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EXISTENCE OF POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC SYSTEM

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Abstract. In this paper, we are concerned with the existence of (component-wise) positive solutions for a semilinear elliptic system, where the nonlinear term is superlinear in one equation and sublinear in the other equation. By constructing a cone $K_1 \times K_2$ which is the Cartesian product of two cones in space $C(\overline{\Omega})$ and computing the fixed point index in $K_1 \times K_2$, we establish the existence of positive solutions for the system. It is remarkable that we deal with our problem on the Cartesian product of two cones, in which the features of two equations can be exploited better.

1. Introduction

In this paper, we consider the existence of (component-wise) positive solutions for the following elliptic system

$$
\begin{cases} -\Delta u = f_1(x,u,v) & \text{in } \Omega, \\ -\Delta v = f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $f_1, f_2 \in C(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a smooth bounded domain.

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In recent years, many authors have studied the existence of nonnegative nontrivial solutions for elliptic system (1.1), see [6], [9], [12], [13], [15] and the references therein, in which (component-wise) positive solutions are also obtained for elliptic systems involving some special nonlinearities. Usually ones change the problem into the fixed point problem of the corresponding compactly continuous mapping on a single cone K in product space $C(\overline{\Omega})\times C(\overline{\Omega})$ and apply the classical fixed point index theory combining with some a priori estimates technique. For instance, in [6] authors established the existence of positive solutions for the case that $f_1 = u^\alpha v^\beta$ and that $f_2 = u^\gamma v^\delta$; in [12] authors obtained positive solutions of the Lane-Emden system ($f_1 = v^p, f_2 = u^q$); in [13] M. A. S. Souto considered nonnegative nontrivial solutions for more general nonlinearities $f_1 = m_{11}(x)u + m_{12}(x)v + f(x,u,v)$, $f_2 = m_{21}(x)u + m_{22}(x)v + g(x,u,v)$ where f has asymptotic behavior at infinity as u^σ and g satisfies some subcritical growth, in particular obtained positive solutions as $f_1 = u^\sigma + v^q, f_2 = u^p$; in [15] H. Zou discussed nonnegative nontrivial solutions for the nonlinearities $f_1 = au^r + bv^q$, $f_2 = cu^p + dv^s$ ($a+b > 0$, $c+d > 0$ and $p,q > 1$) and then dealt with more general cases that f_1 and f_2 have asymptotic behavior at infinity as $a(x)u^r + b(x)v^q$ and $c(x)u^p + d(x)v^s$ (here the coefficients are nonnegative continuous functions) respectively, moreover positive solutions were obtained when $f_1(x,0,v) \neq 0$ for $v > 0$ and $f_2(x,u,0) \neq 0$ for $u > 0$.

More recently, in [1] authors have studied a large class of sublinear and superlinear nonvariational elliptic systems (in detail, see [1, p. 290–291]) and obtained the existence of nonnegative nontrivial solutions under the assumptions that there is an a priori bound on the nonnegative solutions of superlinear system.

Roughly speaking, there is a common ground in the preceding references that they require the coupled nonlinearities in systems have some similar features, e.g. both nonlinearities are suplinear or sublinear. Based on these similar features, they can change the considered problems into the fixed point problems on a single cone in product space and then obtain nonnegative nontrivial solutions, even get positive solutions. Especially, in some of these references, the similarity of features for nonlinearities is essential for their methods.

Consequently, if the coupled nonlinearities in systems have different features, how should we do? For example, see the following system

$$\begin{cases} -\Delta u = \tan^{-1}(1+v)\ u^2 & \text{in } \Omega, \\ -\Delta v = \delta_1 \cot^{-1}(-u)\ |\sin v| & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, δ_1 is the first eigenvalue of the Laplacian subject to Dirichlet data. We can see that the nonlinearities in (1.2) have different features: one nonlinear term is superlinear while the other is sublinear in some sense.

In this paper, we provide a method for some of these problems involving the coupled nonlinearities with different features. Now we are mainly concerned with the existence of positive solutions for system (1.1) involving a new class of nonlinearities in which one is superlinear and the other is sublinear in the sense of the following definition.

DEFINITION 1.1. If f_1, f_2 in system (1.1) satisfy the following assumptions:

(A_1) $\limsup_{u\to 0^+} \max_{x\in\overline{\Omega}} \dfrac{f_1(x,u,v)}{u} < \delta_1 < \liminf_{u\to\infty} \min_{x\in\overline{\Omega}} \dfrac{f_1(x,u,v)}{u}$ uniformly w.r.t. $v \in \mathbb{R}^+$;

(A_2) $\liminf_{v\to 0^+} \min_{x\in\overline{\Omega}} \dfrac{f_2(x,u,v)}{v} > \delta_1 > \limsup_{v\to\infty} \max_{x\in\overline{\Omega}} \dfrac{f_2(x,u,v)}{v}$ uniformly w.r.t. $u \in \mathbb{R}^+$,

where δ_1 is the first eigenvalue of the Laplacian subject to Dirichlet data, then we say that f_1 is superlinear with respect to u at the origin and infinity and that f_2 is sublinear with respect to v at the origin and infinity.

