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DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS: EXISTENCE RESULTS AND TOPOLOGICAL PROPERTIES OF SOLUTION SETS

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Abstract. In this paper, we study the topological structure of solution sets for the first-order differential inclusions with nonlocal conditions:

$$\begin{cases} y'(t) \in F(t, y(t)) & \text{a.e. } t \in [0, b], \\ y(0) + g(y) = y_0, \end{cases}$$

where $F\colon [0, b] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map. Also, some geometric properties of solution sets, R_δ, R_δ-contractibility and acyclicity, corresponding to Aronszajn–Browder–Gupta type results, are obtained. Finally, we present the existence of viable solutions of differential inclusions with nonlocal conditions and we investigate the topological properties of the set constituted by these solutions.

1. Introduction

In this paper we shall prove some existence results and properties of solution sets for ordinary differential inclusions, with nonlocal conditions. Often, nonlocal conditions are motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [18]. As indicated in [18], [19], [24] and the

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references therein, the nonlocal condition $y(0)+g(y) = y_0$ can be more descriptive in physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [24], the author used

$$g(y) = \sum_{k=1}^{p} c_i y(t_i), \tag{1.1}$$

where c_i, $i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, equation (1.1) allows the additional measurements at t_i, $i = 1, \ldots, p$.

Nonlocal Cauchy problems for ordinary differential equations have been investigated by several authors, (see for instance [14], [19], [20], [45]–[49]). Nonlocal Cauchy problems, in the case where F is a multivalued map, were studied by Benchohra and Ntouyas [9]–[12] and Boucherif [14]. For impulsive differential equations and inclusions, we can see the papers [5], [7], [8]. We will consider in this paper the first-order differential inclusion with nonlocal conditions,

$$\begin{cases} y'(t) \in F(t, y(t)) & \text{a.e. } t \in J, \\ y(0) + g(y) = y_0, \end{cases} \tag{1.2}$$

where $J := [0, b]$. $F\colon J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction, $g\colon C(J, \mathbb{R}^n) \to \mathbb{R}^n$ is a given function and $y_0 \in \mathbb{R}^n$.

In 1923, Kneser proved that the Peano existence theorem can be formulated in such a way that the set of all solutions is not only nonempty but is also compact and connected (see [50], [51]). Later, in 1942, N. Aronszajn [2] improved Kneser's theorem by showing that the set of all solutions is even an R_δ-set. It should also be clear that the characterization of the set of fixed points for some operators implies the corresponding result for the solution sets.

J. N. Lasry and R. Robert [44] studied the topological properties of the sets of solutions for a large class of differential inclusions including differential-difference inclusions. The present paper is a continuation of their work but for a general class of differential inclusions with nonlocal conditions. Aronszajn's results for differential inclusions with difference conditions were improved by several authors; for example, see [13], [22], [27], [29], [31], [32], [34].

Our goal of this paper is to examine some properties of solutions sets for differential inclusions with nonlocal conditions.

The paper is organized as follows. We first collect some background material and basic results from multi-valued analysis in Section 2. In Section 3, an existence result in case the nonlinear multi-valued mapping F takes compact convex values is proved under a Nagumo-type growth condition. Section 4 is devoted to the geometric properties of solution sets of the problem (1.2). The existence of viable solutions, topological properties and geometric properties of the set constituted by these solutions can be found in Section 5. We end the paper with some concluding remarks and a rich bibliography.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $J=[0,b]$ be an interval in $\mathbb{R}$ and $C(J,\mathbb{R}^n)$ be the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty=\sup\{\|y(t)\|:0\le t\le b\}.$$

A function $y\colon J\to\mathbb{R}^n$ is called measurable provided for every open subset $U\subset\mathbb{R}^n$, the set $y^{-1}(U)=\{t\in J\colon y(t)\in U\}$ is Lebesgue measurable. In what follows, $L^1(J,\mathbb{R}^n)$ denotes the Banach space of functions $y\colon J\to\mathbb{R}^n$, which are Lebesgue integrable with norm

$$\|y\|_{L^1}=\int_0^b\|y(t)\|\,dt.$$

Denote by $\mathcal{P}(\mathbb{R}^n)=\{Y\subset\mathbb{R}^n : Y\neq\emptyset\}$, $\mathcal{P}_{\rm cl}(\mathbb{R}^n)=\{Y\in\mathcal{P}(\mathbb{R}^n) : Y \text{ closed}\}$, $\mathcal{P}_{\rm b}(\mathbb{R}^n)=\{Y\in\mathcal{P}(\mathbb{R}^n) : Y \text{ bounded}\}$, $\mathcal{P}_{\rm cv}(\mathbb{R}^n)=\{Y\in\mathcal{P}(\mathbb{R}^n) : Y \text{ convex}\}$, $\mathcal{P}_{\rm cp}(\mathbb{R}^n)=\{Y\in\mathcal{P}(\mathbb{R}^n) : Y \text{ compact}\}$.

2.1. Multi-valued analysis. Let (X,d) and (Y,ρ) be two metric spaces and $G\colon X\to\mathcal{P}_{\rm cl}(Y)$ be a multi-valued map. A single-valued map $g\colon X\to Y$ is said to be a selection of G, and we write $g\subset G$, whenever $g(x)\in G(x)$ for every $x\in X$.

G is called *upper semi-continuous* (*u.s.c.* for short) on X if, for each $x_0\in X$, the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M)\subseteq Y$. That is, if the set $G^{-1}(V)=\{x\in X : G(x)\cap V\neq\emptyset\}$ is closed for any closed set V in Y. Equivalently, G is *u.s.c.* if the set $G^{+1}(V)=\{x\in X : G(x)\subset V\}$ is open for any open set V in Y.

The following two results are easily deduced from the limit properties.

Lemma 2.1 (see e.g. [4, Theorem 1.4.13]). *If $G\colon X\to\mathcal{P}_{\rm cp}$ is u.s.c., then for any $x_0\in X$,*

$$\limsup_{x\to x_0}G(x)=G(x_0).$$

Lemma 2.2 (see e.g. [4, Lemma 1.1.9]). *Let $(K_n)_{n\in\mathbb{N}}\subset K\subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{\rm co}\Big(\limsup_{n\to\infty}K_n\Big)=\bigcap_{N>0}\overline{\rm co}\Big(\bigcup_{n\ge N}K_n\Big),$$

where $\overline{\mathrm{co}}\,A$ *refers to the closure of the convex hull of* A*.*

G is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $G(A)$ is relatively compact, i.e. there exists a relatively compact set $K = K(A) \subset X$ such that $G(A) = \bigcup\{G(x) : x \in A\} \subset K$. G is *compact* if $G(X)$ is relatively compact. G is called *locally compact* if, for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $G(U)$ is relatively compact.

Definition 2.3. A multi-valued map $F\colon J = [0,b] \to \mathcal{P}_{\mathrm{cl}}(Y)$ is said to be measurable provided, for every open $U \subset Y$, the set $F^{-1}(U)$ is Lebesgue measurable.

We also have the following lemma.

Lemma 2.4 ([21], [33]). *The mapping* F *is measurable if and only if for each* $x \in Y$*, the function* $\zeta\colon J \to [0,\infty)$ *defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x-y\| : y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski–Ryll–Nardzewski selection theorem.

Lemma 2.5 ([33, Theorem 19.7]). *Let* Y *be a separable metric space and* $F\colon [a,b] \to \mathcal{P}(Y)$ *a measurable multi-valued map with nonempty closed values. Then* F *has a measurable selection.*

We denote the graph of G by the set $\mathcal{G}r(G) = \{(x,y) \in X \times Y,\ y \in G(x)\}$.

Definition 2.6. G is closed if $\mathcal{G}r(G)$ is a closed subset of $X \times Y$, i.e. for all sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ as $n \to \infty$ with $y_n \in F(x_n)$, then $y_* \in G(x_*)$.

The first of the next two results is classical.

Lemma 2.7 ([23, Proposition 1.2]). *If* $G\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ *is u.s.c., then* $\mathcal{G}r(G)$ *is a closed subset of* $X\times Y$*. Conversely, if* G *is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.8. *If* $G\colon X \to \mathcal{P}_{\mathrm{cp}}(Y)$ *is locally compact and has a closed graph, then* G *is u.s.c.*

Proof. Assume that G is not u.s.c. at some point x. Then there exists an open neighbourhood U of $G(x)$ in Y, a sequence $\{x_n\}$ which converges to x, and for every $l \in \mathbb{N}$ there exists $n_l \in \mathbb{N}$ such that $G(x_{n_l}) \not\subset U$. Then for each $l = 1,2,\ldots$, there are y_{n^l} such that $y_{n_l} \in G(x_{n_l})$ and $y_{n_l} \notin U$; this implies that $y_{n_l} \in Y \setminus U$. Moreover, $\{y_{n_l} : l \in \mathbb{N}\} \subset G(\overline{\{x_n : n \geq 1\}})$. Since G is compact, there exists a subsequence of $\{y_{n_l} : l \in \mathbb{N}\}$ which converges to y. G closed implies that $y \in G(x) \subset U$, but this is a contradiction to the assumption that $y_{n_l} \notin U$ for each n_l. $\square$

Lemma 2.9 ([43]). *Given a Banach space X, let $F\colon [a,b]\times X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be an L^1-Carathéodory multi-valued map such that for each $y \in C([a,b],X)$, $S_{F,y} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1([a,b],X)$ into $C([a,b],X)$. Then the operator*

$$\begin{aligned}\Gamma \circ S_F\colon C([a,b],X) &\to \mathcal{P}_{\mathrm{cp,cv}}(C([a,b],X)),\\ y &\mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})\end{aligned}$$

has a closed graph in $C([a,b],X)\times C([a,b],X)$.

