map $A\colon W \to W$ is said to be *positive* if $A(C) \subseteq C$. We denote the *automorphism group* of C by

$$\mathrm{Aut}(C) = \{A \in \mathrm{GL}(W) : A(C) = C\}.$$

A linear subspace U of W is called *C-complemented* if there exists a positive linear projection $P\colon W \to W$ with range U. Note that if U is a subspace of W, then $U \cap C$ is a closed sub-cone of C.

A closed cone C with non-empty interior in W, where $\dim W = n + 1$, is called a *Lorentz cone* if there exists a system of coordinates $x_1, \dots, x_{n+1}$ such that

$$C = \{(x_1, \dots, x_{n+1}) \in W : x_{n+1}^2 - x_1^2 - \dots - x_n^2 \geq 0 \text{ and } x_{n+1} \geq 0\}.$$

Theorem 2.3 (Lyubich–Veitsblit [20]). *If $C \subseteq W$ is a closed cone with non-empty interior, and* $\mathrm{Aut}(C)$ *contains an infinite compact subgroup, then there exists a* 3*-dimensional C-complemented subspace U of W such that $K = C \cap U$ is a Lorentz cone in U.*

We will now prove Theorem 1.1.

Proof of Theorem 1.1. As before identify X with $\{(x, 1) \in \mathbb{R}^{n+1}\colon x \in X\}$ and let $V' = \mathrm{aff}(\Omega_g)$ be the affine span of Ω_g in $\mathbb{R}^{n+1}$. Put $X' = V' \cap X$ and let V be the subspace above V' in $\mathbb{R}^{n+1}$.

Furthermore let C_X denote the closure of the open cone generated by X in $\mathbb{R}^{n+1}$, so

$$C_X = \{\lambda(x, 1) \in \mathbb{R}^{n+1} : x \in \overline{X} \text{ and } \lambda \geq 0\}.$$

Likewise let $C_{X'}$ denote the closure of the open cone generated by X' in V. Let $x^* \in \Omega_g$ be a fixed point of g, and note that x^* is in the interior of $C_{X'}$ in V.

By Proposition 2.2 there exists an $h \in \mathrm{Coll}(X')$ such that h coincides with g on Ω_g. Thus, there exists $A \in \mathrm{Aut}(C_{X'})$ such that $pA = h$, where p denotes the projection. Remark that, as $h(x^*) = g(x^*) = x^*$, we have that $A(x^*) = \sigma x^*$ for some $\sigma > 0$. Putting $B = \frac{1}{\sigma}A$ we see that $B \in \mathrm{Aut}(C_{X'})$, $pB = h$ and $B(x^*) = x^*$.

As x^* in the interior of $C_{X'}$ in V, we know that $(B^k)_{k=0}^{\infty}$ is bounded. Indeed, let $\|x\| = \inf\{\mu > 0\colon -\mu x^* \leq_C x \leq_C \mu x^*\}$ be the order unit norm on V, see [22, p.14]. Now let $x \in V$ and $\|x\| = \tau$. Then $-\tau x^* \leq_C x \leq_C \tau x^*$, so that

$$-\tau x^* = -\tau B(x^*) \leq_C B(x) \leq_C \tau B(x^*) = \tau x^*.$$

This implies that $\|B\| = 1$, and hence $(B^k)_{k=0}^{\infty}$ is bounded. In the same way it can be shown that $\|B^{-1}\| = 1$, and hence the closure of the group generated by $(B^k)_{k=0}^{\infty}$ is a compact subgroup of $\mathrm{Aut}(C_{X'})$.

As there exists no 2-dimensional plane P in $\mathbb{R}^n$ such that $P \cap X$ is the interior of an ellipse, there exists no 3-dimensional subspace U of $\mathbb{R}^{n+1}$ such that $U \cap C_{X'}$ is a Lorentz cone. Hence by the Lyubich–Veitsblit Theorem 2.3 we know that there exists an integer $q \geq 1$ such that $B^q = I$.