For our problem if it is changed into the fixed point problem on the single cone in product space, then our difficulty to be solved is the construction of proper open sets in the single cone. The difficulty essentially results from the different features of nonlinearities in our system. To overcome the difficulty, we will construct a cone $K_1 \times K_2$ which is the Cartesian product of two cones in space $C(\overline{\Omega})$ and choose a proper open set which is the Cartesian product of open sets $O_1(\subset K_1)$ and $O_2(\subset K_2)$, such that the features of nonlinearities can be exploited better. And then we can change the problem into the fixed point problem on the product cone $K_1 \times K_2$. Applying the product formula for the fixed point index on product cone and the classical fixed point index theory together with the "blow up" a priori estimates technique (see [10]), we establish the existence of nontrivial fixed points which belong to $O_1 \times O_2$.

It is possible that the result could also be proved by working in the usual cone and calculating the contribution of the semi-trivial solutions. However, this method is a little more tedious than the method of this paper. It is remarkable that by our way the nontrivial solutions obtained are (component-wise) positive, which is different from the previous references. The main result of this paper is

THEOREM 1.2. *Assume that* f_1 *satisfies* (A_1) *and that* f_2 *satisfies* (A_2). *If there exist* $q \in (1,(n+2)/(n-2))$, $h_1 \in C(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R}^+\setminus\{0\})$ *and* $h_2 \in B_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R}^+)$ *such that*

(H_1) $\lim_{u\to\infty} \dfrac{f_1(x,u,v)}{u^q} = h_1(x,v)$

uniformly with respect to $(x,v)\in\overline{\Omega}\times[0,M]$ (*for all* $M>0$),

(H$_2$) $\limsup\limits_{u\to\infty}\max\limits_{x\in\overline{\Omega}} f_2(x,u,v)=h_2(v)$,

uniformly with respect to $v\in[0,M]$ (*for all* $M>0$),

then system (1.1) *has at least one positive solution.*

Remark 1.3. It is easy to verify that the nonlinearities in system (1.2) satisfy the conditions in Theorem 1.2. Hence, system (1.2) has at least one positive solution.

Remark 1.4. We point out that if f_1 (resp. f_2) is sublinear with respect to u (resp. v) at the origin and infinity in the sense of our definitions, in addition, there exist $g_1,g_2\in B_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R}^+)$ such that

(G$_1$) $\limsup\limits_{v\to\infty}\max\limits_{x\in\overline{\Omega}} f_1(x,u,v)=g_1(u)$

uniformly with respect to $u\in[0,M]$ (for all $M>0$),

(G$_2$) $\limsup\limits_{u\to\infty}\max\limits_{x\in\overline{\Omega}} f_2(x,u,v)=g_2(v)$

uniformly with respect to $v\in[0,M]$ (for all $M>0$),

then system (1.1) has at least one positive solution, which can be obtained by the proof similar to Theorem 1.2. It is remarkable that it seems that the result can not be obtained by computing the fixed point index on the single cone in product space.

Remark 1.5. When has system (1.1) positive solutions under the fundamental assumptions that f_1 (resp. f_2) is superlinear with respect to u (resp. v) at the origin and infinity in the sense of our definitions? It is one of problems we shall be concerned with in future, where the main difficulty results from a priori estimates of solutions to our superlinear system.

The paper is organized as follows: in Section 2, we make some preliminaries; in Section 3, we give some lemmas and finally prove Theorem 1.2.

2. Preliminaries

First, we recall some concepts about the fixed point index (see [7], [14]), which will be used in the proof of our main result. Let X be a Banach space and $P\subset X$ be a closed convex cone. Assume that W is a bounded open subset of X with boundary ∂W, and that $A\colon P\cap\overline{W}\to P$ is a completely continuous operator. If $Au\neq u$ for all $u\in P\cap\partial W$, then the fixed point index $i(A,P\cap W,P)$ is defined. One important fact is that if $i(A,P\cap W,P)\neq 0$, then A has a fixed point in $P\cap W$.

The following lemmas are useful in our proofs.

