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RADIAL SYMMETRY OF n-MODE POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN A DISC AND ITS APPLICATION TO THE HÉNON EQUATION

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Abstract. Let $f \in C((0,1) \times (0,\infty), \mathbb{R})$ and $n \in \mathbb{N}$ with $n \geq 2$ such that for each $u \in (0,\infty)$, $r \mapsto r^{2-2n} f(r,u)\colon (0,1) \to \mathbb{R}$ is nonincreasing and let $D = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$. We show that each positive solution of

$$\Delta u + f(|x|, u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

which satisfies $u(r,\theta) = u(r, \theta + 2\pi/n)$ by the polar coordinates is radially symmetric and $u_r(|x|) < 0$ for each $r = |x| \in (0,1)$. We apply our result to the Hénon equation.

1. Introduction

The radial symmetry of positive solutions of the problem

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

has been studied by many researchers, where B is the open unit ball in $\mathbb{R}^N$ ($N \geq 2$) and $f \in C((0,1) \times (0,\infty), \mathbb{R})$. Gidas, Ni and Nirenberg [12] showed that if f satisfies that for each $u \in (0,\infty)$, $r \mapsto f(r,u)\colon (0,1) \to \mathbb{R}$ is nonincreasing, then

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any $C^2(\overline{B})$-positive solution is radially symmetric. For related results, we refer to [1], [4]–[10], [12], [13], [17]–[20], [22]–[27], [29], [30]. For recent developments of the symmetry of positive solutions of (1.1), see [2], [3], [15], [28], [33] and references therein.

In this paper, we give a radial symmetry result for an n-mode positive solution of

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D \setminus \{0\}, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.2}$$

where $D = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$. We say a function $u\colon \overline{D} \to \mathbb{R}$ is n-mode with $n \in \mathbb{N}$ if it is $2\pi/n$ rotationally invariant, i.e.

$$u(r, \theta) = u\left(r, \theta + \frac{2\pi}{n}\right) \quad \text{for each } (r, \theta) \in [0, 1] \times \mathbb{R}$$

by the polar coordinates. We say u is of class C^n ($n \in \mathbb{N}$) at the origin if u is of class C^{n-1} in a neighbourhood of the origin and each $(n-1)$-th partial derivative is totally differentiable at the origin.

Now, we show our first result.

Theorem 1.1. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $f \in C((0, 1) \times (0, \infty), \mathbb{R})$ such that*:

(a) *for each $u \in (0, \infty)$, $r \mapsto r^{2-2n} f(r, u)\colon (0, 1) \to \mathbb{R}$ is nonincreasing,*
(b) *for each $r_0 \in (0, 1)$ and $M \in (0, \infty)$,*

$$\sup \left\{ \left| \frac{f(r, u_1) - f(r, u_2)}{u_1 - u_2} \right| : (r, u_1, u_2) \in (r_0, 1) \times (0, M]^2, u_1 \neq u_2 \right\} < \infty.$$

Let $u \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ be an n-mode positive solution of (1.2). *Assume that u is of class C^n at the origin or $u(0) > u(x)$ for all $x \in \overline{D} \setminus \{0\}$. Then u is radially symmetric and $u_r(|x|) < 0$ for $r = |x| \in (0, 1)$.*

Remark 1.2. The case $n = 1$ corresponds to Gidas–Ni–Nirenberg's theorem.

Under the condition $u(x) \to \infty$ as $|x| \to 0$, we can show the following.

Theorem 1.3. *Let n and f be as in Theorem* 1.1. *Let $u \in C^2(D \setminus \{0\}) \cap C(\overline{D} \setminus \{0\})$ be an n-mode positive solution of*

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D \setminus \{0\}, \\ u = 0 & \text{on } \partial D, \\ \displaystyle\lim_{|x| \to 0} u(x) = \infty. \end{cases} \tag{1.3}$$

Then u is radially symmetric and $u_r(|x|) < 0$ for $r = |x| \in (0, 1)$.

As an application, we consider the Hénon equation

$$\begin{cases} \Delta u + |x|^\alpha |u|^{p-2} u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases} \tag{1.4}$$

with $\alpha \in (0, \infty)$ and $p \in (2, \infty)$. Smets, Willem and Su [32, Theorem 6] showed that the problem has a nonradial positive solution for sufficiently large $\alpha > 0$ (including the higher dimensional case). We show that the solution they obtained does not have an n-mode symmetry in some sense and that the number of the nonradial positive solutions tends to infinity as $\alpha \to \infty$.