Given a separable Banach space $(E,|\cdot|)$, for a multi-valued map $F\colon J\times E \to \mathcal{P}(E)$, denote

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x)\}.$$

Definition 2.10. A multi-valued map F is called a Carathéodory function if

(a) the function $t \mapsto F(t,x)$ is measurable for each $x \in E$;

(b) for almost every $t \in J$, the map $x \mapsto F(t,x)$ is upper semi-continuous.

Furthermore, F is L^1-Carathéodory if it is locally integrably bounded, i.e. for each positive r, there exists $h_r \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h_r(t), \quad \text{for a.e. } t \in J \text{ and all } |x| \leq r.$$

For each $x \in C(J,E)$, the set

$$S_{F,x} = \{f \in L^1(J,E) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,b]\}$$

is known as the set of *selection functions*.

Remark 2.11. (a) For each $x \in C(J,E)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t,x(t))$ is convex for almost every $t \in J$.

(b) From [54, Theorem 5.10] (see also [43] when E is finite-dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \mapsto \inf\{\|v\| : v \in F(t,x(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t,x(t))\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t,x(t))\}$ belongs to $L^1(J)$; this particularly holds true when F is L^1-Carathéodory. For the sake of completeness, we refer also to Theorem 1.3.5 in [41] which states that $S_{F,x}$ contains a measurable selection whenever x is measurable and F is a Carathéodory function.

For further readings and details on multi-valued analysis, we refer the reader to the books by J. Andres and L. Górniewicz [1], P. Aubin and A. Cellina [3],

P. Aubin and H. Frankowska [4], K. Deimling [23], L. Górniewicz [33], Sh. Hu and N. S. Papageorgiou [38], [39] and M. Kamenskiĭ [41].

3. Existence results

3.1. Convex case. Let $F\colon J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ be a Carathéodory multimap which satisfies some of the following assumptions:

($\mathcal{A}_1$) There exist a function $p \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ such that

$$\|F(t,z)\| \le p(t)\psi(\|z\|) \quad \text{for a.e. } t \in J \text{ and each } z \in \mathbb{R}^n,$$

with

$$\int_0^b p(s)\,ds < \int_1^\infty \frac{du}{\psi(u)}.$$

($\mathcal{A}_2$) $g\colon C(J,\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function and either one of the following conditions holds:

(a) There exist $\alpha \in [0,1)$ and $\gamma,\beta \ge 0$ such that $\|g(y)\| \le \gamma\|y\|_\infty^\alpha + \beta$.

(b) There exist $\sigma \in [0,1)$ and $\xi > 0$ such that $\|g(y)\| \le \sigma\|y\|_\infty + \xi$.

Theorem 3.1. *Assume that F satisfies* ($\mathcal{A}_1$) *and either* ($\mathcal{A}_2$)(a) *or* ($\mathcal{A}_2$)(b). *Then the problem* (1.2) *has at least one solution. Moreover, the the solution set $S(y_0)$ is compact, and the multivalued map $S\colon \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp}}(C(J,\mathbb{R}^n))$ is u.s.c.*

We recall two fundamental results. The first one follows from Leray and Schauder (see [33], [35]).

Lemma 3.2. *Let $(X,\|\cdot\|)$ be a normed space and $F\colon X \to \mathcal{P}_{\mathrm{cl,cv}}(X)$ a compact, u.s.c. multi-valued map. Then either one of the following conditions holds:*

(a) *F has at least one fixed point,*

(b) *the set $\mathcal{M} := \{x \in X,\ x \in \lambda F(x),\ \lambda \in (0,1)\}$ is unbounded.*

The second one is due to Mazur, 1933:

Lemma 3.3 (Mazur's Lemma [55, Theorem 21.4]). *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^m \alpha_{mk}x_k$ with $\alpha_{mk} > 0$, for $k = 1,2,\ldots,m$, and $\sum_{k=1}^m \alpha_{mk} = 1$, with convergence strongly to x.*

Proof of Theorem 3.1. The proof is split into three parts.

Part 1. Under Assumptions ($\mathcal{A}_1$), ($\mathcal{A}_2$)(a) the solutions set is nonempty and compact.

Step 1. $S(y_0) \neq \emptyset$. Consider the operator $N\colon C(J,\mathbb{R}^n) \to \mathcal{P}(C(J,\mathbb{R}^n))$ defined for $y \in C(J,\mathbb{R}^n)$ by

$$N(y) = \left\{h \in C(J,\mathbb{R}^n) : h(t) = y_0 - g(y) + \int_0^t v(s)\,ds,\ t \in [0,b]\right\},$$

where $v \in S_{F,y} = \{u \in L^1(J, \mathbb{R}^n) : u \in F(t, y(t))$, almost every $t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (1.2). Since, for each $y \in C(J, \mathbb{R}^n)$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and so N has convex values. As in [10], [14], [26], we can prove that N maps bounded sets into bounded sets and there exists $M_1 > 0$ such that for every solution y of problem (1.2), we have $\|y\|_\infty \le M_1$. Thus, we only prove that $N(\mathcal{B}_q)$ is relatively compact in $C(J, \mathbb{R}^n)$, where $\mathcal{B}_q = \{y \in C(J, \mathbb{R}^n) : \|y\|_\infty \le q\}$. First, $N(\mathcal{B}_q)$ is an equicontinuous set of $C(J, \mathbb{R}^n)$. To see this, let $0 < \tau_1 < \tau_2 \le b$, $y \in \mathcal{B}_q$, and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h(t) = y_0 - g(y) + \int_0^t v(s)\, ds, \quad t \in [0, b].$$

Then,

$$\|h(\tau_2) - h(\tau_1)\| \le \int_{\tau_1}^{\tau_2} \|v(s)\|\, ds.$$

Hence,

$$\|h(\tau_2) - h(\tau_1)\| \le \psi(q) \int_{\tau_1}^{\tau_2} p(s)\, ds.$$

The terms in the right-hand side tend to zero as $\tau_1 - \tau_2 \to 0$. By the Arzelá–Ascoli theorem, we conclude that $N\colon C(J, \mathbb{R}^n) \to \mathcal{P}_{\mathrm{cp,cv}}(C(J, \mathbb{R}^n))$ is a completely continuous operator. Finally, the nonlinear alternative for multi-valued mappings (Lemma 3.2) implies that $S(y_0) \ne \emptyset$.

Step 2. Compactness of the solution set.

Let $S(y_0) = \{y \in C([0, b], \mathbb{R}^n) : y$ is a solution of (1.2)$\}$. From Step 1, $S_F \ne \emptyset$ and there exists M such that for every $y \in S_F$, $\|y\|_\infty \le M$. Since N is completely continuous, then $N(S(y_0))$ is relatively compact in $C(J, \mathbb{R}^n)$. Let $y \in S(y_0)$; then $y \in N(y)$ and $S(y_0) \subset \overline{N(S(y_0))}$. It remains to prove that $S(y_0)$ is a closed set in $C(J, \mathbb{R}^n)$. Let $y_n \in S(y_0)$ such that y_n converges to y. For every $n \in \mathbb{N}$, there exists $v_n(t) \in F(t, y_n(t))$, for almost every $t \in J$, such that

$$y_n(t) = y_0 - g(y_n) + \int_0^t v_n(s)\, ds.$$

$(\mathcal{A}_1)$ implies that $v_n(t) \in p(t)\psi(M)\overline{B}(0, 1)$, hence $(v_n)_{n\in\mathbb{N}}$ is integrably bounded. Then there exists a subsequence, still denoted by $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v(\cdot) \in L^1(J, \mathbb{R}^n)$. Moreover, the mapping $\Gamma\colon L^1(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ defined by

$$\Gamma(g)(t) = \int_0^t g(s)\, ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [16]. Therefore, for almost every $t \in J$, the sequence $y_n(t)$ converges to $y(t)$, and by the continuity of g, it follows that

$$y(t) = y_0 - g(y) + \int_0^b v(s)\,ds.$$

It remains to prove that $v \in F(t, y(t))$, for almost every $t \in J$. Lemma 3.3 yields the existence of $\alpha_i^n \geq 0$, $i = n, \ldots, k(n)$, such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$, and the sequence of convex combinations $g_n(\,\cdot\,) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\,\cdot\,)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.2, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{g_n(t)\}} \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\{v_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\bigg\{\bigcup_{k\geq n} F(t, y_k(t))\bigg\} = \overline{\mathrm{co}}\Big(\limsup_{k\to\infty} F(t, y_k(t))\Big). \end{aligned} \tag{3.1}$$

Since F is u.s.c. with compact values, then by Lemma 2.1, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t), \quad \text{for a.e. } t \in J.$$

This, with (3.1), implies that $v(t) \in \overline{\mathrm{co}}\,F(t, y(t))$. Since $F(\,\cdot\,,\,\cdot\,)$ has closed, convex values, we deduce that $v(t) \in F(t, y(t))$, for almost every $t \in J$, as claimed. Hence, $y \in S(y_0)$, which implies $S(y_0)$ is closed and hence compact in $C(J, \mathbb{R}^n)$.