Now to prove the convergence to periodic orbits, it suffices to show that $|\omega(x; g)|$ divides q for each $x \in X$, since g is nonexpansive. So, let $x \in X$ and $y \in \omega(y; g)$. By [8] we know that $\omega(x; g) = \omega(y; g)$. As $y \in \Omega_g$, $g^q(y) = h^q(y) = pB^q(y) = y$. Therefore y is a periodic point of g whose period divides q. Thus, $|\omega(x; g)| = |\omega(y; g)|$ divides q, and we are done. □

The proof of Theorem 1.1 does not use the full strength of the Lyubich–Veitsblit Theorem 2.3. In fact, I believe that the hypothesis in Theorem 1.1 can be weakened to the assumption: there exists no 3-dimensional C_X-complemented subspace of $\mathbb{R}^{n+1}$ such that $C_X \cap U$ is a Lorentz cone. For instance, if $C_X = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 : x_4^4 - x_1^4 - x_2^4 - x_3^4 \geq 0 \text{ and } x_4 \geq 0\}$, which corresponds to X being the interior of the unit ball in $\mathbb{R}^3$ with the ℓ_4-norm, then it is known [19] that there exists a 3-dimensional subspace U such that $C_X \cap U$ is a Lorentz cone, but U is not C_X-complemented, see [20, Theorem 3]. A more daring speculation would be to conjecture that there is convergence to periodic orbits for general Hilbert's metric spaces not containing a hyperbolic plane.

It must be noted that there exists an analogous result for finite dimensional strictly convex normed spaces, see [15]. In that case one assumes that there exists no 1-complemented Euclidean plane to ensure convergence to periodic orbits.

4. The simplex

It is known [22] that if X is an open n-simplex, then (X, d_X) is isometric to a normed space $(\mathbb{R}^n, \|\cdot\|_H)$, where $\|\cdot\|_H$ has a polyhedral unit ball. As (X, d_X) and (Y, d_Y) are isometric if X and Y are open n-simplices, we can restrict ourselves to analyzing Hilbert's metric on the *standard n-simplex*,

$$\Delta_n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_i x_i = 1 \text{ and } x_i > 0 \text{ for } i = 1, \dots, n+1\right\}.$$

In that case it is easy to describe the isometry. Let $\mathbb{R}^{n+1}$ be equipped with an equivalence relation $\sim$ given by, $x \sim y$ if $x = y + \lambda(1, \dots, 1)$ for some $\lambda \in \mathbb{R}$. Then $\mathbb{R}^{n+1}/\sim$ is an n-dimensional vector space that can be endowed with the *variation norm*,

$$\|x\|_{\text{var}} = \max_{1 \le i \le n+1} x_i - \min_{1 \le j \le n+1} x_j \quad \text{for } x \in \mathbb{R}^{n+1}/\sim.$$

There exists an isometry of (Δ_n, d_{Δ_n}) onto $(\mathbb{R}^{n+1}/\sim, \|\cdot\|_{\text{var}})$ which is given by,

$$\text{Log}(x) = (\log x_1, \dots, \log x_{n+1}) \quad \text{for } x \in \Delta_n.$$

By taking the representative $x \in \mathbb{R}^{n+1}$ with $x_{n+1} = 0$ in the equivalence class of x in $\mathbb{R}^{n+1}/\sim$, and projecting out the $(n+1)$-th coordinate, we see that (Δ_n, d_{Δ_n}) is isometric to $(\mathbb{R}^n, \|\cdot\|_H)$, where

$$\|z\|_H = \left(0 \vee \max_{1 \le i \le n} z_i\right) - \left(0 \wedge \min_{1 \le j \le n} z_j\right) \quad \text{for } z \in \mathbb{R}^n.$$

Here $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. In dimension $n = 2$ the unit ball is a hexagon, as shown in Figure 2. For $n = 3$ the unit ball is a rhombic-dodecahedron.

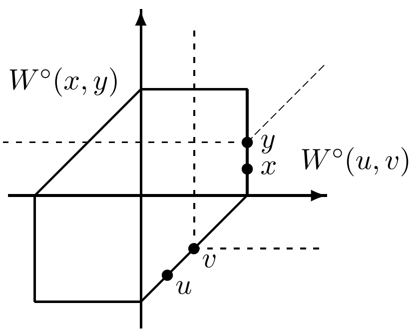

FIGURE 2. The unit ball of $\|\cdot\|_H$

It is clear that there exists an isometry A on $(\mathbb{R}^n, \|\cdot\|_H)$ with period 6. Simply take

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, on (Δ_2, d_{Δ_2}) there exists a nonexpansive mapping that has a period 6 point. In this section it is shown that 6 is the maximal possible period. The proof uses similar methods as the ones developed in [14].