LEMMA 2.1 ([2], [11]). *Let E be a Banach space and $K \subset E$ be a closed convex cone in E, denote $K_r = \{u \in K \mid \|u\| < r\}$, $\partial K_r = \{u \in K \mid \|u\| = r\}$, where $r > 0$. Let $T\colon \overline{K_r} \to K$ be a compact mapping and $0 < \rho \le r$.*

(a) *If $Tx \neq tx$ for all $x \in \partial K_\rho$ and for all $t \ge 1$, then $i(T, K_\rho, K) = 1$.*
(b) *If there exists a compact mapping $H\colon \overline{K_\rho} \times [0,\infty) \to K$ such that*
  (b1) *$H(x,0) = Tx$ for all $x \in \partial K_\rho$,*
  (b2) *$H(x,t) \neq x$ for all $x \in \partial K_\rho$ and all $t \ge 0$,*
  (b3) *there is a $t_0 > 0$, such that $H(x,t) = x$ has no solution $x \in \overline{K_\rho}$, for $t \ge t_0$,*

  then $i(T, K_\rho, K) = 0$.

LEMMA 2.2 ([4]). *Let E be a Banach space and let $K_i \subset E$ $(i = 1,2)$ be a closed convex cone in E. For $r_i > 0$ $(i = 1,2)$, denote $K_{r_i} = \{u \in K_i \mid \|u\| < r_i\}$, $\partial K_{r_i} = \{u \in K_i \mid \|u\| = r_i\}$. Suppose $A_i\colon K_i \to K_i$ is completely continuous. If $u_i \neq A_i u_i$, for all $u_i \in \partial K_{r_i}$, then*

$$i(A, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(A_1, K_{r_1}, K_1) \cdot i(A_2, K_{r_2}, K_2),$$

where $A(u,v) \overset{\text{def}}{=} (A_1 u, A_2 v)$, for all $(u,v) \in K_1 \times K_2$.

Next, we establish the functional analytic framework for the proof of Theorem 1.2 in order to use the results on the fixed point index stated above.

For convenience, we introduce some notations as following:

$$E = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}, \quad K = \{u \in E \mid u(x) \ge 0, \text{ for all } x \in \overline{\Omega}\}.$$

Let us call $S\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ the solution operator of the linear problem

$$\begin{cases} -\Delta u = \psi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\psi \in C(\overline{\Omega})$. It is well known that S takes $C(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ and then S is a linear compact mapping in the space $C(\overline{\Omega})$.

For $\lambda \in [0,1]$ and $u, v \in K$, we define the mappings $T_{\lambda,1}(\,\cdot\,,\,\cdot\,), T_{\lambda,2}(\,\cdot\,,\,\cdot\,)\colon K \times K \to K$ and $T_\lambda(\,\cdot\,,\,\cdot\,)\colon K \times K \to K \times K$ by

$$\begin{aligned} T_{\lambda,1}(u,v) &= S[\lambda f_1(x,u,v) + (1-\lambda) f_1(x,u,0)], \\ T_{\lambda,2}(u,v) &= S[\lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v)], \\ T_\lambda(u,v) &= (T_{\lambda,1}(u,v), T_{\lambda,2}(u,v)). \end{aligned} \tag{2.2}$$

It is easy to see that mappings $T_{\lambda,1}(\,\cdot\,,v)$, $T_{\lambda,2}(u,\,\cdot\,)$ and $T_\lambda(\,\cdot\,,\,\cdot\,)$ are compact.

3. Proof of Theorem 1.2

Before proving Theorem 1.2, let us state our main idea of proof. First, we deal with the single equations $-\Delta u = f_1(x,u,0)$ and $-\Delta v = f_2(x,0,v)$ with Dirichlet boundary conditions, and then consider the following parameterized system

$$\begin{cases} -\Delta u = \lambda f_1(x,u,v) + (1-\lambda) f_1(x,u,0) & \text{in } \Omega, \\ -\Delta v = \lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where parameter $\lambda \in [0,1]$. Based on the preceding preliminaries, we only need to consider the fixed point index of compact mapping T_λ corresponding to system (3.1). Applying the homotopy invariance and product formula (see Lemma 2.2) of the fixed point index together with some fixed point index results (see Lemmas 3.1 and 3.2), we can compute the fixed point index of compact mapping T_1 corresponding to system (1.1), and establish the existence of positive solutions.

LEMMA 3.1. *Assume that f_1 satisfies* (A$_1$) *and* (H$_1$)*, then there exist $R_0 > r_0 > 0$ such that for all $r \in (0, r_0]$ and $R \in [R_0, \infty)$,*

$$i(T_{0,1}(\,\cdot\,, v), K_R \setminus \overline{K_r}, K) = -1.$$

PROOF. From the definition of $T_{\lambda,1}$, we know that $T_{0,1}(u,v) = S[f_1(x,u,0)]$.