Part 2. The u.s.c. of $S(\,\cdot\,)$.

From Step 2, we have $S(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(C(J, \mathbb{R}^n))$. Set $\Gamma_S := \{(x, y) \mid y \in S(x)\}$ and let $(x_n, y_n) \in \Gamma_S$, i.e. $y_n \in S(x_n)$, and $(x_n, y_n) \to (x, y)$ as $n \to \infty$, where $y_n \in S(x_n)$. Then there exists $v_n \in L^1(J, \mathbb{R}^n)$ such that

$$y_n(t) = x_n - g(y_n) + \int_0^t v_n(s)\,ds, \quad t \in [0, b].$$

Since (x_n, y_n) converge to (x, y) then there exists $M > 0$ such that

$$\|x_n\| \leq M, \quad \text{for all } n \in \mathbb{N}.$$

By using $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, we can easily prove that there exist $\overline{M} > 0$ such that

$$\|y_n\|_\infty \leq \overline{M}, \quad \text{for all } n \in \mathbb{N}.$$

By the definition of y_n we have $y_n'(t) = v_n(t)$, for almost every $t \in [0, b]$. Then

$$\|v_n(t)\| \leq p(t)\psi(M), \quad t \in [0, b].$$

Thus, $v_n(t) \in p(t)\psi(M)\overline{B}(0, 1) := \chi(t)$, for almost every $t \in [0, b]$. It is clear that $\chi\colon [0, b] \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ is a multivalued map that is integrably bounded. Since

$\{v_n(\,\cdot\,) : n \geq 1\} \in \chi(\,\cdot\,)$, we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w([0,b],\mathbb{R}^n)$. From Mazur's lemma, there exists

$$v \in \overline{\operatorname{conv}}\{v_n(t) : n \geq 1\},$$

so there exists a subsequence $\{\overline{v}_n(t) : n \geq 1\}$ in $\overline{\operatorname{conv}}\{v_n(t) : n \geq 1\}$, such that $\overline{v}_n$ converges strongly to $v \in L^1([0,b],\mathbb{R}^n)$. Since $F(t,\,\cdot\,)$ is u.s.c. with convex values, then we can easily prove that (see Step 2)

$$v(t) \in F(t, y(t)), \quad \text{a.e. } t \in [0,b].$$

Let

$$z(t) = x - g(y) + \int_0^t v(s)\,ds, \quad t \in [0,b].$$

Since the function g is continuous, we obtain the estimates

$$||y_n - z\|_\infty \leq \|x_n - x\| + \|g(y_n) - g(y)\|_\infty + \int_0^b \|\overline{v}_n(s) - v(s)\|\,ds.$$

The right-hand side terms tend to 0 as $n \to \infty$. Hence,

$$y(t) = x - g(y) + \int_0^t v(s)\,ds, \quad t \in [0,b].$$

So $y \in S(x)$. Now, we show that $S(\,\cdot\,)$ maps bounded sets into relatively compact sets of $C(J,\mathbb{R}^n)$. Let B be a bounded set in $\mathbb{R}^n$ and $\{y_n\} \subset S_\varphi(B)$. Then there exists $\{x_n\} \subset B$ such that

$$y_n(t) = x_n - g(y_n) + \int_0^t v_n(s)\,ds, \quad t \in [0,b],$$

where $v_n \in S_{F,y_n}$, $n \in \mathbb{N}$. Since $\{x_n\}$ is a bounded sequence, then there exists a subsequence of $\{x_n\}$ converging to x, and from $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, there exists $M_* > 0$ such that

$$\|y_n\|_\infty \leq M_*, \quad n \in \mathbb{N}.$$

From Part 1, $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in $C(J,\mathbb{R}^n)$. As a consequence of the Arzéla–Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in $C(J,\mathbb{R}^n)$. By the above arguments, we can prove that

$$y(t) = x - g(y) + \int_0^t v(s)\,ds, \quad t \in [0,b],$$

where $v \in S_{F,y}$. Then $y \in S(x)$. This implies that $S(\,\cdot\,)$ is u.s.c. □

3.2. The nonconvex case. In this section, we present a second existence result for problem (1.2) when the multi-valued nonlinearity is not necessarily convex. In the proof, we will make use of the nonlinear alternative of Leray–Schauder type [35] combined with a selection theorem due to Bressan and Colombo [15] for lower semicontinuous multi-valued maps with decomposable values. The main ingredients are presented hereafter. We first start with some definitions (see e.g. [4]). Consider a topological space E and a family A of subsets of E.

Definition 3.4. A is called a σ-algebra if it satisfies the following properties:

(a) $\emptyset \in A$.
(b) If $\mathcal{O} \in A$ then $E \setminus \mathcal{O} \in A$.
(c) If $\mathcal{O}_n \in A$, $n = 1, 2, \ldots$ then $\bigcup_{n\geq 1} \mathcal{O}_n \in A$.

Definition 3.5. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$, where I is Lebesgue measurable in J and D is Borel measurable in E.

Definition 3.6. A subset $A \subset L^1(J, E)$ is decomposable if, for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, we have

$$u\chi_I + v\chi_{J\setminus I} \in A,$$

where χ_A stands for the characteristic function of the set A.

Let $F: J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty closed values. Assign to F the multi-valued operator $\mathcal{F}: C(J, E) \to \mathcal{P}(L^1(J, E))$ defined by $\mathcal{F}(y) = S_{F,y}$. The operator $\mathcal{F}$ is called the *Nemyts'kiĭ operator associated to* F.

Definition 3.7. Let $F: J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty compact values. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 3.8 (see [15], [38]). *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multi-valued operator $N: X \to \mathcal{P}_{\rm cl}(L^1(J, E))$ with closed decomposable values has a continuous selection, i.e. there exists a continuous single-valued function $f: X \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis:

($\mathcal{B}_1$) $F: [0, b] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty, compact valued, multi-valued map such that
(a) the mapping $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
(b) the mapping $y \mapsto F(t, y)$ is lower semi-continuous for almost every $t \in [0, b]$.

The following lemma is crucial in the proof of our existence theorem.

Lemma 3.9 (see e.g. [28]). *Let $F: J\times\mathbb{R}^n \to \mathcal{P}_{\rm cp}(\mathbb{R}^n)$ be an integrably bounded multi-valued map satisfying $(\mathcal{B}_1)$. Then F is of lower semi-continuous type.*

The single-valued version of the Nonlinear Alternative of Leary and Schauder may be stated as follows:

Lemma 3.10. *Let X be a Banach space and $C \subset X$ a nonempty bounded, closed, convex subset. Assume U is an open subset of C with $0 \in U$ and let $G: \overline{U} \to C$ be a continuous compact map. Then*

(a) *either there is a point $u \in \partial U$ and $\lambda \in (0,1)$, with $u = \lambda G(u)$,*
(b) *or G has a fixed point in $\overline{U}$.*

We now present our existence result.