Let $(\mathbb{R}^n, \|\cdot\|)$ be a polyhedral normed space, i.e. it unit ball is a polyhedron. A sequence $x^1, \dots, x^k \in \mathbb{R}^n$ is called an *additive chain* if

$$\|x^1 - x^k\| = \sum_{i=1}^{k-1} \|x^i - x^{i+1}\|.$$

As $\|\cdot\|$ is a polyhedral norm, $x^1, x^2, \dots, x^k$ need not lie on a straight-line in order to be an additive chain. For $x, y \in \mathbb{R}^n$ define

$$W(x,y) = \{z \in \mathbb{R}^n \colon x, y, z \text{ is an additive chain}\}$$

and denote its interior by $W^\circ(x,y)$. Given a polyhedral norm on $\mathbb{R}^n$ whose unit ball has m facets, i.e. m faces of dimension $n-1$, there exist m linear functionals $\phi_i, \dots, \phi_m$ such that

$$\|x\| = \max_{i=1,\dots,m} \langle \phi_i, x\rangle \quad \text{for all } x \in \mathbb{R}^n.$$

Let $I(x,y) = \{i \colon \|x-y\| = \phi_i(x-y)\}$.

LEMMA 4.1. *If $x^1, \dots, x^k$ is a sequence in a polyhedral normed vector space $(\mathbb{R}^n, \|\cdot\|)$ and $\|x^1 - x^k\| = \phi_i(x^1 - x^k)$, then $x^1, \dots, x^k$ is an additive chain if and only if $\|x^j - x^{j+1}\| = \phi_i(x^j - x^{j+1})$ for all $j = 1, \dots, k-1$.*

PROOF. Clearly

$$\|x^1 - x^k\| = \sum_{j=1}^{k-1} \|x^j - x^{j+1}\| \geq \sum_{j=1}^{k-1} \phi_i(x^j - x^{j+1}) = \phi_i(x^1 - x^k) = \|x^1 - x^k\|.$$

Thus, $\phi_i(x^j - x^{j+1}) = \|x^j - x^{j+1}\|$ for all $j = 1, \dots, k-1$. Conversely,

$$\|x^1 - x^k\| \leq \sum_{j=1}^{k-1} \|x^j - x^{j+1}\| = \sum_{j=1}^{k-1} \phi_i(x^j - x^{j+1}) = \phi_i(x^1 - x^k) \leq \|x^1 - x^k\|. \qquad \square$$

It follows from Lemma 4.1 that $x^1, x^2, \dots, x^k$ is an additive chain if and only if $\bigcap_{j=1}^{k-1} I(x^j, x^{j+1}) \neq \emptyset$.

LEMMA 4.2. *For $x \neq y$ in a polyhedral normed vector $(\mathbb{R}^n, \|\cdot\|)$ we have that*

$$W^\circ(x,y) = \{z \in W(x,y) \colon I(y,z) \subseteq I(x,y)\}.$$

PROOF. Suppose that x, y, z is an additive chain and $I(y,z) \not\subseteq I(x,y)$. Let $i \in I(y,z) \setminus I(x,y)$. For $\varepsilon > 0$ there exists $z' \in \mathbb{R}^n$ such that $I(y,z') = \{i\}$ and $\|z' - z\| \leq \varepsilon$. Note that x, y, z' is not an additive chain, as $I(x,y) \cap I(y,z') = \emptyset$. Hence $z \in \partial W(x,y)$, which shows that

$$W^\circ(x,y) \subseteq \{z \in W(x,y) \colon I(y,z) \subseteq I(x,y)\}.$$

On the other hand, given an additive chain x, y, z, we let

$$\varepsilon = \min_{i,j} \phi_i(y-z) - \phi_j(y-z) > 0,$$

where the minimum is taken over all $i \in I(y,z)$ and $j \notin I(y,z)$.