In view of assumption (A_1), there exist $\varepsilon \in (0,\delta_1)$ and $r_0 > 0$, such that

$$f_1(x,u,0) \le (\delta_1 - \varepsilon)u, \quad \text{for all } (x,u) \in \overline{\Omega} \times [0,r], \text{ where } r \in (0,r_0]. \tag{3.2}$$

We claim that $T_{0,1}(u,v) \ne tu$ for all $t \ge 1$ and all $u \in \partial K_r$. In fact, if there exist $t_0 \ge 1$ and $u_0 \in \partial K_r$ such that $T_{0,1}(u_0,v) = t_0 u_0$, then u_0 satisfies the following equation

$$\begin{cases} -\Delta u_0 = t_0^{-1} f_1(x,u_0,0) & \text{for all } x \in \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

Multiplying both sides of the equation above by a positive eigenfunction φ_1 associated to the first eigenvalue δ_1 of $(-\Delta\,, H_0^1(\Omega))$ and integrating on Ω, we get that

$$\int_\Omega (-\Delta u_0)\varphi_1 = \int_\Omega t_0^{-1} f_1(x,u_0,0)\varphi_1.$$

Combining with (3.2), we have

$$\delta_1 \int_\Omega u_0\varphi_1 \le (\delta_1 - \varepsilon) \int_\Omega u_0 \varphi_1,$$

which is a contradiction. Hence, applying conclusion (a) of Lemma 2.1 we obtain that

$$i(T_{0,1}(\,\cdot\,, v), K_r, K) = 1 \quad \text{for all } r \in (0, r_0]. \tag{3.3}$$

By virtue of assumption (A_1) and continuity of f_1, there exist $\varepsilon > 0$ and $C > 0$ such that

$$f_1(x, u, 0) \geq (\delta_1 + \varepsilon)u - C, \quad \text{for all } (x, u) \in \overline{\Omega} \times \mathbb{R}^+. \tag{3.4}$$

Next, we show that there exists $R_0 > r_0$ such that

$$i(T_{0,1}(\,\cdot\,, v), K_R, K) = 0 \quad \text{for all } R \in [R_0, \infty). \tag{3.5}$$

For this matter, we need to construct the homotopy $H: \overline{K_R} \times \mathbb{R}^+ \to K$ as following:

$$H(u, t) = S[f_1(x, u + t, 0)].$$

Now we verify all the conditions of (b) in Lemma 2.1 which yields (3.5).

First, it is obvious that condition (b1) of Lemma 2.1 holds.

Second, we prove that there exists a $t_0 > 0$ such that equation $H(u, t) = u$ does not have solutions for $t \geq t_0$, which implies condition (b3) of Lemma 2.1. Actually, let u be a solution for the following equation

$$\begin{cases} -\Delta u = f_1(x, u + t, 0) & \text{for all } x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

In combination with (3.4), we have

$$-\Delta u \geq (\delta_1 + \varepsilon)(u + t) - C.$$

Multiplying both sides of the inequality above by φ_1 and integrating on Ω, we obtain that

$$\int_\Omega (-\Delta u)\varphi_1 \geq (\delta_1 + \varepsilon) \int_\Omega (u + t)\varphi_1 - C \int_\Omega \varphi_1.$$

From the inequality above, it is easy to see that $t \leq C/(\delta_1 + \varepsilon)$. As a result, choosing $t_0 = C/(\delta_1 + \varepsilon) + 1$ we can conclude the desired conclusion.

Finally, we only need to verify condition (b2) of Lemma 2.1. In fact, by the growth condition (H_1), we know that for all $t \in [0, t_0]$, the solutions for equation $H(u, t) = u$ have a uniform a priori bound R_0^* (based on the "blow up" a priori estimates in [10]). Hence, for all $R \geq R_0 \equiv \max\{r_0, R_0^*\} + 1$, we have $H(u, t) \neq u$ for all $u \in \partial K_R$.

Noticing (3.3) and (3.5), for all $r \in (0, r_0]$ and $R \in [R_0, \infty)$ we have

$$i(T_{0,1}(\,\cdot\,, v), K_R \backslash \overline{K_r}, K) = -1. \tag{3.6}$$

□

Lemma 3.2. *Assume that f_2 satisfies* (A_2)*, then there exist $\overline{R}_0 > \overline{r}_0 > 0$ such that for all $\overline{r} \in (0, \overline{r}_0]$ and $\overline{R} \in [\overline{R}_0, \infty)$,*

$$i(T_{0,2}(u, \,\cdot\,), K_{\overline{R}} \setminus \overline{K_{\overline{r}}}, K) = 1.$$

Proof. By the definition of $T_{\lambda,2}$, we get that $T_{0,2}(u, v) = S[f_2(x, 0, v)]$.

