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COMPETITION SYSTEMS WITH STRONG INTERACTION ON A SUBDOMAIN

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Abstract. We study the large-interaction limit of an elliptic system modelling the steady states of two species u and v which compete to some extent throughout a domain Ω but compete strongly on a subdomain $A \subset \Omega$. In the strong-competition limit, u and v segregate on A but not necessarily on $\Omega \setminus A$. The limit problem is a system on $\Omega \setminus A$ and a scalar equation on A and in general admits an interesting range of types of solution, not all of which can be the strong-competition limit of coexistence states of the original system.

1. Introduction

Elliptic systems of the form

$$
\begin{aligned}
-\Delta u &= f(u) - kuv && \text{in } \Omega,\\
-\Delta v &= g(v) - kuv && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,\\
v &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

are well-known to arise in modelling the steady states of populations with densities u and v that compete in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. The self-interaction functions f and g are assumed to be continuously differentiable and such that $f(0) = g(0) = 0$ and $f(y) < 0$ and $g(y) < 0$ for large positive y. A key

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example that satisfies these conditions is the logistic function $f(u) = u(1-u)$. The positive parameter k is a measure of the strength of the competition between u and v in Ω, and since u and v represent densities, interest is in non-negative solutions of (1.1). If both u and v are strictly positive throughout Ω, the two species co-exist in Ω and (u,v) is said to be a *coexistence state*. The strong-interaction ($k \to \infty$) limit of such coexistence states for the system (1.1) was first studied in E. N. Dancer and Y. Du [6]. Under certain conditions, k-dependent solutions (u^k, v^k) of (1.1) converge to the positive and negative parts respectively of a sign-changing solution w of the scalar limit problem

$$
\begin{aligned}
-\Delta w &= f(w^+) - g(-w^-) && \text{in } \Omega, \\
w &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

where $w^+ := \max(0,w)$ and $w^- := \min(0,w)$, whereas a solution w of (1.2) yields a coexistence state (u,v) near $(w^+, -w^-)$ when k is sufficiently large. See [6] for details.

The model (1.1) assumes a similar form for the competitive interaction throughout the domain Ω, and in [6] and later related work, it is supposed that the species interact strongly on the whole of Ω. But it is a natural biological question to ask what happens if u and v may compete to some extent in the whole of a region Ω, but compete strongly on a subdomain A. This gives rise to a new k-dependent system, of the form

$$
\begin{aligned}
-\Delta u &= f(u) - suv - k\chi_A uv && \text{in } \Omega, \\
-\Delta v &= g(v) - ruv - \alpha k\chi_A uv && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega, \\
v &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.3}
$$

where k is again a positive competition parameter, f and g are as above, and here A is a non-empty open subset of Ω with smooth boundary such that $\overline{A} \subset \Omega$, though this could be weakened. Both boundaries $\partial\Omega$ and ∂A are supposed to be of class $C^{2,\mu}$ for some $\mu > 0$. The parameters r and s are assumed to be non-negative, and the parameter α strictly positive.

Our aim is to study the effect of large, positive values of k on solutions of (1.3), corresponding to strong interaction of the populations on A. We will see in Section 2 that the large interaction $k \to \infty$ limit gives rise to a limit problem with interesting dependence on the limiting behaviour of u and v on the internal boundary ∂A. Note that no conditions on ∂A, other than its location, are imposed *a priori*. The limiting problem is in fact a system on part of the domain $(\Omega \setminus A)$ and a scalar equation on the remainder (A), and we find that the population densities u and v segregate on $\overline{A}$ but not in general on $\Omega \setminus A$. This problem seems to be of a type not previously seen and it would be of interest to understand it rather better. In Section 3, we make some initial remarks both on the existence of solutions of this new limit problem, and on when solutions of this problem arise as the limit of coexistence states of (1.3). As part of the description of the limit problem in Section 2, Lemma 2.9 relates the normal derivatives of the limits of u and v on either side of the internal boundary ∂A. Since we could not locate a reference guaranteeing the existence of normal derivatives into $\Omega \setminus A$ almost everywhere on ∂A, a short proof to establish this is included in an Appendix.

Problems in which the effect of competition varies across the spatial domain have been investigated by many authors; see, for example, [1]–[3], [7], [12] and the references therein. In particular, S. Cano-Casanova and J. López-Gómez [1], Y. Du and X. Liang [7] and J. López-Gómez [12] allow species to have a "refuge" or "protection zone" in some species-dependent part of the domain and consider, among other results, consequences of strong interaction between species in the presence of such refuges. Note that [1], [7], [12] study the effect of competition tending to infinity in some but not all equations, which is rather different from our situation here. A further strong-competition limit is derived by E. C. M. Crooks and E. N. Dancer [3] for the case of an inhomogeneous modification of (1.1) in which the terms $-kuv$ are multiplied by functions $\alpha_1(x)$ and $\alpha_2(x)$, with the crucial assumption, excluding our system (1.3), that α_1 and α_2 are bounded below by a positive constant on Ω. N. Igbida and F. Karami [11] study localized strong interaction in a reaction-diffusion system with a different, non-competitive type of coupling.

Note that the study of strong-competition limits of elliptic systems is of interest not only for questions of spatial segregation and coexistence in population dynamics, as here and in [1]–[4], [6], [7], [12], but is also key to the understanding of phase separation in Hartree–Fock type approximations of systems of modelling Bose–Einstein condensates. See, for instance, the recent articles by B. Noris, H. Tavares, S. Terracini and G. Verzini [13], S. Terracini and G. Verzini [14], and J. Wei and T. Weth [15].

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2. Derivation of the strong-interaction limit problem

The presence of the characteristic function χ_A in (1.3) means that one cannot expect classical solutions in general. By a solution of (1.3), we will mean a pair of functions (u,v) such that $u,v \in W^{2,p}(\Omega)$, $p > N$, and satisfy (1.3) almost everywhere, with Δu, Δv being understood in a weak sense. Note that such u,v in fact belong to $C^{1,\lambda}(\overline{\Omega})$ for $\lambda < 1 - N/p$.

We first prove some *a priori* bounds and basic $k \to \infty$ convergence results on the full domain Ω, before turning to the limiting behaviour on A, $\Omega \setminus \overline{A}$ and ∂A.