Theorem 3.11. *Suppose that the hypotheses $(\mathcal{A}_1)$, $(\mathcal{A}_2)$(a) or $(\mathcal{A}_2)$(b), and $(\mathcal{B}_1)$ are satisfied. Then problem (1.2) has at least one solution.*

Proof. $(\mathcal{A}_1)$ and $(\mathcal{B}_1)$ imply, by Lemma 3.9, that F is of lower semi-continuous type. From Lemma 3.8, there is a continuous selection $f: C(J,\mathbb{R}^n) \to L^1([0,b],\mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$, for all $y \in C(J,\mathbb{R}^n)$. Consider the problem

$$\begin{cases} y'(t) = f(y)(t), & t \in [0,b], \\ y(0) + g(y) = y_0, \end{cases} \tag{3.2}$$

and the operator $G: C(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ defined by

$$G(y)(t) = y_0 - g(y) + \int_0^t f(y)(s)\,ds, \quad t \in [0,b].$$

As in Theorem 3.1, we can prove that the single-valued operator G is compact and there exists $M_* > 0$ such that for all possible solutions y, we have $\|y\|_\infty < M_*$. Now, we only check that G is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,b],\mathbb{R}^n)$ as $n \to \infty$. Then

$$\|G(y_n(t)) - G(y(t))\| \le \|g(y_n) - g(y)\|_\infty + \int_0^t \|f(y_n(s)) - f(y(s))\|\,ds.$$

Since the functions f and g are continuous, we have

$$\|G(y_n) - G(y)\|_\infty \le \|g(y_n) - g(y)\| + \|f(y_n) - f(y)\|_{L^1}$$

which, by continuity of f and g, tends to 0, as $n \to \infty$. Let

$$U = \{y \in C(J,\mathbb{R}^n) : \|y\|_\infty < M_*\}.$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda G y$ for in $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of the Leray–Schauder type (Lemma 3.10), we deduce that G has a fixed point $y \in U$, which is a solution of problem (3.2), and hence a solution to the problem (1.2). $\square$

3.3. A further result. In this part, we present a second existence result to problem (1.2) with a nonconvex valued right-hand side. First, consider the Hausdorff pseudo-metric distance

$$H_d\colon \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}^+ \cup \{\infty\}$$

defined by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{\mathrm{b,cl}}(E), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [42]). In particular, H_d satisfies the triangle inequality.

Definition 3.12. A multi-valued operator $N\colon E \to \mathcal{P}_{\mathrm{cl}}(E)$ is called:

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in E,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that, if N is γ-Lipschitz, then for every $\gamma' > \gamma$,

$$N(x) \subset N(y) + \gamma' d(x,y) B(0,1), \quad \text{for all } x, y \in E.$$

Our proofs are based on the following classical fixed point theorem for contraction multi-valued operators proved by Covitz and Nadler in 1970 (see also K. Deimling, [23, Theorem 11.1]).

Lemma 3.13. *Let (X,d) be a complete metric space. If $G\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} G \neq \emptyset$.

Let us introduce the following hypotheses:

($\mathcal{H}_1$) $F\colon J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R}^n)$; $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}^n$.

($\mathcal{H}_2$) There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t,x), F(t,y)) \le l(t)\|x-y\|, \quad \text{for a.e. } t \in J \text{ and all } x, y \in \mathbb{R}^n,$$

with $H_d(0, F(t,0)) \le l(t)$ for almost every $t \in J$.

($\mathcal{H}_3$) There exist $c \in [0,1)$ such that

$$\|g(y) - g(x)\| \le c\|y-x\|_\infty, \quad \text{for every } x, y \in C(J, \mathbb{R}^n).$$

Theorem 3.14. *Let Assumptions ($\mathcal{H}_1$)–($\mathcal{H}_3$) be satisfied. Then problem* (1.2) *has at least one solution.*

Proof. In order to transform the problem (1.2) into a fixed point problem, let the multi-valued operator $N\colon C(J,\mathbb{R}^n) \to \mathcal{P}(C(J,\mathbb{R}^n))$ be as defined in Theorem 3.1. We shall show that N satisfies the assumptions of Lemma 3.13.

(a) $N(y) \in \mathcal{P}_{\rm cl}(C(J,\mathbb{R}^n))$ for each $y \in C(J,\mathbb{R}^n)$.

Indeed, let $y_n \in N(y)$ converge to y in $C(J,\mathbb{R}^n)$. Then there exists $v_n \in S_{F,y}$ such that

$$y_n(t) = y_0 - g(y) + \int_0^t v_n(s)\,ds, \quad t \in [0,b].$$

Using the fact that F has compact values and from $(\mathcal{H}_2)$, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J,\mathbb{R}^n)$ and hence $v \in S_{F,y}$.

(b) There exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_\infty, \quad \text{for all } y, \overline{y} \in C(J,\mathbb{R}^n).$$

Let $y, \overline{y} \in C(J,\mathbb{R}^n)$ and $h \in N(y)$. Then there exists $v(t) \in F(t, y(t))$ (v is a measurable selection) such that, for each $t \in J$,

$$h(t) = y_0 - g(y) + \int_0^t v(s)\,ds.$$

$(\mathcal{H}_2)$ tells us that $H_d(F(t,y(t)), F(t,\overline{y}(t))) \le l(t)\|y(t) - \overline{y}(t)\|$ for almost every $t \in J$. Hence, there is $w \in F(t, \overline{y}(t))$ such that

$$\|g(t) - w\| \le l(t)\|y(t) - \overline{y}(t)\|, \quad t \in J.$$

Consider the mapping $U\colon J \to P(\mathbb{R}^n)$ given by

$$U(t) = \{w \in \mathbb{R}^n : \|g(t) - w\| \le l(t)\|y(t) - \overline{y}(t)\|\}, \quad t \in J,$$

that is, $U(t) = \overline{\mathcal{B}}(g(t), l(t)\|y(t) - \overline{y}(t)\|)$. Since g, l, y, $\overline{y}$ are measurable, Theorem III.4.1 in [21] tells us that the closed ball U is measurable. In addition, $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ imply that, for each $y \in C(J,\mathbb{R}^n)$, $F(t, y(t))$ is measurable. Finally, the set $V(t) = U(t) \cap F(t, \overline{y}(t))$ is nonempty since it contains w. Therefore, the intersection multi-valued operator V is measurable with nonempty, closed values (see [4], [21], [33]). By Lemma 2.5, there exists a function $\overline{v}(t)$, which is a measurable selection for V. Thus, $\overline{v}(t) \in F(t, \overline{y}(t))$ and

$$\|v(t) - \overline{v}(t)\| \le l(t)\|y(t) - \overline{y}(t)\|, \quad \text{for a.e. } t \in J.$$

Let us define, for almost every $t \in J$.

$$\overline{h}(t) = y_0 - g(\overline{y}) + \int_0^t \overline{v}(s)\,ds.$$

Then

$$\begin{aligned} \|h(t) - \overline{h}(t)\| &\le c\|y - \overline{y}\|_\infty + \int_0^t l(s)\|y(s) - \overline{y}(s)\|\,ds \\ &\le c\|y - \overline{y}\|_\infty + \int_0^t l(s)e^{\tau L(s)}e^{-\tau L(s)}\|y(s) - \overline{y}(s)\|\,ds \\ &\le c\|y - \overline{y}\|_\infty + \frac{1}{\tau}e^{\tau L(t)}\|y - \overline{y}\|_{B_*}, \end{aligned}$$

and so

$$\|h-\overline{h}\|_{B_*} \leq \left(c+\frac{1}{\tau}\right)\|y-\overline{y}\|_{B_*}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at

$$H_d(N(y),N(\overline{y})) \leq \left(c+\frac{1}{\tau}\right)\|y-\overline{y}\|_{B_*},$$

where

$$\|y\|_{B_*} = \sup\{e^{-\tau L(t)}\|y(t)\| : t \in [0,b]\} \quad \text{and} \quad L(t) = \int_0^t l(s)\,ds.$$

Since $c<1$, there exists $\tau>0$ such that $c+1/\tau<1$. So, N is a contraction and thus, by Lemma 3.13, N has a fixed point y, which is a solution to (1.2). □

Arguing as in Theorem 3.1, we can also establish the following result, the proof of which is omitted.

Theorem 3.15. *Suppose that all conditions of Theorem* 3.14 *are satisfied and* $F\colon J\times\mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$. *Then the solution set of problem* (1.2) *is nonempty and compact.*

4. Geometric structure of solution sets

4.1. Background in geometric topology. We start with some elementary notions and notations from geometric topology. For details, we recommend [13], [30], [31], [33], [35], [37], [44]. In what follows (X,d) and (Y,d') denote two metric spaces.

Definition 4.1. A set $A \in \mathcal{P}(X)$ is called a contractible space provided there exists a continuous homotopy $h\colon A\times[0,1]\to A$ and $x_0\in A$ such that

(a) $h(x,0)=x$, for every $x\in A$,
(b) $h(x,1)=x_0$, for every $x\in A$,

i.e. if the identity map $A\to A$ is homotopic to a constant map (A is homotopically equivalent to a point).

Note that if $A\in\mathcal{P}_{\mathrm{cv,cl}}(X)$, then A is contractible. Also, the class of contractible sets is much larger than the class of closed convex sets.

Definition 4.2. A compact nonempty space X is called an R_δ-set provided there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}$ such that $X=\bigcap_{n=1}^\infty X_n$.

Definition 4.3. A space X is called an absolute retract (in short $X\in\mathrm{AR}$) provided that, for every space Y, every closed subset $B\subseteq Y$ and any continuous map $f\colon B\to X$, there exists a continuous extension $\widetilde{f}\colon Y\to X$ of f over Y, i.e.