From assumption (A_2), there exist $\varepsilon > 0$ and $\overline{r}_0 > 0$ such that

$$f_2(x,0,v) \geq (\delta_1+\varepsilon)v \quad \text{for all } (x,v)\in\overline{\Omega}\times[0,\overline{r}], \text{ where } \overline{r}\in(0,\overline{r}_0]. \tag{3.7}$$

Now we show that

$$i(T_{0,2}(u,\,\cdot\,),K_{\overline{r}},K)=0 \quad \text{for all } \overline{r}\in(0,\overline{r}_0]. \tag{3.8}$$

In fact, we only need to make the homotopy $H^*\colon \overline{K_{\overline{r}}}\times\mathbb{R}^+\to K$ as following:

$$H^*(v,t)=S[f_2(x,0,v)]+\frac{t}{\delta_1}\varphi_1,$$

and then prove that H^* satisfies all the conditions of (b) in Lemma 2.1.

First, it is clear that condition (b1) of Lemma 2.1 is valid.

Second, we consider solutions for equation $H^*(v,t)=v$. Assume that v is a solution for it, then v satisfies the following equation

$$\begin{cases} -\Delta v = f_2(x,0,v)+t\varphi_1 & \text{for all } x\in\Omega,\\ v|_{\partial\Omega}=0. \end{cases}$$

Noticing (3.7), we have

$$-\Delta v \geq (\delta_1+\varepsilon)v+t\varphi_1.$$

Multiplying both sides of the above inequality by φ_1 and integrating on Ω, we know that

$$\int_\Omega(-\Delta v)\varphi_1 \geq (\delta_1+\varepsilon)\int_\Omega v\varphi_1 + t\int_\Omega \varphi_1^2,$$

which implies a contradiction $\delta_1\geq\delta_1+\varepsilon$. As a result, conditions (b2) and (b3) of Lemma 2.1 also hold.

By assumption (A_2) and continuity of f_2, there exist $\varepsilon\in(0,\delta_1)$ and $C>0$ such that

$$f_2(x,0,v)\leq(\delta_1-\varepsilon)v+C \quad \text{for all } (x,v)\in\overline{\Omega}\times\mathbb{R}^+. \tag{3.9}$$

Next, we show that there exists $\overline{R}_0>\overline{r}_0$ such that

$$i(T_{0,2}(u,\,\cdot\,),K_{\overline{R}},K)=1 \quad \text{for all } \overline{R}\in[\overline{R}_0,\infty). \tag{3.10}$$

On that purpose, suppose that there exist $t\geq 1$ and $v\in\partial K_{\overline{R}}$ such that $T_{0,2}(u,v)=tv$, that is,

$$\begin{cases} -\Delta v = t^{-1}f_2(x,0,v) & \text{for all } x\in\Omega,\\ v|_{\partial\Omega}=0. \end{cases} \tag{3.11}$$

In what follows, we prove that there exists a positive constant C (independent of t) such that $\|v\|_\infty\leq C$ for all solutions v of (3.11). From (3.9) and (3.11), it follows that

$$\int_\Omega|\nabla v|^2\leq(\delta_1-\varepsilon)\int_\Omega v^2+C\int_\Omega v,$$

combining with Pioncàre's inequality and Hölder's inequality, which implies that

$$\|v\|_{L^2} \le C, \quad \|v\|_{L^1} \le C \quad \text{and} \quad \|v\|_{H_0^1} \le C. \tag{3.12}$$

Furthermore, by (3.9) and Sobolev embedding theorem, we know that

$$\|f_2(x,0,v)\|_{L^{2^*}} \le C\|v\|_{H_0^1} + C. \tag{3.13}$$

By L^p-theory about elliptic equations, we get that $v \in W^{2,2^*}(\Omega)$ and

$$\|v\|_{W^{2,2^*}} \le C\|f_2(x,0,v)\|_{L^{2^*}}. \tag{3.14}$$

In combination with (3.9), (3.12)–(3.14) and boot-strap technique, it is not difficult to show that there is a positive constant C (independent of t) such that $\|v\|_{L^\infty} \le C$, that is, all solutions of (3.11) have a uniform bound C. Choosing $\overline{R}_0 = \max\{\overline{r}_0, C\} + 1$, we have that $T_{0,2}(u,v) \ne tv$ for all $t \ge 1$ and $v \in \partial B_{\overline{R}}$, for all $\overline{R} \ge \overline{R}_0$. As a result, applying conclusion (a) of Lemma 2.1 we conclude that (3.10) is valid.

By (3.8) and (3.10), for all $\overline{r} \in (0, \overline{r}_0]$ and $\overline{R} \in [\overline{R}_0, \infty)$ we have

$$i(T_{0,2}(u,\cdot), K_{\overline{R}} \setminus \overline{K_{\overline{r}}}, K) = 1. \tag{3.15}$$

□

LEMMA 3.3. *Suppose that f_1 satisfies* (H$_1$) *and that f_2 satisfies* (A$_2$) *and* (H$_2$). *Let $(u(x), v(x))$ be a positive solution of system* (3.1), *then there exists some uniform constant C (independent of λ, u and v) such that $\|u\|_{L^\infty} \le C$ and $\|v\|_{L^\infty} \le C$.*

PROOF. We prove that there exist positive constants C_1 and C_2 (independent of λ, u and v) such that $\|v\|_{L^\infty} \le C_1$ and $\|u\|_{L^\infty} \le C_2$ according to the following two steps.