2.1. Convergence on Ω. Given a solution (u^k, v^k) of (1.3), define

$$w^k := \alpha u^k - v^k. \tag{2.1}$$

Then subtracting the second equation in (1.3) from α times the first equation gives that w^k satisfies the equation

$$\begin{aligned} -\Delta w^k &= \alpha f(u^k) - g(v^k) - (\alpha s - r) u^k v^k && \text{in } \Omega, \\ w^k &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

which has no explicit dependence on k.

Lemma 2.1. *Suppose that (u^k, v^k) is a non-negative solution of (1.3) for some $k \in \mathbb{N}$. Then*

(a) *there exists $M > 0$, independent of $k \in \mathbb{N}$, such that $0 \le u^k, v^k \le M$;*
(b) *there exists $K > 0$, independent of $k \in \mathbb{N}$, such that*

$$\int_\Omega |\nabla u^k|^2 \, dx, \quad \int_\Omega |\nabla v^k|^2 \, dx \le K;$$

(c) *the function w^k defined in (2.1) is bounded independently of k in $W^{2,p}(\Omega)$ for each $p \in [1, \infty)$, and in $C^{1,\lambda}(\overline{\Omega})$ for each $\lambda \in (0,1)$.*

Proof. Let $M > 0$ be such that $f(s) < 0$ and $g(s) < 0$ when $s > M$. If (u^k, v^k) is a non-negative solution of (1.3) and $u^k(x) \ge M$ for $x \in \Omega' \subset \Omega$, then $-\Delta u^k \le f(u^k) < 0$ in Ω' and so u^k cannot attain a maximum in Ω' unless it is constant, from which it follows that $u^k \le M$ in Ω; similarly $v^k \le M$ in Ω. To prove (b), note that multiplication of the first equation in (1.3) by u^k and integration over Ω gives

$$-\int_\Omega |\nabla u^k|^2 \, dx + \int_\Omega u^k f(u^k) \, dx \ge 0,$$

since $u^k(x) = 0$ for all $x \in \partial\Omega$ and u^k, v^k and r, s, k and α are non-negative. The estimate for ∇u^k then follows from (a), and the corresponding estimate for ∇v^k can be proved likewise. Part (a) and (2.2) yield that Δw^k is bounded in $L^\infty(\Omega)$ independently of k and $w^k = 0$ on $\partial\Omega$ for all k, which implies (c). □

The following corollary is immediate from Lemma 2.1.

Corollary 2.2. *Given a sequence of non-negative solutions* $(u^k, v^k)_{k\in\mathbb{N}}$ *of* (1.3), *there exist subsequences* $\{u^{k_n}\}$, $\{v^{k_n}\}$ *and non-negative functions* $\overline{u}, \overline{v} \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ *such that as* $k_n \to \infty$,

$$\begin{aligned} &u^{k_n} \rightharpoonup \overline{u}, \quad v^{k_n} \rightharpoonup \overline{v} \quad \text{in } W_0^{1,2}(\Omega), \\ &u^{k_n} \to u, \quad v^{k_n} \to v \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega, \end{aligned}$$

and

$$w^{k_n} = \alpha u^{k_n} - v^{k_n} \to \overline{w} := \alpha\overline{u} - \overline{v} \quad \text{in } C^{1,\lambda}(\overline{\Omega}) \text{ for each } \lambda \in (0,1). \tag{2.3}$$

2.2. Convergence on A**.** To identify the limit problem satisfied by $\overline{u}$ and $\overline{v}$, we focus first on the strong-interaction region $A \subset \Omega$ and use a blow-up technique to prove that the product $u^k v^k$ converges uniformly to zero in $\overline{A}$ as $k \to \infty$.

Lemma 2.3. *Let* $\varepsilon > 0$. *Then there exists* $k_0 \in \mathbb{N}$ *such that if* $k \geq k_0$ *and* (u^k, v^k) *is a non-negative solution of* (1.3), *then given* $x \in \overline{A}$,

$$u^k(x) \leq \varepsilon \quad \text{or} \quad v^k(x) \leq \varepsilon.$$

Proof. Suppose, for contradiction, that there exist $\varepsilon_0 > 0$ and sequences $k_j \to \infty$, $x_{k_j} \in \overline{A}$ such that

$$u^{k_j}(x_{k_j}) \geq \varepsilon_0 \quad \text{and} \quad v^{k_j}(x_{k_j}) \geq \varepsilon_0.$$

For each j, define $x' = \sqrt{k_j}(x - x_{k_j})$ and sets Ω_j, A_j such that $x' \in \Omega_j$, A_j whenever $x \in \Omega$, A respectively. Then for $x' \in \Omega_j$, the functions U^{k_j}, V^{k_j} defined by $(U^{k_j}, V^{k_j})(x') = (u^{k_j}, v^{k_j})(x)$ satisfy

$$\begin{aligned} -\Delta U^{k_j} &= k_j^{-1}[f(U^{k_j}) - rU^{k_j}V^{k_j}] - \chi_{A_j}U^{k_j}V^{k_j} \quad \text{a.e. in } \Omega_j, \\ -\Delta V^{k_j} &= k_j^{-1}[g(V^{k_j}) - sU^{k_j}V^{k_j}] - \alpha\chi_{A_j}U^{k_j}V^{k_j} \quad \text{a.e. in } \Omega_j, \end{aligned} \tag{2.4}$$

$$0 \in A_j, \quad U^{k_j}(0) \geq \varepsilon_0 \quad \text{and} \quad V^{k_j}(0) \geq \varepsilon_0. \tag{2.5}$$