$\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f: X \to Y$, the set $f(X)$ is a retract of Y.

From [1, Proposition 2.15], if $X \in \mathrm{AR}$, then it is a contractible space.

Definition 4.4. A space A is closed acyclic if

(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$,

where $H_* = \{H_n\}_{n \geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j: \{p\} \to X$, $j(p) = x_0 \in A$, induces an isomorphism $j_*: H_*(\{p\}) \to H_*(A)$.

Definition 4.5. An u.s.c. map $F: X \to \mathcal{P}(Y)$ is called acyclic if for each $x \in X$, the image set $F(x)$ is compact and acyclic.

From the continuity of Čech-homology functors, we have the following lemma.

Lemma 4.6 ([30]). *Let X be a compact metric space. Then X is an acyclic space if its structure corresponds to one of the following types:*

(a) *X is convex,*
(b) *X is contractible,*
(c) *X is* AR*,*
(d) *X is an R_δ set.*

The next definitions were introduced in [31].

Definition 4.7. A metric space X is called acyclically contractible if there exists an acyclic homotopy $\Pi: X \times [0,1] \to \mathcal{P}(X)$ such that:

(a) $x_0 \in \Pi(x,1)$, for every $x \in X$ and for some $x_0 \in X$.
(b) $x \in \Pi(x,0)$, for every $x \in X$.

Notice that any contractible space and any acyclic, compact metric space are acyclically contractible (see [1, Theorem 19]). Also from [33], any acyclically contractible space is acyclic.

Definition 4.8. A metric space X is called R_δ-contractible if there exists a multivalued homotopy $\Pi: X \times [0,1] \to \mathcal{P}(X)$ which is u.s.c. and satisfies:

(a) $x \in \Pi(x,1)$, for every $x \in X$,
(b) $\Pi(x,0) = B$ for every $x \in X$ and for some $B \subset X_n$,
(c) $\Pi(x,\alpha)$ is an R_δ-set, for every $\alpha \in [0,1]$ and $x \in X$.

Next, we present a result about the topological structure of the set of solutions of some nonlinear functional equations due to N. Aronszajn and developed by F. Browder and C. P. Gupta in [17] (see also [1, Theorem 1.2]).

Theorem 4.9. *Let X be a space, $(E,\|\cdot\|)$ a Banach space and $f\colon X\to E$ a proper map, i.e. f is continuous and for every compact $K\subset E$, the set $f^{-1}(K)$ is compact. Assume further that, for each $\varepsilon>0$, a proper map $f_\varepsilon\colon X\to E$ is given, and the following two conditions are satisfied:*

(a) $\|f_\varepsilon(x)-f(x)\|<\varepsilon$, *for every $x\in X$,*
(b) *for every $\varepsilon>0$ and $u\in E$ in a neighbourhood of the origin such that $\|u\|\le\varepsilon$, the equation $f_\varepsilon(x)=u$ has exactly one solution x_ε.*

Then the set $S=f^{-1}(0)$ is R_δ-set.

The following Lasota–Yorke Approximation theorem (see [33]) will be needed in this section.

Lemma 4.10. *Let E be a normed space, X a metric space and $f\colon X\to E$ be a continuous map. Then, for each $\varepsilon>0$, there is a locally Lipschitz map $f_\varepsilon\colon X\to E$ such that*

$$\|f(x)-f_\varepsilon(x)\|<\varepsilon,\quad \textit{for every } x\in X.$$

4.2. Application. Consider the first-order impulsive single-valued problem,

$$\begin{cases} y'(t)=f(t,y(t)) & \text{for a.e. } t\in J=[t_0,b],\\ y(t_0)+g(y)=y_0, \end{cases}\tag{4.1}$$

where $f\colon J\times\mathbb{R}^n\to\mathbb{R}^n$ and $g\colon C(J,\mathbb{R}^n)\to\mathbb{R}^n$ are a given functions, and $y_0\in\mathbb{R}^n$.

Denote by $S(f,y_0)$ the set of all solutions of problem (4.1). We are in a position to state and prove an Aronsajn-type result for this problem. First, we list two assumptions:

($\mathcal{C}_1$) $f\colon J\times\mathbb{R}^n\to\mathbb{R}^n$ is an Carathéodory function.
($\mathcal{C}_2$) There exist a function $p\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi\colon[t_0,\infty)\to[0,\infty)$ such that

$$\|f(t,x)\|\le p(t)\psi(\|x\|)\quad \text{for a.e. } t\in J \text{ and each } x\in\mathbb{R}^n,$$

with

$$\int_{t_0}^b p(s)\,ds<\int_{\|y_0\|}^\infty\frac{du}{\psi(u)}.$$

Then, our first result in this section is the following.

Theorem 4.11. *Assume that ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{H}_3$) hold. Then the set $S(f,y_0)$ is R_δ and hence an acyclic space.*

Proof. Let $F\colon C(J,\mathbb{R}^n)\to C(J,\mathbb{R}^n)$ be defined by

$$F(y)(t)=y_0-g(y)+\int_{t_0}^t f(s,y(s))\,ds,\quad t\in[t_0,b].$$

Thus, $\operatorname{Fix} F = S(f, y_0)$. From Theorem 3.11, we know that $S(f, y_0) \neq \emptyset$, and there exists $\overline{M} > 0$ such that

$$\|y\|_\infty \leq \overline{M}, \quad \text{for every } y \in S(f, y_0).$$

Define

$$\widetilde{f}(t, y(t)) = \begin{cases} f(t, y(t)) & \text{if } \|y(t)\| \leq \overline{M}, \\ f\left(t, \dfrac{\overline{M} y(t)}{\|y(t)\|}\right) & \text{if } \|y(t)\| \geq \overline{M}. \end{cases}$$

Since f is L^1-Carathéodory, the function $\widetilde{f}$ is Carathéodory and is integrably bounded by $(\mathcal{C}_2)$. So there exists $h \in L^1(J, \mathbb{R}^+)$ such that

$$\|\widetilde{f}(t, x)\| \leq h(t), \quad \text{for a.e. } t \text{ and all } x \in \mathbb{R}^n. \tag{4.2}$$

Consider the modified problem

$$\begin{cases} y'(t) = \widetilde{f}(t, y(t)) & \text{for a.e. } t \in J, \\ y(0) - g(y) = y_0. \end{cases}$$

We can easily prove that $S(f, y_0) = S(\widetilde{f}, y_0) = \operatorname{Fix} \widetilde{F}$, where $\widetilde{F}: C(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ is as defined by

$$\widetilde{F}(y)(t) = y_0 - g(y) + \int_{t_0}^t \widetilde{f}(s, y(s))\, ds, \quad t \in [t_0, b].$$

By the inequality (4.2) and the continuity of g, we deduce that

$$\|\widetilde{F}(y)\|_\infty \leq \|y_0\| + \overline{M}\gamma + \beta + \|h\|_{L^1} := R.$$

Then $\widetilde{F}$ is uniformly bounded. As in Theorem 3.11, we can prove that

$$\widetilde{F}: C(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$$

is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \widetilde{F}(y)$ which is a proper map. From the compactness of $\widetilde{F}$ and the Lasota–Yorke approximation theorem, we can easily prove that all conditions of Theorem 4.9 are met. Therefore, the solution set $S(\widetilde{f}, y_0) = \widetilde{G}^{-1}(0)$ is an R_δ set and hence an acyclic space by Lemma 4.6. □

4.3. σ-selectionable multivalued maps. The following definitions and theorem can be found in [33], [36] (see also [3, p. 86]). Let (X, d) and (Y, d') be two metric spaces.

Definition 4.12. We say that a map $F: X \to \mathcal{P}(Y)$ is σ-Ca-selectionable if there exists a decreasing sequence of compact valued u.s.c. maps $F_n: X \to Y$ satisfying:

(a) F_n has a Carathédory selection, for all $n \geq 0$ (F_n are called Ca-selectionable),
(b) $F(x) = \bigcap_{n \geq 0} F_n(x)$, for all $x \in X$.

Definition 4.13.

(a) A single-valued map $f\colon [0,a]\times X\to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\,\cdot\,,x)$ is measurable for every $x\in X$, and for every $x\in X$, there exists a neighbourhood V_x of $x\in X$ and an integrabe function $L_x\colon [0,a]\to[0,\infty)$ such that

$$d'(f(t,x_1),f(t,x_2))\le L_x(t)d(x_1,x_2)\quad \text{for every } t\in[0,a] \text{ and } x_1,x_2\in V_x.$$

(b) A multi-valued mapping $F\colon [0,a]\times X\to\mathcal{P}(Y)$ is mLL-selectionable if it has an mLL-selection.