Step 1. Show that there exists a positive constant C_1 (independent of λ, u and v) such that $\|v\|_{L^\infty} \le C_1$, which is based on L^p-theory and boot-strap technique. Furthermore, there is a positive constant C^* (independent of λ, u and v) such that $\|v\|_{C^{1,\alpha}} \le C^*$, here $\alpha \in (0,1)$.

Noticing that $(u(x), v(x))$ satisfies the following equation

$$\begin{cases} -\Delta v(x) = \lambda f_2(x, u(x), v(x)) + (1-\lambda) f_2(x, 0, v(x)) & \text{for all } x \in \Omega, \\ v|_{\partial\Omega} = 0. \end{cases} \tag{3.16}$$

By assumptions (A$_2$) and (H$_2$), there exist $\varepsilon \in (0, \delta_1)$ and $C > 0$ such that

$$\lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v) \le (\delta_1 - \varepsilon)v + C, \tag{3.17}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+$. By (3.16) and (3.17), it follows that

$$\int_\Omega |\nabla v(x)|^2 \le (\delta_1 - \varepsilon) \int_\Omega v^2(x) + C \int_\Omega v(x),$$

combining with Pioncàre's inequality and Hölder's inequality, which implies that

$$\|v\|_{L^2} \le C, \ \|v\|_{L^1} \le C \text{ and } \|v\|_{H_0^1} \le C. \tag{3.18}$$

Furthermore, by (3.17) and Sobolev embedding theorem, we know that

$$\|\lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v)\|_{L^{2^*}} \le C\|v\|_{H_0^1} + C. \tag{3.19}$$

By L^p-theory about elliptic equations, we get that $v \in W^{2,2^*}(\Omega)$ and

$$\|v\|_{W^{2,2^*}} \le C\|\lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v)\|_{L^{2^*}}. \tag{3.20}$$

In combination with (3.17)–(3.20) and boot-strap technique, it is not difficult to show that there is a positive constant C_1 (independent of λ, u and v) such that $\|v\|_{L^\infty} \le C_1$.

In addition, by L^p-theory and Sobolev embedding theorem, it is easy to prove that there is a positive constant C^* (independent of λ, u and v) such that $\|v\|_{C^{1,\alpha}} \le C^*$, here $\alpha \in (0,1)$.

Step 2. Prove that there exists a positive constant C_2 (independent of λ, u and v) such that $\|u\|_{L^\infty} \le C_2$, which is based on the "blow up" a priori estimates technique in [10].

Suppose, by contradiction, that there is no such a priori bound. That is, there exist a sequence of numbers $\{\lambda_k\}_{k=1}^\infty \subset [0,1]$ and a sequence of positive solutions $\{(u_k, v_k)\}_{k=1}^\infty$ to a family of systems

$$\begin{cases} -\Delta u_k = \lambda_k f_1(x,u_k,v_k) + (1-\lambda_k) f_1(x,u_k,0) & \text{for all } x \in \Omega, \\ -\Delta v_k = \lambda_k f_2(x,u_k,v_k) + (1-\lambda_k) f_2(x,0,v_k) & \text{for all } x \in \Omega, \\ u_k|_{\partial\Omega} = v_k|_{\partial\Omega} = 0, \end{cases} \tag{3.21}$$

such that $\lim_{k\to\infty} \|u_k\|_{L^\infty} = \infty$.

By maximum principle, there exists a sequence of points $\{P_k\}_{k=1}^\infty \subset \Omega$ such that

$$M_k \equiv \sup_{x\in\Omega} u_k(x) = u_k(P_k) \to \infty \quad \text{as } k \to \infty. \tag{3.22}$$

We may assume that $\lambda_k \to \lambda \in [0,1]$ and $P_k \to P \in \overline{\Omega}$ as $k \to \infty$. The proof breaks down into two cases depending on whether $P \in \Omega$ or $P \in \partial\Omega$.