For each $z' \in \mathbb{R}^n$ with $\|z - z'\| < \varepsilon/2$, we have that $I(y,z') \subseteq I(y,z)$. Indeed, if $j \notin I(y,z)$ and $i \in I(y,z)$, then

$$\begin{aligned}\phi_j(y-z') = \phi_j(y-z) + \phi_j(z-z') &\leq \phi_i(y-z) + \phi_j(z-z') - \varepsilon < \phi_i(y-z) - \varepsilon/2 \\ &\leq \phi_i(y-z') + \phi_i(z'-z) - \varepsilon/2 < \phi_i(y-z'),\end{aligned}$$

and hence $j \notin I(y,z')$.

As $I(y,z) \subseteq I(x,y)$, we know that $I(y,z') \subseteq I(x,y)$. This implies that $I(y,z') \cap I(x,y)$ is non-empty, and hence x, y, z' is an additive chain. □

Recall that if $\mathcal{O} = \{\xi, g(\xi), \ldots, g^{p-1}(\xi)\}$ is a periodic orbit of a nonexpansive mapping g on a metric space (Y,d), then the iterates $\Gamma = \{g^k \colon k = 0, \ldots p-1\}$ form a cyclic group of isometries on $\mathcal{O}$ that acts transitively on $\mathcal{O}$, i.e. for each $x, y \in \mathcal{O}$ there exists $g^k \in \Gamma$ such that $g^k(x) = y$. The following lemma generalizes [14, Lemma 2.2].

LEMMA 4.3. *If S is a compact set in a polyhedral normed space $(\mathbb{R}^n, \|\cdot\|)$ and S has a transitive commutative group of isometries, then*

$$W^\circ(x,y) \cap S = \emptyset \quad \textit{for all } x \neq y \in S.$$

PROOF. Let $x, y, z \in S$ be such that $x \neq y$ and $z \in W^\circ(x,y)$. Put $\varepsilon = \min\{\|x-y\|, \|y-z\|\} > 0$. Denote the collection of all additive chains in S starting with x, y, z and which are such that the distance between consecutive points in the chain is at least ε by $\mathcal{F}$.

Note that, since S is compact, there exists an upper bound on the length of the additive chains in $\mathcal{F}$. Let $x^1 = x, x^2 = y, x^3 = z, \ldots, x^r$ be an additive chain in $\mathcal{F}$ of maximal length.

For $1 \leq k, l \leq r$ let $g_{k,l}$ be an isometry on S in the commutative group that acts transitively on S such that $g_{k,l}(x^k) = x^l$. Denote $x^{r+1} = g_{1,2}(x^r)$. We now show that $x^2, x^3, \dots, x^{r+1}$ is also an additive chain in S in which the distance between consecutive points is at least ε. Indeed, as the group is commutative,

$$\|x^r - x^{r+1}\| = \|g_{1,r}(x^1) - g_{1,r}(g_{1,2}(x^1))\| = \|x^1 - x^2\| \geq \varepsilon.$$

This equality implies,

$$\|x^2 - x^{r+1}\| = \|g_{1,2}(x^1) - g_{1,2}(x^r)\| = \sum_{j=1}^{r-1} \|x^j - x^{j+1}\| = \sum_{j=2}^{r} \|x^j - x^{j+1}\|,$$

and hence $\bigcap_{j=2}^r I(x^j, x^{j+1}) \neq \emptyset$. As $z \in W^\circ(x, y)$, we know that $I(x^2, x^3) \subseteq I(x^1, x^2)$, and hence $\bigcap_{j=1}^r I(x^j, x^{j+1}) \neq \emptyset$. But this implies that $x^1, \dots, x^{r+1}$ is an additive chain in $\mathcal{F}$, which contradicts the maximality of r. □

We can now prove Theorem 1.2.

PROOF OF THEOREM 1.2. If g is a nonexpansive mapping on $(\mathbb{R}^2, \|\cdot\|_H)$ with a periodic orbit $\mathcal{O}$ and fixed point x^*, then $\mathcal{O}$ is contained in the boundary of the ball $B(x^*) = \{x \in \mathbb{R}^2 : \|x - x^*\|_H \leq R\}$ for some $R \geq 0$. Without loss of generality we may assume that $x^* = 0$.