Now we can assume, without loss of generality, that there exists $\overline{x} \in \overline{A} \subset \Omega$ such that $x_j \to \overline{x}$ as $j \to \infty$. Then $\operatorname{dist}(\overline{x}, \partial\Omega) > 0$, so given an arbitrary compact set $K \subset \mathbb{R}^N$, $K \subset \Omega_j$ for j sufficiently large, and it is immediate from (2.4) and Lemma 2.1(a) that ΔU^{k_j} and ΔV^{k_j} are bounded in $L^\infty(\Omega_j)$ independently of j. So U^{k_j}, V^{k_j} are bounded in $W^{2,p}(K)$ for every $p \in [1,\infty)$ and thus in $C^{1,\lambda}(K)$ for each $\lambda \in (0,1)$. Hence given $\lambda \in (0,1)$, there are subsequences, not relabelled, of U^{k_j}, V^{k_j} that converge strongly in $C^{1,\lambda}(K)$ for each compact set $K \subset \mathbb{R}^N$ to limit functions $U, V \in C^{1,\lambda}(\mathbb{R}^N)$ which satisfy the weak form of the system

$$\begin{aligned} \Delta U - \chi_T UV = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta V - \alpha\chi_T UV = 0 \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{2.6}$$

where T is either a half-space or $\mathbb{R}^N$, depending on whether or not $k_j^{1/2}\mathrm{dist}(x_j, \partial A)$ is bounded independently of j. (See [4] for details explaining the convergence of A_j to a half space in the case that $k_j^{1/2}\mathrm{dist}(x_j, \partial A)$ is bounded.) It follows from (2.5) that the solution (U, V) of (2.6) satisfies

$$0 \le U, V \le M, \quad U(0) \ge \varepsilon_0 \quad \text{and} \quad V(0) \ge \varepsilon_0. \tag{2.7}$$

Now $\alpha U - V$ is bounded on $\mathbb{R}^N$, by (2.7), and harmonic on $\mathbb{R}^N$, by (2.6). So for some constant C, $\alpha U - V \equiv C$ on $\mathbb{R}^N$. Suppose that $C \ge 0$. Then

$$\Delta V = \chi_T V(V + C) \quad \text{in } \mathbb{R}^N. \tag{2.8}$$

Since $\Delta V \ge 0$ and V is bounded on $\mathbb{R}^N$, $V(x) \to \sup_{\mathbb{R}^N} V =: m$ as $|x| \to \infty$ along almost all directions in the unit sphere (in fact, except on a set of capacity zero; see W. K. Hayman and P. B. Kennedy [10, Theorem 3.21]), and in particular, along almost all directions into T, both when T is $\mathbb{R}^N$ and when T is a half-space. Since ΔV must tend to zero in a weak sense along such directions, it follows from (2.8) that $m(m+C) = 0$ and hence $m := \sup V = 0$, since $C \ge 0$ and $m \ge 0$. But this contradicts the fact that $V(0) \ge \varepsilon_0$, by (2.7).

A similar argument shows that if the constant C is negative, then $U \equiv 0$ on $\mathbb{R}^N$, which again contradicts (2.7). □

Here and in the following, we use the notation

$$w^+ := \max(0, w) \quad \text{and} \quad w^- := \min(0, w),$$

so that $w = w^+ + w^-$.

Corollary 2.4. *Let $\overline{u}$, $\overline{v}$ and $\overline{w}$ be as in the statement of Corollary* 2.2. *Then*

$$\overline{u}\,\overline{v} = 0 \quad \textit{a.e. in } \overline{A}, \tag{2.9}$$

$$\alpha\overline{u} = \overline{w}^+ \quad \textit{and} \quad -\overline{v} = \overline{w}^- \quad \textit{a.e. in } \overline{A}. \tag{2.10}$$

Proof. Lemmas 2.3 and 2.1(a) imply that $u^k v^k$ tends to zero uniformly in $\overline{A}$ as $k \to \infty$, which yields (2.9), from which (2.10) follows immediately using the definition $\overline{w} = \alpha\overline{u} - \overline{v}$. □

Corollary 2.2 and Lemma 2.3 together give the following uniform convergence result for the sequence u^{k_n} and v^{k_n} on $\overline{A}$.

Lemma 2.5. *Let $\{u^{k_n}\}$, $\{v^{k_n}\}$ and $\overline{u}$, $\overline{v}$ be as in the statement of Corollary* 2.2. *Then*

$$u^{k_n} \to \overline{u}, \quad v^{k_n} \to \overline{v} \quad \textit{uniformly on } \overline{A}, \tag{2.11}$$

and $\overline{u}$ and $\overline{v}$ are Lipschitz continuous on $\overline{A}$.

Proof. Let $\varepsilon > 0$. Then Lemma 2.3 yields the existence of k_0 such that for each $x \in \overline{A}$ and $k_n \geq k_0$, either $\alpha u^{k_n}(x) \leq \varepsilon$ or $v^{k_n}(x) \leq \varepsilon$. If $w^{k_n}(x) \geq 0$, then $(w^{k_n})^+(x) = w^{k_n}(x)$, $(w^{k_n})^-(x) = 0$ and $\alpha u^{k_n}(x) \geq v^{k_n}(x) \geq 0$, and hence $v^{k_n}(x) \leq \varepsilon$, from which it follows both that $|(w^{k_n})^+(x) - \alpha u^{k_n}(x)| = |v^{k_n}(x)| \leq \varepsilon$ and $|(w^{k_n})^-(x) + v^{k_n}(x)| = |v^{k_n}(x)| \leq \varepsilon$. If $w^{k_n}(x) \leq 0$, then $(w^{k_n})^+(x) = 0$, $(w^{k_n})^-(x) = w^{k_n}(x)$, and similar arguments apply with the rôles of αu^{k_n} and v^{k_n} reversed. Hence as $k_n \to \infty$,

$$(w^{k_n})^+ - \alpha u^{k_n} \to 0 \quad \text{and} \quad (w^{k_n})^- + v^{k_n} \to 0 \quad \text{uniformly on } \overline{A}. \tag{2.12}$$

Since Corollary 2.2 implies that

$$(w^{k_n})^+ - \overline{w}^{\,+} \to 0 \quad \text{and} \quad (w^{k_n})^- - \overline{w}^{\,-} \to 0 \quad \text{uniformly on } \Omega,$$

the uniform convergence (2.11) follows from (2.12) and (2.10). To see that $\overline{u}$ and $\overline{v}$ are Lipschitz continuous on $\overline{A}$, note that for all $x, y \in \Omega$,

$$|\overline{w}^{\,+}(x) - \overline{w}^{\,+}(y)| \leq |\overline{w}(x) - \overline{w}(y)| \quad \text{and} \quad |\overline{w}^{\,-}(x) - \overline{w}^{\,-}(y)| \leq |\overline{w}(x) - \overline{w}(y)|.$$

Thus $\overline{w}^{\,+}$ and $\overline{w}^{\,-}$ are Lipschitz continuous on Ω, since $\overline{w} \in C^{1,\lambda}(\overline{\Omega})$, $\lambda \in (0,1)$, by Corollary 2.2. So the result follows from (2.10). □

We define the boundary function $\psi := \alpha\overline{u} - \overline{v}|_{\partial A}$. Then $\psi \in C^{1,\lambda}(\partial\Omega)$ for each $\lambda \in (0,1)$ and

$$\overline{u} = \alpha^{-1}\psi^+, \quad \overline{v} = -\psi^- \quad \text{on } \partial A. \tag{2.13}$$

Note, in particular, that ψ is Lipschitz continuous on ∂A.