For each $\alpha > 0$, we denote by $\lceil \alpha \rceil$ and $\lfloor \alpha \rfloor$ the smallest integer greater than or equal to α and the largest integer less than or equal to α, respectively. For each $\alpha \geq 0$ and $p > 2$, we set

$$R_{\alpha,p}(u) = \frac{\displaystyle\int_D |\nabla u|^2 \, dx}{\left(\displaystyle\int_D |x|^\alpha |u|^p \, dx \right)^{2/p}} \quad \text{for each } u \in H_0^1(D) \setminus \{0\}.$$

THEOREM 1.4. *There hold the following*:

(a) *If $\alpha \in (0, \infty)$, $p \in (2, \infty)$ and u is an n-mode, positive solution of* (1.4) *with $n \geq 1 + \lceil \alpha/2 \rceil$, then u is radially symmetric.*

(b) *For each $\alpha, p \in (2, \infty)$, if $n_\alpha \geq 1$ then problem* (1.4) *has a nonradial, n-mode positive solution u_n for $n = 1, \ldots, n_\alpha$ such that*

$$R_{\alpha,p}(u_1) < \ldots < R_{\alpha,p}(u_{n_\alpha}), \tag{1.5}$$

where n_α is the greatest integer less than

$$\left(\frac{\alpha + 2}{2\alpha} \right)^{4/(p-2)} \left(\frac{\alpha - 2}{\alpha} \right)^{2\alpha/(p-2)} \left(1 + \frac{\alpha}{2} \right). \tag{1.6}$$

In particular, there hold the following:

(i) *For each $\alpha \in (2, \infty)$, if $p \in (2, \infty)$ is large enough, then $n_\alpha = \lceil \alpha/2 \rceil$, that is, problem* (1.4) *has a nonradial, n-mode positive solution for $n = 1, \ldots, \lceil \alpha/2 \rceil (= n_\alpha)$ satisfying* (1.5).

(ii) *For each $p \in (2, \infty)$ and $m \in \mathbb{N}$, if $\alpha \in (2, \infty)$ is large enough, problem* (1.4) *has a nonradial, n-mode positive solution u_n for $n = 1, \ldots, m$ such that $R_{\alpha,p}(u_1) < \ldots < R_{\alpha,p}(u_m)$. In particular, for each $p \in (2, \infty)$, the number of nonradial positive solutions of* (1.4) *tends to infinity as $\alpha \to \infty$.*

REMARK 1.5. In the theorem above, we can easily see $n_\alpha \leq \lceil \alpha/2 \rceil$.

REMARK 1.6. In the case that $\alpha > 0$ is sufficiently small, Kajikiya [16, Theorem 2.4] showed that the least energy solution of (1.4) is radially symmetric.

We study further the symmetry of n-mode positive solutions for (1.1). In some of radial symmetry results for positive solutions of semilinear elliptic equations, geometry plays an important role. Naito, Nishimoto and Suzuki [25] considered the case that $(1-r^2)^2 f(r,u)\colon (0,1)\to\mathbb{R}$ is decreasing for each $u\in(0,\infty)$. Using hyperbolic geometry, they showed each positive solution of (1.1) is radially symmetric; see also [26] for the higher dimensional case. The authors [31] studied the case that $a\in(-1,1]$ and $(1+ar^2)^2 f(r,u)\colon(0,1)\to\mathbb{R}$ is nonincreasing for each $u\in(0,\infty)$, and they showed that each positive solution (1.1) is radially symmetric. We generalize Theorems 1.1 and 1.3 by using elliptic and hyperbolic geometry. In the following, the case $a=0$ corresponds to Theorem 1.1.