Case 1. ($P \in \Omega$) Let $2d$ denote the distance of P to $\partial\Omega$, and $B_r(a)$ the ball of radius r and center $a \in \mathbb{R}^n$. Let μ_k be a sequence of positive numbers (to be defined below) and $y = (x - P_k)/\mu_k$. Define the scaled function

$$\overline{u}_k(y) = \mu_k^{2/(q-1)} u_k(x). \tag{3.23}$$

Choose μ_k such that

$$\mu_k^{2/(q-1)} M_k = 1. \tag{3.24}$$

Since $M_k \to \infty$, we have $\mu_k \to 0$ as $k \to \infty$. For large k, $\overline{u}_k(y)$ is well defined in $B_{d/\mu_k}(0)$ and

$$\sup_{y \in B_{d/\mu_k}(0)} \overline{u}_k(y) = \overline{u}_k(0) = 1. \tag{3.25}$$

Moreover, $\overline{u}_k(y)$ satisfies in $B_{d/\mu_k}(0)$

$$\begin{aligned} -\Delta \overline{u}_k(y) = \mu_k^{2q/(q-1)} [\lambda_k f_1(\mu_k y + P_k, \overline{u}_k(y), v_k(\mu_k y + P_k)) \\ + (1-\lambda_k) f_1(\mu_k y + P_k, \overline{u}_k(y), 0)]. \end{aligned} \tag{3.26}$$

Note that v_k are uniformly bounded (see Step 1), and by assumption (H_1)

$$\begin{aligned} &\lim_{k\to\infty} |\mu_k^{2q/(q-1)} f_1(\mu_k y + P_k, \mu_k^{-2/(q-1)} \overline{u}_k(y), v_k(\mu_k y + P_k)) \\ &\qquad - h_1(\mu_k y + P_k, v_k(\mu_k y + P_k))(\overline{u}_k(y))^q| = 0, \\ &\lim_{k\to\infty} |\mu_k^{2q/(q-1)} f_1(\mu_k y + P_k, \mu_k^{-2/(q-1)} \overline{u}_k(y), 0) \\ &\qquad - h_1(\mu_k y + P_k, 0)(\overline{u}_k(y))^q| = 0. \end{aligned} \tag{3.27}$$

Therefore, given any radius R such that $B_R(0) \subset B_{d/\mu_k}(0)$, by L^p-theory we can find uniform bounds for $\|\overline{u}_k\|_{W^{2,p}(B_R(0))}$.

Choosing $p > n$ large, by Sobolev compact embedding theorem we obtain that $\{\overline{u}_k\}$ is precompact in $C^{1,\alpha}(B_R(0))(0 < \alpha < 1)$. It follows that there exists a subsequence $\overline{u}_{k_j}$ converging to $\overline{u}$ in $W^{2,p}(B_R(0)) \cap C^{1,\alpha}(B_R(0))$. By Hölder continuity $\overline{u}(0) = 1$. From the result obtained in Step 1 and the Arzelá–Ascoli Theorem, there exists a subsequence of $v_k(\mu_k y + P_k)$, relabel $v_{k_j}(\mu_{k_j} y + P_{k_j})$ which converges to $v(P)$ in $C(B_R(0))$. Furthermore, since

$$\begin{cases} \lambda_{k_j} \to \lambda, \quad v_{k_j}(\mu_{k_j} y + P_{k_j}) \to v(P), \\ h_1(\mu_{k_j} y + P_{k_j}, v_{k_j}(\mu_{k_j} y + P_{k_j})) \to h_1(P, v(P)), \\ h_1(\mu_{k_j} y + P_{k_j}, 0) \to h_1(P, 0), \end{cases} \tag{3.28}$$

as $k_j \to \infty$, $\overline{u}(y)$ is a solution of

$$-\Delta \overline{u}(y) = [\lambda h_1(P, v(P)) + (1-\lambda) h_1(P, 0)] \overline{u}^q(y). \tag{3.29}$$

We claim that $\overline{u}$ is well defined in all of $\mathbb{R}^n$ and $\overline{u}_{k_j} \to \overline{u}$ in $W^{2,p} \cap C^{1,\alpha}(p > n)$ on any compact subset. To show this we consider $B_{R'}(0) \supset B_R(0)$. Repeating the above argument with $B_{R'}(0)$, the subsequence $\overline{u}_{k_j}$ has a convergent subsequence $\overline{u}_{k'_j} \to \overline{u}'$ on $B_{R'}(0)$. $\overline{u}'$ satisfies (3.24), and necessarily $\overline{u}'|_{B_R(0)} = \overline{u}$. By unique continuation the entire original subsequence $\overline{u}_{k_j}$ converges, so that $\overline{u}$ is well-defined. By the global result of Liouville type (see Theorem 1.2 in [10, p. 886]), we have $\overline{u} = 0$, a contradiction, since $\overline{u}(0) = 1$.

Case 2. ($P \in \partial\Omega$) By arguments similar to Case 1, we can reduce the problem of a priori bounds to the global results of Liouville type (see Theorems 1.2 and 1.3 in [10, p. 886]) and deduce a contradiction. □

Proof of Theorem 1.2. Now we can seek the fixed points of T_1 in one certain open set $(K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}})$, where $r_1 \in (0, r_0]$, $R_1 \in [R_0, \infty)$, $r_2 \in (0, \overline{r}_0]$ and $R_2 \in [\overline{R}_0, \infty)$ will be determined later.