The next lemma identifies the equation satisfied by the limit function $\overline{w}$ in the region A.

Lemma 2.6. *Let $\overline{w}$ be as defined in* (2.3). *Then the restriction of $\overline{w}$ to A, $\overline{w}|_A$, satisfies the equation*

$$-\Delta\overline{w} = \alpha f(\alpha^{-1}\overline{w}^{\,+}) - g(-\overline{w}^{\,-}) \quad \text{a.e. in } A, \tag{2.14}$$

together with the boundary condition

$$\overline{w} = \psi \quad \text{on } \partial A, \tag{2.15}$$

where ψ is as in (2.13).

Proof. Let $\{u^{k_n}\}$, $\{v^{k_n}\}$ be as in the statement of Corollary 2.2. It follows from (2.2) that for each $\phi \in W_0^{1,2}(A)$,

$$\int_A \nabla w^{k_n} \cdot \nabla\phi\, dx = \int_A [\alpha f(u^{k_n}) - g(v^{k_n}) - (\alpha s - r)u^{k_n}v^{k_n}]\phi\, dx,$$

and then letting $k_n \to \infty$ using Corollary 2.2 gives that

$$\int_A \nabla \overline{w} \cdot \nabla \phi \, dx = \int_A [\alpha f(\overline{u}) - g(\overline{v}) - (\alpha s - r)\overline{u}\,\overline{v}]\phi \, dx. \tag{2.16}$$

Then (2.14) follows from (2.16), (2.9) and (2.10). (2.15) is immediate from the definition of ψ in (2.13). □

2.3. Convergence on $\Omega \setminus \overline{A}$. In the region $\Omega \setminus \overline{A}$, the strong competition terms in (1.3) are absent and we can easily pass to the limit as $k \to \infty$ to identify the limit problem satisfied by $\overline{u}$ and $\overline{v}$.

Lemma 2.7. *Let $\overline{u}$, $\overline{v}$ be as in the statement of Corollary 2.2. Then the restrictions $\overline{u}|_{\overline{\Omega}\setminus A}$ and $\overline{v}|_{\overline{\Omega}\setminus A}$ belong to $C^2(\Omega \setminus \overline{A}) \cap C(\overline{\Omega} \setminus A)$ and satisfy the system*

$$\begin{aligned} -\Delta \overline{u} &= f(\overline{u}) - s\,\overline{u}\,\overline{v} \quad \text{in } \Omega \setminus \overline{A}, \\ -\Delta \overline{v} &= g(\overline{v}) - r\,\overline{u}\,\overline{v} \quad \text{in } \Omega \setminus \overline{A}, \end{aligned} \tag{2.17}$$

together with the boundary conditions

$$\begin{aligned} &\overline{u} = \overline{v} = 0 && \text{on } \partial\Omega, \\ &\overline{u} = \alpha^{-1}\psi^+, \quad \overline{v} = -\psi^- && \text{on } \partial A, \end{aligned} \tag{2.18}$$

where ψ is as in (2.13).

Proof. Let $\{u^{k_n}\}$ and $\{v^{k_n}\}$ be as in the statement of Corollary 2.2. It follows from (1.3) that, for each $\phi \in W_0^{1,2}(\Omega \setminus \overline{A})$,

$$\begin{aligned} \int_{\Omega\setminus\overline{A}} \nabla u^{k_n} \cdot \nabla \phi \, dx &= \int_{\Omega\setminus\overline{A}} [f(u^{k_n}) - s u^{k_n} v^{k_n}]\phi \, dx, \\ \int_{\Omega\setminus\overline{A}} \nabla v^{k_n} \cdot \nabla \phi \, dx &= \int_{\Omega\setminus\overline{A}} [g(v^{k_n}) - r u^{k_n} v^{k_n}]\phi \, dx, \end{aligned}$$

and passing to the limit as $k_n \to \infty$ using Corollary 2.2 then yields

$$\begin{aligned} \int_{\Omega\setminus\overline{A}} \nabla \overline{u} \cdot \nabla \phi \, dx &= \int_{\Omega\setminus\overline{A}} [f(\overline{u}) - s\overline{u}\,\overline{v}]\phi \, dx, \\ \int_{\Omega\setminus\overline{A}} \nabla \overline{v} \cdot \nabla \phi \, dx &= \int_{\Omega\setminus\overline{A}} [g(\overline{v}) - r\overline{u}\,\overline{v}]\phi \, dx. \end{aligned}$$

Since $\overline{u}, \overline{v} \in L^\infty(\Omega \setminus \overline{A})$, it then follows by standard regularity theory that $\overline{u}, \overline{v} \in C^2(\Omega \setminus \overline{A})$ and satisfy (2.17). The fact that $\overline{u}, \overline{v} \in C(\overline{\Omega} \setminus A)$ follows from the continuity of the boundary data (2.18), by [9, Corollary 8.28]. □

2.4. Conditions of the limit problem on ∂A. It follows from Lemmas 2.7 and 2.5 that $\overline{u}$ and $\overline{v}$ are continuous on the whole of $\overline{\Omega}$, and in particular, across ∂A, with

$$\overline{u} = \alpha^{-1}\psi^+, \quad \overline{v} = -\psi^- \quad \text{on } \partial A.$$

A further condition on ∂A arises from the fact that $\alpha\overline{u} - \overline{v} \in C^1(\overline{\Omega})$, by Corollary 2.2. The normal derivative of $\alpha\overline{u} - \overline{v}$ is thus continuous across ∂A, which has implications for the normal derivatives of $\overline{u}$ and $\overline{v}$ on either side of ∂A whenever these derivatives exist. To give a meaning to the normal derivatives of $\overline{u}$ and $\overline{v}$ on the side of ∂A *into* A, we need the following simple lemma.