Definition 4.14. We say that a multivalued map $\phi\colon [0,a]\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ with closed values is upper-Scorza–Dragoni if, given $\delta>0$, there exists a closed subset $A_\delta\subset[0,1]$ such that the measure $\mu([0,a]\setminus A_\delta)\le\delta$ and the restriction ϕ_δ of ϕ to $A_\delta\times\mathbb{R}^n$ is u.s.c.

Theorem 4.15 (see [33, Theorem 19.19]). *Let E and E_1 be two separable Banach spaces and let $F\colon [a,b]\times E\to\mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza–Dragoni map. Then F is σ-Ca-selectionable, the maps $F_n\colon [a,b]\times E\to\mathcal{P}(E_1)$ $(n\in\mathbb{N})$ are almost upper semicontinuous, and we have*

$$F_n(t,e)\subset\overline{\operatorname{conv}}\bigg(\bigcup_{x\in E}F_n(t,x)\bigg).$$

Moreover, if F is integrably bounded, then F is σ-mLL-selectionable.

Let $S(y_0)$ denote the set of all solutions of problem (1.2). Now, we are in position to state and prove another characterization of the geometric structure of $S(y_0)$.

Theorem 4.16. *Let $F\colon J\times\mathbb{R}^n\to\mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be a Carathéodory and an mLL-selectionable multi-valued map which satisfies conditions $(\mathcal{A}_1)$ and $(\mathcal{H}_3)$. Then, for every $y_0\in\mathbb{R}^n$, the set $S(y_0)$ is contractible.*

Proof. Let $f\subset F$ be a measurable, locally Lipschitz selection and consider the single-valued problem

$$\begin{cases} y'(t)=f(t,y(t)) & \text{for a.e. } t\in J,\\ y(0)+g(y)=y_0.\end{cases}\tag{4.3}$$

Since f and g are Lipschitz function, by the Banach fixed point theorem, we can prove that problem (4.3) has exactly one solution for every $y_0\in\mathbb{R}^n$. Define the homotopy $h\colon S(y_0)\times[0,1]\to S(y_0)$ by

$$h(y,\alpha)(t)=\begin{cases} y(t) & \text{for } t\in[0,\alpha b],\\ \overline{x}(t) & \text{for } \alpha b<t\le b,\end{cases}$$

where $\overline{x} = S(f, y_0)$ is the unique solution of problem (4.3). In particular,

$$h(y,\alpha) = \begin{cases} y & \text{for } \alpha = 1, \\ \overline{x} & \text{for } \alpha = 0. \end{cases}$$

We will prove that h is a continuous homotopy. Let $(y_n, \alpha_n) \in S(y_0) \times [0,1]$ be such that $(y_n, \alpha_n) \to (y, \alpha)$, as $n \to \infty$. We need to show that $h(y_n, \alpha_n) \to h(y, \alpha)$. We have

$$h(y_n, \alpha_n)(t) = \begin{cases} y_n(t) & \text{for } t \in [0, \alpha_n b], \\ \overline{x}(t) & \text{for } t \in (\alpha_n b, b]. \end{cases}$$

(a) If $\lim_{n\to\infty} \alpha_n = 0$, then

$$h(y,0)(t) = \begin{cases} y_0 - g(y) & \text{for } t = 0, \\ \overline{x}(t) & \text{for } t \in (0, b]. \end{cases}$$

From $(\mathcal{A}_1)$ and $(\mathcal{H}_3)$, there exists $M > 0$ such that

$$\|y_n\|_\infty \le M, \quad \text{for each } n \in \mathbb{N}.$$

Hence,

$$\|h(y_n, \alpha_n) - h(y, \alpha)\|_\infty \le \|g(y_n) - g(y)\|_\infty + \psi(M) \int_0^{\alpha_n b} p(s)\, ds,$$

which tends to 0 as $n \to \infty$. The case when $\lim_{n\to\infty} \alpha_n = 1$ is treated similarly.

(b) If $\alpha_n \neq 0$ and $0 < \lim_{n\to\infty} \alpha_n = \alpha < 1$, then we may distinguish between two sub-cases:

(i) $y_n \in S(y_0)$ implies the existence of $v_n \in S_{F,y_n}$ such that for $t \in [0, \alpha_n b]$

$$y_n(t) = y_0 - g(y_n) + \int_0^t v_n(s)\, ds.$$

Since $F(t, \cdot\,)$ is u.s.c. for every $\varepsilon > 0$, there exists $n_0 \ge 0$ such that for any $n \ge n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \quad \text{for a.e. } t \in [0, \alpha b].$$

In addition, $F(\cdot\,, \cdot\,)$ has compact convex values, so there exists a subsequence $v_{n_m}(\cdot\,) \in \overline{\operatorname{conv}}\{v_n\}$ such that $v_{n_m}(\cdot\,)$ converges to a limit $v(\cdot\,)$ satisfying

$$v(t) \in F(t, y(t)) + \varepsilon \overline{B}(0,1), \quad \text{for all } \varepsilon > 0.$$

Therefore,

$$v(t) \in F(t, y(t)), \quad \text{for a.e. } t \in [0, \alpha b].$$

Now $\{y_n\}$ converges to y so $\|y_n\|_\infty \le R$ for some $R > 0$. Then Assumption $(\mathcal{A}_1)$ implies that

$$\|v_{n_m}(t)\| \le p(t)\psi(M), \quad \text{for a.e. } \ t \in [0, b].$$

By the Lebesgue dominated convergence theorem, it follows that

$$v \in L^1([0,b],\mathbb{R}^n) \Rightarrow v \in S_{F,y}.$$

Using the continuity of g, we find that for $t \in [0,b]$,

$$y(t) = y_0 - g(y) + \int_0^t v(s)\,ds.$$

(ii) If $t \in (\alpha_n b, b]$, then

$$h(y_n,\alpha_n)(t) = h(y,\alpha)(t) = \overline{x}(t).$$

Thus,

$$\|h(y_n,\alpha_n) - h(y,\alpha)\|_\infty \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, h is a continuous function, proving that $S(y_0)$ is contractible to the point $\overline{x} = S(f,y_0)$. □

A further precise result is given in the following theorem.

THEOREM 4.17. *Let $F\colon J\times\mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ be a Carathéodory and a Ca-selectionable multi-valued map. Assume the additional conditions in Theorem* 4.16 *are satisfied. Then the solution set $S(y_0)$ is R_δ-contractible and acyclic.*

PROOF. Replace the singlevalued homotopy $h\colon S(y_0)\times[0,1] \to S(y_0)$ in Theorem 4.16 by the multivalued homotopy $\Pi\colon S(y_0)\times[0,1] \to \mathcal{P}(S(y_0))$ defined by

$$\Pi(x,\alpha) = \{y \in S(f,\alpha b,x)\},$$

where $f \subset F$, and $S(f,\alpha b,x)$ is the solution set of the problem,

$$\begin{cases} y' = f(t,y(t)) & \text{for a.e. } t \in (\alpha b, b], \\ y(0) - g(y) = x(0) & \text{for } t \in [0,\alpha b]. \end{cases}$$

From the definition of Π, $\Pi(x,0) = S(f,0,x)$ and $x \in \Pi(x,1)$ for every $x \in S(y_0)$. It remains to prove that $\Pi(\,\cdot\,,\,\cdot\,)$ is u.s.c.; this can be done as in [31], [25].

Next, we claim that $\Pi(x,\alpha)$ is an R_δ set for each fixed $\alpha \in [0,1]$ and $x \in S(x)$. Clearly $\Pi(x,\alpha) = S(x)$. Since F is σ-Ca-selectionable, there exists a decreasing sequence of multivalued maps $F_k\colon [0,b]\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ $(k \in \mathbb{N})$ that have Carathéodory selections and satisfy

$$F_{k+1}(t,u) \subset F_k(t,u), \quad \text{for almost all } t \in [0,b],\ u \in \mathbb{R}^n$$

and

$$F(t,u) = \bigcap_{k=0}^{\infty} F_k(t,u), \quad u \in \mathbb{R}^n.$$

Then

$$\Pi(y_0,\alpha) = \bigcap_{k=0}^{\infty} S(F_k,y_0),$$

where $S(F_k, x)$ is solution of the problem

$$\begin{cases} y'(t) \in F_k(t, y(t)) & \text{for a.e. } t \in J, \\ y(0) + g(y) = y_0. \end{cases}$$

Theorem 3.1 implies that $\Pi(x, \alpha)$ and $S(F_k, y_0)$ are compact sets. Moreover, from Theorem 4.16, the sets $S(F_k, y_0)$ are contractible sets. Therefore, $\Pi(x, \alpha)$ is an R_δ set.

As a consequence, all properties in Definition 4.8 are satisfied, so the set $S(y_0)$ is R_δ-contractible. This completes the proof of the theorem. □

We next derive some additional results regarding the topological structure of the solution sets.