Partition $\partial B(x^*)$ as follows:

$$\begin{aligned}
A_1 &= \{x \in \mathbb{R}^2 : x_1 = R \text{ and } 0 \leq x_2 < R\},\\
A_2 &= \{x \in \mathbb{R}^2 : x_2 = R \text{ and } 0 < x_1 \leq R\},\\
A_3 &= \{x \in \mathbb{R}^2 : x_2 - x_1 = R \text{ and } 0 < x_2 < R\},\\
A_4 &= \{x \in \mathbb{R}^2 : x_1 = -R \text{ and } -R < x_2 \leq 0\},\\
A_5 &= \{x \in \mathbb{R}^2 : x_2 = -R \text{ and } -R \leq x_1 < 0\},\\
A_6 &= \{x \in \mathbb{R}^2 : x_1 - x_2 = R \text{ and } -R \leq x_2 < 0\}.
\end{aligned}$$

Each A_i corresponds to a facet of $\partial B(x^*)$ with one of its vertices removed.

By Lemma 4.3 each A_i can contain at most two points of $\mathcal{O}$, see also Figure 2. Moreover, if A_i contains 2 points of $\mathcal{O}$, then $A_{i+1} \cap \mathcal{O}$ is empty. (Here we are counting modulo 6.) Indeed, if ϕ_i denotes the facet defining functional of $\partial B(x^*)$ corresponding to A_i, then it is easy to verify that if $x \neq y$ in A_i, then $I(x, y) = \{i+1, i+2\}$, so that $W^\circ(x, y) \supseteq A_{i+1}$. Thus, $A_{i+1} \cap \mathcal{O}$ is empty, if $x \neq y$ in $A_i \cap \mathcal{O}$, and hence $\mathcal{O}$ has at most 6 points. □

The example of a period 6 orbit of a nonexpansive mapping on Δ_2 can be generalized to Δ_n. To do this it is convenient to work in $(\mathbb{R}^{n+1}/\sim, \|\cdot\|_{\text{var}})$. Let $\mathcal{X}$ be the subset of all points $x \in \mathbb{R}^{n+1}/\sim$ such that $x_i = 1$ or $x_i = 0$ for each i. Note that $(0, \dots, 0) = (1, \dots, 1)$ and that for each $x \in \mathcal{X}$ we have $-x = (1, \dots, 1) - x \in \mathcal{X}$. If $\mathcal{S} \subseteq \mathcal{X}$ is such that there exists no $x \neq y$ in $\mathcal{S}$ with $x \leq y$, where the inequality holds coordinate-wise, then $\|x - y\|_{\text{var}} = 2$ for all $x \neq y$ in S. For such sets $\mathcal{S}$ it is known [16, p. 867] that there exists a nonexpansive mapping $g_{\mathcal{S}}$ on $\mathbb{R}^{n+1}/\sim$ which has $\mathcal{S}$ as a periodic orbit. Now suppose that n is even and let $\mathcal{S}_n \subseteq \mathcal{X}$ be the set of points with exactly $n/2$ coordinates equal to 1. Then $-\mathcal{S}_n$ is the set of points with exactly $n/2+1$ coordinates equal to 1 in $\mathcal{X}$. Clearly the reflection in the origin, $R(x) = -x$, is an isometry on $\mathbb{R}^{n+1}/\sim$. Now if $|\mathcal{S}_n| = \binom{n+1}{n/2}$ is odd, the nonexpansive mapping $R \circ g_{\mathcal{S}_n}$ has a periodic orbit of length $2\binom{n+1}{n/2}$. For $n = 2$, this gives a period 6 orbit Δ_2. On the other hand, if $\binom{n+1}{n/2}$ is even, we can drop one point of $\mathcal{S}_n$ and obtain a periodic orbit of length $2\binom{n+1}{n/2} - 2$. A similar construction exists for n odd. This shows, for general n, that there exists a nonexpansive mapping on (Δ_n, d_{Δ_n}) that has a periodic point with period

$$2\binom{n+1}{\lfloor n/2 \rfloor} - 2\delta_n,$$

where $\delta_n = 1$ if $\binom{n+1}{\lfloor n/2 \rfloor}$ is even and $\delta_n = 0$ otherwise. It seems unlikely that this lower bound is optimal, but no better examples are known.

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