Lemma 2.8. *Let $B \subset \mathbb{R}^N$ be a bounded domain and suppose that $w \in C^1(B)$. Then for each $x_0 \in B$ and each direction $v \in S^{n-1} = \{\xi : |\xi| = 1\}$, the directional derivatives of w^+ and w^- at x_0 in direction ξ exist.*

Proof. We will consider w^+; similar arguments apply for w^-. If $w(x_0) \neq 0$, then w^+ is either identically equal to w on a neighbourhood of x_0 or identically equal to 0 on a neighbourhood of x_0, so the result follows from the fact that w is continuously differentiable.

Suppose now that $w(x_0) = 0$, and let $\xi \in S^{n-1}$. If $\nabla w(x_0) \cdot \xi > 0$, then $w(x_0 + t\xi) > 0$ for $t > 0$ sufficiently small, and so for such t

$$\frac{w^+(x_0 + t\xi) - w^+(x_0)}{t} = \frac{w(x_0 + t\xi) - w(x_0)}{t} \to \nabla w(x_0) \cdot \xi \quad \text{as } t \to 0.$$

If $\nabla w(x_0) \cdot \xi < 0$, then $w(x_0 + t\xi) < 0$ for $t > 0$ sufficiently small, so

$$\frac{w^+(x_0 + t\xi) - w^+(x_0)}{t} = \frac{0 - 0}{t} = 0 \quad \text{for all } t > 0,$$

and if $\nabla w(x_0) \cdot \xi = 0$, then for $t > 0$,

$$\begin{aligned} \frac{w^+(x_0 + t\xi) - w^+(x_0)}{t} &= \begin{cases} \dfrac{w(x_0 + t\xi) - w(x_0)}{t} & \text{if } w(x_0 + t\xi) \geq 0, \\ 0 & \text{if } w(x_0 + t\xi) < 0, \end{cases} \\ &\to 0 = \nabla w(x_0) \cdot \xi \quad \text{as } t \to 0. \end{aligned}$$

So the directional derivative of w^+ at x_0 exists for all directions $\xi \in S^{n-1}$. □

The existence at each point of ∂A of the normal derivatives $\partial\overline{u}/\partial\nu$, $\partial\overline{v}/\partial\nu$ in the normal direction *into* A is then immediate from Lemma 2.8 and (2.10).

To give sense to the corresponding normal derivatives almost everywhere on ∂A *from inside* $\Omega \setminus A$, note that we can consider instead the question of the existence of such derivatives for a function y such that

$$\Delta y = 0 \quad \text{in } \Omega \setminus \overline{A}, \qquad y = \alpha^{-1}\psi^+ \quad \text{on } \partial A,$$

where $y = \overline{u} - z$ and $z \in C^1(\overline{\Omega} \setminus A)$ satisfies $\Delta z = f(\overline{u}) - s\overline{u}\,\overline{v}$ in $\Omega \setminus \overline{A}$ and $z = 0$ on $\partial(\Omega \setminus \overline{A})$, since the existence of $\partial z/\partial\nu$ on ∂A is clear and so the existence of $\partial\overline{u}/\partial\nu$ follows from that of $\partial y/\partial\nu$. Recall from Lemma 2.5 and (2.13) that $\alpha^{-1}\psi^+$ is Lipschitz continuous on ∂A and thus in $W^{1,p}(\partial A)$ for each $p \in [1, \infty)$. To prove that $\partial y/\partial\nu$ exists almost everywhere on ∂A is probably folklore, but since we could not find a reference, we include a short proof in the Appendix.

Note that it is much easier to prove that $\partial y/\partial\nu$ exists at points $\overline{x} \in \partial A$ when $\psi(\overline{x}) \neq 0$, since ψ^+ and ψ^- are then $C^{1,\lambda}$ near $\overline{x}$.

Now that meaning has been given to the normal derivatives of $\overline{u}$, $\overline{v}$ almost everywhere on ∂A, the fact that the normal derivative of $\alpha\overline{u} - \overline{v}$ is continuous across ∂A immediately implies that the jump in $\partial\overline{v}/\partial\nu$ across ∂A must equal α times the jump in $\partial\overline{u}/\partial\nu$ across ∂A. We thus have the following conditions for $\overline{u}$ and $\overline{v}$ on ∂A.

Lemma 2.9. *Let $\overline{u}$ and $\overline{v}$ be as in the statement of Corollary* 2.2. *Then*

(a) *$\overline{u}$ and $\overline{v}$ are continuous across ∂A;*

(b) *for almost every $x \in \partial A$ (in the sense of $(n-1)$-dimensional Lebesgue measure),*
$$\alpha\left[\frac{\partial\overline{u}}{\partial\nu}(x)\right] = \left[\frac{\partial\overline{v}}{\partial\nu}(x)\right],$$
where $[\partial\cdot/\partial\nu]$ denotes the difference between the normal derivative into A and the normal derivative from inside $\Omega\setminus\overline{A}$.

3. The limit problem

We summarise the limit equations derived in Section 2. The pair $(\overline{u},\overline{v})$ given by Corollary 2.2 and the function $\overline{w} = \alpha\overline{u} - \overline{v}$ satisfy the problem

$$(\mathrm{P}^{\mathrm{limit}}) \qquad \begin{cases} -\Delta\overline{w} = \alpha f(\alpha^{-1}\overline{w}^{\,+}) - g(-\overline{w}^{\,-}) & \text{a.e. in } A,\\ \overline{w} = \psi & \text{on } \partial A,\\ \overline{u} = \alpha^{-1}\overline{w}^{\,+}, \quad \overline{v} = -\overline{w}^{\,-} & \text{a.e. in } A,\\ -\Delta\overline{u} = f(\overline{u}) - s\,\overline{u}\,\overline{v} & \text{in } \Omega\setminus\overline{A},\\ -\Delta\overline{v} = g(\overline{v}) - r\,\overline{u}\,\overline{v} & \text{in } \Omega\setminus\overline{A},\\ \overline{u} = \overline{v} = 0 & \text{on } \partial\Omega,\\ \overline{u} = \alpha^{-1}\psi^+, \quad \overline{v} = -\psi^- & \text{on } \partial A,\\ \alpha\dfrac{\partial\overline{u}}{\partial\nu} - \dfrac{\partial\overline{w}^{\,+}}{\partial\nu} = \dfrac{\partial\overline{v}}{\partial\nu} - \dfrac{\partial(-\overline{w}^{\,-})}{\partial\nu} & \text{on } \partial A,\\ \overline{u} \geq 0, \quad \overline{v} \geq 0 & \text{in } \Omega, \end{cases}$$

where the boundary function ψ is as introduced in (2.13) and ν denotes the normal direction to ∂A pointing into A.