Combining Lemma 2.2 with (3.6) and (3.15), we know that

$$i(T_0, (K_{R_1} \setminus \overline{K_{r_1}}\,) \times (K_{R_2} \setminus \overline{K_{r_2}}\,), K \times K) = -1.$$

In order to seek the nontrivial fixed points of T_1, we want to prove that

$$\begin{aligned} i(T_1, (K_{R_1} \setminus \overline{K_{r_1}}\,) \times (K_{R_2} \setminus \overline{K_{r_2}}\,), K \times K)& \\ = i(T_0, (K_{R_1} \setminus \overline{K_{r_1}}\,) &\times (K_{R_2} \setminus \overline{K_{r_2}}\,), K \times K). \end{aligned}$$

By the homotopy invariance of fixed point index, we only need to verify that

$$(u, v) \neq T_\lambda(u, v) \tag{3.30}$$

for all $\lambda \in [0, 1]$ and $(u, v) \in \partial[(K_{R_1} \setminus \overline{K_{r_1}}\,) \times (K_{R_2} \setminus \overline{K_{r_2}}\,)]$.

First, by condition (A_1) there are $\varepsilon \in (0, \delta_1)$ and $r_1 \in (0, r_0]$ such that

$$\lambda f_1(x, u, v) + (1 - \lambda) f_1(x, u, 0) \leq (\delta_1 - \varepsilon) u, \tag{3.31}$$

for all $x \in \overline{\Omega}$, $u \in [0, r_1]$ and $v \in \mathbb{R}^+$. We claim that

$$(u, v) \neq T_\lambda(u, v), \quad \text{for all } \lambda \in [0, 1] \text{ and } (u, v) \in \partial K_{r_1} \times K. \tag{3.32}$$

In fact, if there exist $\lambda_0 \in [0, 1]$ and $(u_0, v_0) \in \partial K_{r_1} \times K$, such that $(u_0, v_0) = T_{\lambda_0}(u_0, v_0)$, then (u_0, v_0) satisfies the following equation

$$\begin{cases} -\Delta u_0 = \lambda_0 f_1(x, u_0, v_0) + (1 - \lambda_0) f_1(x, u_0, 0) & \text{for all } x \in \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{3.33}$$

By (3.31) and (3.33), we have

$$-\Delta u_0 \leq (\delta_1 - \varepsilon) u_0.$$

Multiplying both sides of the inequality above by φ_1 and integrating on Ω, we get that

$$-\int_\Omega \Delta u_0 \varphi_1 \leq (\delta_1 - \varepsilon) \int_\Omega u_0 \varphi_1,$$

which yields a contradiction $\delta_1 \leq \delta_1 - \varepsilon$.

Second, by assumption (A_2) we know that there exist $\varepsilon > 0$ and $r_2 \in (0, \overline{r}_0]$ such that

$$\lambda f_2(x, u, v) + (1 - \lambda) f_2(x, 0, v) \geq (\delta_1 + \varepsilon) v, \quad \text{for all } x \in \overline{\Omega},\ v \in [0, r_2] \text{ and } u \in \mathbb{R}^+. \tag{3.34}$$

By (3.34) and the proof similar to (3.32), we can obtain that

$$(u, v) \neq T_\lambda(u, v), \quad \text{for all } \lambda \in [0, 1] \text{ and } (u, v) \in K \times \partial K_{r_2}. \tag{3.35}$$

Finally, we consider the equation $T_\lambda(u,v) = (u,v)$, that is

$$(3.36)\qquad \begin{cases} -\Delta u = \lambda f_1(x,u,v) + (1-\lambda) f_1(x,u,0) & \text{for all } x \in \Omega, \\ -\Delta v = \lambda f_2(x,u,v) + (1-\lambda) f_2(x,0,v) & \text{for all } x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases}$$

From Lemma 3.3, all the solutions for (3.36) have a uniform a priori bound C independent of λ, u and v. Hence, choosing $R_1 \geq \max\{R_0, C+1\}$ and $R_2 \geq \max\{\overline{R}_0, C+1\}$ we have

$$(3.37)\qquad \begin{cases} (u,v) \neq T_\lambda(u,v) & \text{for all } \lambda \in [0,1] \text{ and } (u,v) \in \partial K_{R_1} \times K, \\ (u,v) \neq T_\lambda(u,v) & \text{for all } \lambda \in [0,1] \text{ and } (u,v) \in K \times \partial K_{R_2}. \end{cases}$$

In combination with (3.32), (3.35) and (3.37), it is easy to see that (3.30) is valid. □

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