Theorem 4.18. *Let $F: J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ be a Carathéodory and a σ-Ca-selectionable multi-valued map. Assume that all additional conditions of Theorem* 4.16 *are satisfied. Then the solution set $S(y_0)$ is an R_δ set.*

Proof. Since F is σ-Ca-selectionable, there exists a decreasing sequence of multivalued maps $F_k: [0, b] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ $(k \in \mathbb{N})$ that have Carathéodory selections and satisfy

$$F_{k+1}(t, u) \subset F_k(t, u), \quad \text{for almost all } t \in [0, b],\ u \in \mathbb{R}^n,$$

and

$$F(t, x) = \bigcap_{k=0}^{\infty} F_k(t, x), \quad x \in \mathbb{R}^n.$$

Then,

$$S(y_0) = \bigcap_{k=0}^{\infty} S(F_k, y_0).$$

From Theorem 4.17, the set $S(F_k, y_0)$ is contractible for each $k \in \mathbb{N}$. Hence $S(y_0)$ is an R_δ set. □

Theorem 4.19. *Let $F: J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ be a Carathéodory and a σ-mLL-selectionable map. Assume that all conditions of Theorem* 4.16 *are fulfilled. Then the solution set $S(y_0)$ is an R_δ set.*

Proof. It is enough to prove that F is a σ-mLL-selectionable and then apply Theorem 4.16. □

Theorem 4.20. *Let $F: J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ be upper-Scorza–Dragoni. Assume that all conditions of Theorem* 4.16 *are satisfied. Then the solution set $S(y_0)$ is an R_δ.*

Proof. Since F is upper-Scorza–Dragoni, then from Theorem 4.15, F is a σ-Ca-selection map. Therefore $S(y_0)$ is an R_δ-set. □

5. Existence results in a closed set

Let K be a closed subset of $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, we define

$$T_K(x) = \left\{ y \in \mathbb{R}^n : \lim_{t \to 0^+} \inf \frac{1}{t} d(x + ty, K) = 0 \right\},$$

which is called the *Bouligand tangent cone* to K at x. A nonempty closed subset $K \subset \mathbb{R}^n$ is called a *proximate retract* provided there exists an open neighbourhood U of K in $\mathbb{R}^n$ and a retraction $r: U \to \mathbb{R}^n$ such the following two conditions satisfied:

(i) $r(x) = x$ for all $x \in K$,
(ii) $\|x - r(x)\| = d(x, K)$, for every $x \in U$.

It is well known that the class of all proximate retracts is quite rich; in particular, it contains convex sets and C^2-manifolds. It is easy to see that, for given K, if $r: U \to K$ exists, then it is unique. Since one can take a sufficiently small U, for example by restricting U to $U \cap \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) < \delta\}$, $\delta > 0$, we may assume that $\|r(x) - x\| \leq \delta$, for a given $\delta > 0$ and $x \in U$. The following lemmas play an important role in our considerations.

Lemma 5.1 ([34]). *Let K be a proximate retract. Then*

$$T_K(r(x)) \subseteq \{y \in \mathbb{R}^n : \langle y, x - r(x) \rangle \leq 0\}, \quad \textit{for any } x \in U,$$

where $\langle \cdot\,, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Proof. If $x \in K$, then $r(x) = x$. Hence, for every $y \in T_K(r(x))$, we have

$$\langle y, x - r(x) \rangle = 0.$$

This implies that $T_K(r(x)) \subseteq \{y \in \mathbb{R}^n : \langle y, x - r(x) \rangle \leq 0\}$. Hence, we assume that $x \in U \setminus K$ and $y \in \mathbb{R}^n$ are such that $\langle y, x - r(x) \rangle > 0$; then

$$\lim_{t \to 0^+} \frac{d(r(x) + ty, \mathbb{R}^n \setminus B(x, \|x - r(x)\|))}{t} > 0.$$

In fact,

$$\begin{aligned}
&\lim_{t \to 0^+} \frac{d(r(x) + ty, \partial B(x, \|x - r(x)\|))}{t} \\
&\qquad \geq \lim_{t \to 0^+} \frac{\|x - r(x)\| - \|x + ty - r(x)\|}{t} \\
&\qquad = \lim_{t \to 0^+} \frac{(\|x - r(x)\| - \|x - ty - r(x)\|)\|x - r(x) - ty\|}{t\|x - r(x) - ty\|} \\
&\qquad \geq \lim_{t \to 0^+} \frac{-\langle -ty, x - r(x) - ty \rangle}{t\|x - r(x) - ty\|} = \lim_{t \to 0^+} \frac{-t^2\|y\| + t\langle y, x - r(x) \rangle}{t\|x - r(x) + ty\|} \\
&\qquad = \lim_{t \to 0^+} \frac{\langle y, x - r(x) \rangle}{\|x - r(x) + ty\|} = \lim_{t \to 0^+} \frac{\langle y, x - r(x) \rangle}{\|x - r(x)\|} > 0.
\end{aligned}$$

On other hand, $K \subset \mathbb{R}^n \setminus B(x, \|x - r(x)\|)$, and so

$$d(r(x) + ty, K) \geq d(r(x) + ty, \quad \mathbb{R}^n \setminus B(x, \|x - r(x)\|).$$

In conclusion, $y \notin T_K(r(x))$. □

Lemma 5.2 ([34]). *Let K be a proximate retract, $r: U \to K$ a metric retraction and $s > 0$ be such that $K \cap \overline{B}(0, s) \neq \emptyset$, where $\overline{B}(0, s)$ is the closure of $B(0, s)$ in $\mathbb{R}^n$. Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0$, there exist subsets $K \subset K_\varepsilon \subset U$ of $\mathbb{R}^n$, K_ε closed and U_ε open, and a continuous retraction $r_\varepsilon: U_\varepsilon \to K_\varepsilon$ such that the following conditions are satisfied:*

(a) $\bigcap_{0<\varepsilon\leq\varepsilon_0} K_\varepsilon = K$,
(b) $\|r_\varepsilon(x) - x\| = \operatorname{dist}(x, K_\varepsilon)$, *for all* $x \in U_\varepsilon \cap \overline{B(O, s)}$,
(c) $T_{K_\varepsilon}(x) \subseteq \{y \in \mathbb{R}^n : \langle y, y - r(x)\rangle \leq 0\}$, *for any* $x \in K_\varepsilon \cap \overline{B(0, s)}$.

Lemma 5.3 ([52], [53]). *Let K be a proximate retract, let U be an open neighbouirhood of K in $\mathbb{R}^n$, and let $r: U \to K$ be a metric retraction. Assume further that $\varepsilon > 0$ is chosen in such a way that $\overline{O}_{2\varepsilon}(K) \subset U$. Then we have:*

($\overline{\mathcal{A}}_1$) $\overline{O}_{2\varepsilon}(K)$ *is a approximate retract;*
($\overline{\mathcal{A}}_2$) $\{y \in \mathbb{R}^n : \langle y, x - r(x)\rangle \leq 0\} \subset T_{\overline{O}_\varepsilon(K)}(x)$, *for all* $x \in \overline{O}_\varepsilon(K)$;
($\overline{\mathcal{A}}_3$) $T_K(r(x)) \subset \{y \in \mathbb{R}^n : \langle y, x - r(x)\rangle \leq 0\}$, *for all* $x \in \overline{O}_\varepsilon(K)$,

where

$$O_{2\varepsilon}(K) = \{x \in \mathbb{R}^n : d(x, a) < \varepsilon \text{ for some } a \in K\}.$$

5.1. Viable solutions on proximate retracts. In this subsection we shall discuss the existence of viable solutions of some classes of differential inclusions with with nonlocal conditions. Our approach here is based on [27], [29], [32], [34]; we give natural generalizations of some of the results contained, therein, for

$$\begin{cases} y'(t) \in F(t, y(t)) & \text{for a.e. } t \in [0, b], \\ y(0) - g(y) = y_0, \end{cases} \tag{5.1}$$

where $F: [0, b] \times K \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map and $g: C([0, b], K) \to y_0 + K$ is a given function.

Definition 5.4. A function $y \in \mathrm{AC}([0, b], \mathbb{R}^n)$ is called a solution of (5.1) (or a viable solution), if there exists $v \in L^1([0, b], K)$ such that $y'(t) = v(t)$ for almost every $t \in [0, b]$, and $y'(t) \in F(t, y(t))$ for almost all (a.a.) $t \in [0, b]$ such that $y(t) \in K$ for each $t \in [0, b]$ and $y(0) - g(y) = y_0$.