It would be of interest to understand this rather complicated limit problem better. In particular, the question of the existence of solutions of $(\mathrm{P}^{\mathrm{limit}})$ and the issue of which of these solutions arise as a limit of coexistence states of (1.3) for large k are important both mathematically and from the point of view of applications of (1.3) in population dynamics. Here we make some initial observations.

Remark 3.1. We briefly consider conditions on the limit problem which guarantee the existence of a coexistence state for all large k. Note first that for such a state to exist, it is prerequisite that there exist strictly positive solutions of both $-\Delta u = f(u)$ and $-\Delta v = g(v)$ on Ω with Dirichlet boundary conditions, because a coexistence state for (1.3) and a pair of large positive constants respectively yield sub and super-solutions for these equations. We assume that $f(y) = ay(1-y)$ where $a > \lambda_1$, the least eigenvalue $-\Delta$ on Ω with Dirichlet conditions on $\partial\Omega$, and that g has a similar form. This ensures that there is a unique solution of (1.3) of the form $(\widehat{u}, 0)$ with $\widehat{u}$ strictly positive in Ω, and this solution is stable in the subspace with $v = 0$, and likewise, that there is a unique solution of (1.3) of the form $(0, \widehat{v})$ with $\widehat{v}$ strictly positive in Ω, and this solution is stable in the subspace with $u = 0$. Our argument is in fact valid for more general nonlinearities provided that these properties hold; otherwise, the problem seems more complicated.

If the solutions $(\widehat{u}, 0)$ and $(0, \widehat{v})$ are both linearly stable or both linearly unstable for all large k, degree theory in cones can be used, much as in [5], to prove the existence of a coexistence state. It thus remains to check when the stability condition holds. Since the linearisation of the mapping

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} -\Delta u - f(u) + suv + k\chi_A uv \\ -\Delta v - g(v) + ruv + k\alpha\chi_A uv \end{pmatrix}, \quad u, v \in W_0^{1,2}(\Omega),$$

about the state $(\widehat{u}, 0)$ has the form

$$\mathcal{L}(\widehat{u}, 0)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\Delta - f'(\widehat{u}) & s\widehat{u} + k\chi_A\widehat{u} \\ 0 & -\Delta - g'(0) + r\widehat{u} + k\alpha\chi_A\widehat{u} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix},$$

for $y, z \in W_0^{1,2}(\Omega)$, the spectrum of this linearisation is the union of the spectra of $-\Delta - f'(\widehat{u})$ and $-\Delta - g'(0) + r\widehat{u} + k\alpha\chi_A\widehat{u}$ in $W_0^{1,2}(\Omega)$. Since $(\widehat{u}, 0)$ is a stable solution of (1.3) in the subspace $v = 0$, the spectrum of $-\Delta - f'(\widehat{u})$ is strictly positive and it thus follows that $(\widehat{u}, 0)$ is a linearly stable solution of the system (1.3) in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ if the least eigenvalue $\lambda_1(k)$ of the eigenvalue problem

$$\begin{aligned} -\Delta z - g'(0)z + r\widehat{u}z + k\alpha\chi_A\widehat{u}z &= \lambda z \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

is strictly positive, and is linearly unstable if $\lambda_1(k)$ is strictly negative. Using the variational characterisation

$$\lambda_1(k) = \inf_{z \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla z|^2 - g'(0)z^2 + r\widehat{u}z^2 + k\alpha\chi_A\widehat{u}z^2 \, dx}{\int_\Omega z^2 \, dx},$$

it is easy to see that $\lambda_1(k)$ is strictly increasing in k and that as $k \to \infty$, $\lambda_1(k) \to \widetilde{\lambda}_1$, where $\widetilde{\lambda}_1$ is the least eigenvalue of the operator $-\Delta - g'(0)I + r\widehat{u}I$ on $\Omega \setminus \overline{A}$ with Dirichlet boundary conditions on $\partial A \cup \partial\Omega$. Clearly there is an analogous result for the solution $(0,\widehat{v})$ where $\lambda_2(k)$ is the least eigenvalue of the eigenvalue problem

$$\begin{aligned} -\Delta y - f'(0)y + s\widehat{v}y + k\chi_A\widehat{v}y &= \lambda\, y \quad \text{in } \Omega, \\ y &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

$\lambda_2(k) \to \widetilde{\lambda}_2$ as $k \to \infty$ and $\widetilde{\lambda}_2$ is the least eigenvalue of $\Delta - f'(0)I + s\widehat{v}I$ on $\Omega \setminus \overline{A}$ with Dirichlet boundary conditions on $\partial A \cup \partial\Omega$.

Hence there is coexistence for all large k if either $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are both non-positive or both strictly positive. In the former case, both $\lambda_1(k)$ and $\lambda_2(k)$ are strictly negative for large k, since $\lambda_1(k)$ and $\lambda_2(k)$ are each strictly increasing in k, and so both $(\widehat{u},0)$ and $(0,\widehat{v})$ are linearly stable for all large k; in the latter, both $\lambda_1(k)$ and $\lambda_2(k)$ are strictly positive for large k and so both $(\widehat{u},0)$ and $(0,\widehat{v})$ are linearly unstable for large k.

Remark 3.2. It is unclear whether or not $(\overline{u},\overline{v})$ being a limit of coexistence states as $k \to \infty$ implies either that $\overline{u}$ and $\overline{v}$ are both strictly positive on $\Omega \setminus A$ or that $\overline{w}$ must change sign on $\overline{A}$. Note that if $\Omega \setminus A$ is connected, then either $\overline{u} > 0$ on $\Omega \setminus \overline{A}$ or $\overline{u} \equiv 0$ on $\Omega \setminus \overline{A}$, and either $\overline{v} > 0$ on $\Omega \setminus \overline{A}$ or $\overline{v} \equiv 0$ on $\Omega \setminus \overline{A}$, by the maximum principle. See also Remark 3.4 below.

Remark 3.3. It is difficult to have a solution $(\overline{u},\overline{v})$ of $(\mathrm{P}^{\mathrm{limit}})$ that is a limit of coexistence states (u^k, v^k) with $\overline{u}$ vanishing identically on Ω but $\overline{v}$ not vanishing identically on Ω. In fact, this can only happen if $f'(0)$ is an eigenvalue of the linear problem

$$\begin{aligned} -\Delta y + s\overline{v}y &= \lambda\, y \quad \text{in } \Omega \setminus A, \\ y &= 0 \qquad \text{on } \partial(\Omega \setminus A), \end{aligned}$$

to which there corresponds a non-negative eigenfunction. This can be proved via arguments from [6] using limits of $u^k_\sharp := u^k/\|u^k\|_\infty$ instead of u^k; see, for example, ideas in the proof of [6, Theorem 2.2]. Note that since $(0,\overline{v})$ is a solution of $(\mathrm{P}^{\mathrm{limit}})$ *whenever* $\overline{v}$ is a positive solution of $-\Delta\overline{v} = g(\overline{v})$ on Ω with $\overline{v} = 0$ on $\partial\Omega$, this shows, in particular, that not all solutions of $(\mathrm{P}^{\mathrm{limit}})$ are limits of coexistence states. The analogous result for $\overline{v}$ is that in a solution $(\overline{u},\overline{v})$ of $(\mathrm{P}^{\mathrm{limit}})$ that is a limit of coexistence states, $\overline{v}$ can only vanish identically on Ω with $\overline{u}$ not vanishing identically on Ω if $g'(0)$ is an eigenvalue of the linear problem

$$\begin{aligned} -\Delta z + r\overline{u}z &= \lambda\, z \quad \text{in } \Omega \setminus A, \\ z &= 0 \qquad \text{on } \partial(\Omega \setminus A), \end{aligned}$$

to which there corresponds a non-negative eigenfunction. From the point of view of population dynamics, this suggests that in the strong-competition limit, it is rare for one species to survive somewhere in Ω if the other species vanishes everywhere in Ω.

If $\overline{u} = \overline{v} \equiv 0$ on Ω and $(\overline{u}, \overline{v})$ is a limit of coexistence solutions (u^k, v^k), one can obtain new limit systems, again using the ideas in [6]. As in [6], there are two cases, depending on whether or not $k\|u^k\|_\infty$ and $k\|v^k\|_\infty$ are bounded as $k \to \infty$. There is an interesting point here. In [6], the condition that $(f'(0), g'(0))$ does not belong to the Fučik spectrum (that is, that the only solution of the equation

$$\begin{aligned} -\Delta z &= f'(0)z^+ + g'(0)z^- && \text{in } \Omega, \\ z &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

is $z \equiv 0$) appears naturally in the case when $k\|u^k\|_\infty$ and $k\|v^k\|_\infty \to \infty$ as $k \to \infty$ (see, for instance, [6, Theorems 2.2 and 2.3]). Our analogue is that there is no non-trivial solution of the system

$$\begin{aligned} -\Delta u &= f'(0)u && \text{in } \Omega \setminus A, \\ -\Delta v &= g'(0)v && \text{in } \Omega \setminus A, \\ -\Delta w &= f'(0)w^+ + g'(0)w^- && \text{in } A, \\ u &= \alpha^{-1}w^+,\ v = -w^- && \text{on } \overline{A}, \\ &u \text{ and } v \text{ are continuous} && \text{on } \partial A, \\ &\alpha u - v \text{ is } C^1 && \text{across } \partial A, \\ u &= v = 0 && \text{on } \partial\Omega, \\ u,&v \geq 0 && \text{on } \Omega. \end{aligned}$$

This appears much more complicated to study than (3.1). There is one simple case when this condition holds, namely if $f'(0) > \lambda_1$, $g'(0) > \lambda_1$, where λ_1 is the least eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions, and A is connected with sufficiently small diameter that $f'(0)$ and $g'(0)$ are also greater than the least eigenvalue of $-\Delta$ on $\Omega \setminus A$. Then there is no non-trivial nonnegative solution of $-\Delta u = f'(0)u$ or $-\Delta v = g'(0)v$ on $\Omega \setminus A$, so that $u = v \equiv 0$ on $\Omega \setminus A$ and then $w \equiv 0$ on A by unique continuation.

Remark 3.4. There may sometimes be solutions $(\overline{u}, \overline{v})$ of $(\mathrm{P}^{\mathrm{limit}})$ where $\overline{u}$ vanishes either on A or on $\Omega \setminus A$ but not on both, and similar solutions with $\overline{v}$ vanishing either on A or on $\Omega \setminus A$. We give two illustrative examples.

Example 3.5. Suppose that $f = g$, $r = s$, $\alpha = 1$ and $f'(0) > \lambda_1(\Omega \setminus \overline{A})$, the least eigenvalue of $-\Delta$ on $\Omega \setminus \overline{A}$ with Dirichlet boundary conditions. Then there is a solution $(\overline{u}, \overline{v})$ of $(\mathrm{P}^{\mathrm{limit}})$ with $\overline{u} = \overline{v} \equiv 0$ in A and $\overline{u} \equiv \overline{v} > 0$ in $\Omega \setminus \overline{A}$, since on $\Omega\setminus\overline{A}$, $\overline{u}=\overline{v}$ can be taken to equal the unique positive solution of the equation

$$\begin{aligned} -\Delta u &= f(u) - ru^2 && \text{in } \Omega\setminus\overline{A}, \\ u &= 0 && \text{on } \partial\Omega\cup\partial A, \end{aligned}$$

and the normal derivative $(\partial/\partial\nu)(\overline{u}-\overline{v})=0$ on both sides of the boundary ∂A.