To solve problem (5.1), we consider the following auxiliary problem:

$$\begin{cases} y'(t) \in \widetilde{F}(t, y(t)) & \text{for a.e. } t \in [0, b], \\ y(0) - g(y) = y_0, \end{cases} \tag{5.2}$$

where $\widetilde{F}\colon [0,b]\times\mathbb{R}^n\to\mathbb{R}^n$ is defined by

$$\widetilde{F}(t,y)=\begin{cases}\alpha(y)F(t,r(y)) & \text{if } y\in U \text{ and } t\in[0,b],\\ 0 & \text{if } y\notin U \text{ and } t\in[0,b],\end{cases}$$

where $r\colon U\to K$ is the metric retraction and $\alpha\colon\mathbb{R}^n\to[0,1]$ is a continuous Uryshon function such that $\alpha|_K\equiv 1$ and $\alpha|_{\mathbb{R}^n\setminus K}\equiv 0$. Obviously, $\widetilde{F}$ is unique up to the choice of the Uryshon function α.

Proposition 5.5. *If $F\colon[0,b]\times K\to\mathcal{P}(\mathbb{R}^n)$ is a Carathédory multifunction in K, then $\widetilde{F}\colon[0,b]\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a Carathédory function on $\mathbb{R}^n$.*

Definition 5.6. A map $F\colon[0,b]\times K\to\mathcal{P}(\mathbb{R}^n)$ tangent to K, is called weakly tangent (tangent) to K, if $F(t,y)\cap T_K(y)\neq\emptyset$, $(F(t,y)\subset T_K(y))$, for $y\in K$ and almost all $t\in[0,b]$.

Lemma 5.7. *Let $F\colon[0,b]\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ tangent to K, be a proximate retract in $\mathbb{R}^n$. If $y\in \mathrm{AC}([0,b],\mathbb{R}^n)$ is a solution of the problem* (5.2) *and $y_0+g(y)\in K$, then $y(t)\in K$, for each $t\in[0,b]$.*

Proof. Let $d\colon[0,b]\to\mathbb{R}_+$ be defined by $d(t)=d(y(t),K)$, $t\in[0,b]$. We show that $d(t)=0$ for all $t\in[0,b]$. Since $y(0)=g(y)+y_0$, we have $d(0)=0$, and from the definition of d, we see that

$$|d(t+h)-d(t)|\leq\|y(t+h)-y(t)\|,\quad t\in[0,b],$$

and so d is an absolutely continuous function. Let $t_0\in[0,b]$ be such that $y'(t_0)\in F(t_0,y(t_0))$. If $y(t_0)\in U$, then $y'(t_0)\in T_K(r(y(t_0)))$, and

$$\lim_{t\to 0^+}\inf\frac{d(r(y(t_0))+hy'(t_0),K)}{h}=0. \tag{5.3}$$

We have

$$d(y(t_0+h),K)-d(y(t_0),K)\leq\|y(t_0+h)-y(t_0)-hy'(t_0)\|+d(y(t_0)+hy'(t_0),K),$$

and from (5.3), we obtain

$$\lim_{h\to0^+}\inf\frac{d(t_0+h)-d(t_0)}{h}\leq 0.$$

If $y(t_0)\notin U$, then $y'(t_0)=0$, and

$$d(t_0+h)-d(t_0)\leq\|y(t_0+h)-y(t_0)-hy'(t_0)\|\Rightarrow\lim_{h\to0^+}\inf\frac{d(t_0+h)-d(t_0)}{h}\leq 0.$$

Since d is differentiable almost everywhere and its derivative $d'(t)\leq 0$ for almost every $t\in[0,b]$, it is nonincreasing. But $d(0)=0$ and hence $d(t)=0$ for every $t\in[0,b]$. □

Now, we present the first result of this section.

Theorem 5.8. *Let* $F\colon [0,b]\times K \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ *be a Carathéodory multifunction, tangent to* K*, an approximate retract to* $\mathbb{R}^n$*, and let* g *satisfy* $(\mathcal{H}_3)$*. Assume further the following conditions hold:*

$(\mathcal{R}_1)$ *there exist* $p \in L^1([0,b],\mathbb{R}_+)$ *and* $\psi\colon [0,\infty)\to(0,\infty)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(\|u\|) \quad \text{for all } u \in \mathbb{R}^n,$$

$(\mathcal{R}_2)$ *the function* $g\colon C([0,b],\mathbb{R}^n)\to\mathbb{R}^n$ *is continuous and* $L(C([0,b],\mathbb{R}^n))\subset K$*, where* $L(y)=g(y)+y_0$.

Then the problem (5.1) *has at least one viable solution, the solution set is compact and is an* R_δ*-set.*

Proof. We consider the modified problem

$$\begin{cases} y'(t)\in \widetilde{F}(t,y(t)) & \text{for a.e. } t\in[0,b],\\ y(0)-g(y)=y_0, \end{cases} \tag{5.4}$$

where $\widetilde{F}$ is defined in problem (5.2). From Theorem 3.1, the problem (5.4) has at least one solution $y(t)$ and

$$S(\widetilde{F})=\{y \mid y \text{ is a solution of the problem (5.2)}\}$$

is compact. By Lemma 5.7, we have $y(t)\in K$ for all $t\in[0,b]$. Hence,

$$\widetilde{F}(t,y(t))=F(t,y(t)),\quad t\in[0,b].$$

This implies that $y(t)$ is a solution of problem (5.1) and

$$S(F,K)=\{y\mid y \text{ is a solution of the problem (5.1)}\}=S(\widetilde{F}).$$

Now, we show that $S(\widetilde{F})$ is an R_δ-set. From Theorem 3.1, there exist $M>0$ such that for every y solution of the problem (5.2) we have $\|y\|_\infty < M$. We set

$$\widetilde{F}_M(t,y)=\begin{cases}\widetilde{F}(t,y) & \text{if } \|y\|\le M \text{ and } t\in[0,b],\\ \widetilde{F}\Big(t,\dfrac{My}{\|y\|}\Big) & \text{if } \|y\|\ge M \text{ and } t\in[0,b].\end{cases}$$

It is clear that $\widetilde{F}_M$ is an integrably bounded Carathéodory map and $S(\widetilde{F}) = S(\widetilde{F}_M)$. As in [22] (see Lemma 1), there exists an almost upper semicontinuous, integrably bounded map $G\colon [0,b]\times\mathbb{R}^n\to\mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ such the $S(G)=S(\widetilde{F}_M)$. From theorem [33], G is σ-mLL-selectionable; then there exists a sequence of multivalued maps $\{G_k\}_{k=1}$, such that

$$G_{k+1}(t,u)\subset G_k(t,u)\quad \text{for almost every } t\in[0,b],\ u\in\mathbb{R}^n,$$

and $G(t,u)=\bigcap_{k=1}^\infty G_k(t,u)$. By Theorems 3.1 and 4.16, $S(G_k)$ is compact and contractible, and so $S(F)$ is R_δ-set. $\square$

Theorem 5.9. *Let $F\colon [0,b]\times K \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be a Carathéodory multifunction, weakly tangent to K, an approximate retract to $\mathbb{R}^n$, and assume g satisfies $(\mathcal{H}_3)$. Also assume that all the conditions of Theorem* 5.8 *hold. Then the solution of problem* (5.1) *is a nonempty set and an R_δ-set.*

Proof. Let $r\colon U \to K$ be the metric retraction. According $(\overline{\mathcal{A}}_1)$, we choose $\varepsilon > 0$ such that $O_{2\varepsilon}(K) \subset U$ and $\overline{O_{2\varepsilon}(K)}$ is a proximate neighbourhood retract. We consider the multivalued map $T\colon \overline{O_\varepsilon(K)} \to \mathcal{P}_{\rm cl}(\mathbb{R}^n)$ defined by

$$T(x) = \{y \in \mathbb{R}^n : \langle y, x - r(x)\rangle \leq 0\}.$$

We easily prove that T has a closed graph in $\overline{O_\varepsilon(K)} \times \mathbb{R}^n$. The multivalued mapping

$$F_\varepsilon\colon [0,b] \times \overline{O_\varepsilon(K)} \to \mathcal{P}_{\rm cl}(\mathbb{R}^n)$$

defined by

$$F_\varepsilon(t,y) = F(t, r(y)) \cap T(r(y))$$

is Carathéodory integrably bounded. From Lemma 5.3, F_ε satisfies the tangent to K condition. From Theorem 5.8, $S(F_\varepsilon)$ is R_δ. Finally, we can observe that, for every $y_0 \in K$, we have

$$S(F) = \bigcap_{n=1}^{\infty} S(F_{1/n}).$$

Hence, $S(F)$ is R_δ-set. □

6. Concluding remarks

In this paper, we have investigated problem (1.2) under various assumptions on the right hand side multivalued nonlinearity, and we have obtained a number of new results regarding existence of solutions. The main assumptions on the nonlinearity are the Carathéodory and the Lipschitz conditions with respect to the Hausdorff distance in generalized metric spaces. When the multivalued nonlinearity is also σ-Ca- or σ-mLL-selectionable, based on Aronszajn type results, we investigated the geometric properties of the solution set, proving that it is R_δ, contractible, or acyclic. Also, the existence of viable solutions of differential inclusions with nonlocal conditions and their topological and geometric structures were investigated.

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