(Note, on the other hand, that if $\overline{u}$ and $\overline{v}$ both vanish on $\Omega\setminus A$ instead of on A, then it can easily be proved by unique continuation that $\overline{u}$ and $\overline{v}$ both vanish identically on Ω, since both $\overline{w}$ and $\nabla\overline{w}$ then vanish on ∂A, implying that $\overline{w}$ is identically zero on A.)

Example 3.6. Let Ω be an origin-centered ball, radius R, and restrict attention to radial functions. Suppose that $f=g$ are such that $f'(0)=g'(0)$ is the second eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. Then bifurcation from the eigenvalue $f'(0)$ in the space of radial functions yields that the equation

$$\begin{aligned} -\Delta w &= f(w) && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has a radial solution close to the second radial eigenfunction of $-\Delta$, and hence that is positive at the centre of Ω and changes sign exactly once in the radial direction, at $r=r_0\in(0,R)$. Choose the set A such that

$$\{x:|x|\le r_0\}\subset\subset A\subset\subset\Omega,$$

and let $\overline{u}=\alpha^{-1}w^{+}$, $\overline{v}=-w^{-}$ in Ω. Then $(\overline{u},\overline{v})$ is a solution of $(\mathrm{P}^{\mathrm{limit}})$ with $\overline{u}\equiv 0$ on $\Omega\setminus\overline{A}$, but $\overline{u}$ does not vanish on A.

Note that it is not clear whether or not the solutions of $(\mathrm{P}^{\mathrm{limit}})$ in Examples 3.5 and 3.6 can arise as the limit of coexistence states of (1.3). The fact that any solution of $(\mathrm{P}^{\mathrm{limit}})$ for which $\overline{u}$ and $\overline{v}$ both vanish on $\Omega\setminus A$ must vanish identically on Ω, as remarked at the end of Example 3.5, has a natural interpretation if $(\overline{u},\overline{v})$ is the limit of coexistence states since intuitively one would not expect both species to choose to avoid the region $\Omega\setminus A$ and concentrate in the strong-competition region A. The existence of the solutions in Example 3.5 that both vanish on A but not on $\Omega\setminus A$ seems biologically reasonable because one might expect species to be able to coexist in the $k\to\infty$ limit by concentrating in the area $\Omega\setminus A$ and thus such solutions in fact to be the limit of coexistence states, but we have not proved this. It is more difficult to understand the solutions in Example 3.6 biologically and tempting to conjecture that these solutions cannot be the limit of coexistence states.

4. Appendix

We prove that if D is a bounded $C^{1,\mu}$-domain for some $\mu > 0$ and $f \in W^{1,p}(\partial D)$ for some $1 < p < \infty$, then the solution Lf of

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } D, \\ u &= f \quad \text{on } \partial D, \end{aligned}$$

has normal derivative *almost everywhere* on ∂D. We stress that this is almost certainly folklore, but we could not find a reference.

Denote the inward normal to ∂D at $x \in \partial D$ by ν_x and note that Lf is smooth in D. Given $\delta > 0$, to be fixed later, and $x \in \partial D$, let

$$T_x = \{x + t\,\nu_x : 0 < t \leq \delta\},$$

and given a function $q : \overline{\Omega} \to \mathbb{R}$, let $q^*(x) = \sup\{|q(t)| : t \in T_x\}$. By [8, Theorem 1.7], there exist $\delta, K > 0$, independent of f, such that

$$\left\| \left(\frac{\partial (Lf)}{\partial \nu_x} \right)^* \right\|_{L^p(\partial D)} \leq K \|f\|_{W^{1,p}(\partial D)}. \tag{4.1}$$

(This is, in fact, a special case of their maximal function estimate.)

Now choose $f_j \in W^{1,p}(\partial D)$ smooth (at least in $C^{1,\lambda}(\partial\Omega)$ for some $\lambda > 0$) such that

$$\|f_j - f\|_{W^{1,p}(\partial D)} \to 0 \quad \text{as } j \to \infty.$$

By [9, Theorem 8.34], $Lf_j \in C^1(\overline{\Omega})$, and by the estimate (4.1),

$$\left\| \left(\frac{\partial}{\partial \nu_x} (L(f_j - f)) \right)^* \right\|_{L^p(\partial D)} \to 0 \quad \text{as } j \to \infty.$$

We can then choose a subsequence so that

$$\left(\frac{\partial}{\partial \nu_x} (L(f_j - f)) \right)^* (x) \to 0 \quad \text{as } j \to \infty \text{ for each } x \in S,$$

where $S \subset \partial D$ and $\partial D \setminus S$ has zero $(n-1)$-dimensional Lebesgue measure.

It remains to prove that the inward normal derivative of Lf exists at x for each $x \in S$. To see this, first note that it suffices to prove that given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$t^{-1}(Lf(x + t\nu_x) - Lf(x))$$

lies within a set of diameter at most ε whenever $0 < t < \eta$. By the mean value theorem, this will follow if we can show that given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\frac{\partial}{\partial \nu_x} Lf(x + t\nu_x)$$

lies in a set of diameter at most ε whenever $0 < t < \eta$. Now, by (4.1),

$$\frac{\partial}{\partial \nu_x} L(f_j - f)(x + t\nu_x) \to 0 \quad \text{as } j \to \infty$$

uniformly in $0 < t \le \delta$. Choose j_0 such that

$$\left| \frac{\partial}{\partial \nu_x} L(f_{j_0} - f)(x + t\nu_x) \right| < \frac{\varepsilon}{2} \quad \text{for all } 0 < t \le \delta. \tag{4.2}$$

But $Lf_{j_0} \in C^1(\overline{\Omega})$, and thus by shrinking δ if necessary, we can assume that

$$\left| \frac{\partial}{\partial \nu_x} Lf_{j_0}(x + t\nu_x) - \frac{\partial}{\partial \nu_x} Lf_{j_0}(x) \right| < \frac{\varepsilon}{2} \quad \text{for all } 0 < t \le \delta. \tag{4.3}$$

Hence it follows from (4.2) and (4.3) that

$$\left| \frac{\partial}{\partial \nu_x} Lf(x + t\nu_x) - \frac{\partial}{\partial \nu_x} L(f_{j_0})(x) \right| < \varepsilon \quad \text{for all } 0 < t \le \delta,$$

as required. □

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