Theorem 1.7. *Let $n\in\mathbb{N}$ with $n\geq 2$, $a\in[-1,1]$ and $f\in C((0,1)\times(0,\infty),\mathbb{R})$ such that*

(a) *for each $u\in(0,\infty)$, $r\mapsto(1+ar^{2n})^2 r^{2-2n} f(r,u)\colon(0,1)\to\mathbb{R}$ is nonincreasing in the case of $a\in(-1,1]$, and it is decreasing in the case of $a=-1$,*

(b) *for each $r_0\in(0,1)$ and $M\in(0,\infty)$,*

$$\sup\left\{\left|\frac{f(r,u_1)-f(r,u_2)}{u_1-u_2}\right| : (r,u_1,u_2)\in(r_0,1)\times(0,M]^2, u_1\neq u_2\right\}<\infty.$$

Let $u\in C^2(D\setminus\{0\})\cap C(\overline{D})$ be an n-mode positive solution of (1.2)*. Assume that u is of class C^n at the origin or $u(0)>u(x)$ for all $x\in\overline{D}\setminus\{0\}$. Then u is radially symmetric and $u_r(|x|)<0$ for $r=|x|\in(0,1)$.*

Remark 1.8. The case $n=1$ and $a=-1$ corresponds to [25, Theorem 1] (see also [26, Theorem 1]), and the case $n=1$ and $a\in(-1,0)\cup(0,1]$ corresponds to [31, Theorem 1].

As a consequence of Theorem 1.7, we have the following:

Corollary 1.9. *Let n, a and f be as in Theorem* 1.7*. Assume $f(r,u)\geq 0$ for each $(r,u)\in(0,1)\times(0,\infty)$. Let $u\in C^2(D\setminus\{0\})\cap C(\overline{D})$ be a positive solution of* (1.2) *which is of class C^k $(k\in\mathbb{N}\cup\{\infty\})$ at the origin. If u is m-mode with $n\leq m\leq k$, then u is radially symmetric.*

In the following, the case $a=0$ corresponds to Theorem 1.3.

Theorem 1.10. *Let n, a and f be as in Theorem* 1.7*. Let $u\in C^2(D\setminus\{0\})\cap C(\overline{D}\setminus\{0\})$ be an n-mode positive solution of* (1.3)*. Then u is radially symmetric and $u_r(|x|)<0$ for $r=|x|\in(0,1)$.*

This paper is organized as follows. In Section 2, we give the proofs of Theorems 1.1 and 1.3, and in Section 3, we give the proof of Theorem 1.4. Although we need some devices to prove Theorems 1.7 and 1.10, we can prove them similarly. So we give a proof of Theorem 1.7 in Appendix and we omit the proof of Theorem 1.10.

2. Proofs of Theorems 1.1 and 1.3

First, we give the proof of Theorem 1.1. Let n, f and u be as in Theorem 1.1. By using the polar coordinates, we define $\widetilde{u}\colon \overline{D} \to \mathbb{R}$ by $\widetilde{u}(r,\theta) = u(r^{1/n}, \theta/n)$ for $(r,\theta) \in \overline{D}$. Since u is n-mode and u satisfies (1.2), we can see that $\widetilde{u} \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ and $\widetilde{u}$ satisfies

$$\begin{cases} \Delta\widetilde{u} + \widetilde{f}(|x|, \widetilde{u}) = 0 & \text{in } D \setminus \{0\}, \\ \widetilde{u} = 0 & \text{on } \partial D, \end{cases} \tag{2.1}$$

where $\widetilde{f} \in C((0,1) \times (0,\infty), \mathbb{R})$ is given by $\widetilde{f}(r,t) = n^{-2} r^{(2-2n)/n} f(r^{1/n}, t)$ for $(r,t) \in (0,1) \times (0,\infty)$, and $\widetilde{f}$ satisfies

$$\text{for each } t > 0,\ r \mapsto \widetilde{f}(r,t) \text{ is nonincreasing.} \tag{2.2}$$

Indeed, from

$$\begin{aligned} \Delta\widetilde{u}(r,\theta) + \widetilde{f}(r, \widetilde{u}(r,\theta)) = \frac{1}{n^2}\, r^{(2-2n)/n} \biggl(u_{rr}\biggl(r^{1/n}, \frac{\theta}{n}\biggr) + \frac{1}{r^{1/n}}\, u_r\biggl(r^{1/n}, \frac{\theta}{n}\biggr) \\ + \frac{1}{r^{2/n}}\, u_{\theta\theta}\biggl(r^{1/n}, \frac{\theta}{n}\biggr) + f\biggl(r^{1/n}, u\biggl(r^{1/n}, \frac{\theta}{n}\biggr)\biggr)\biggr) = 0, \end{aligned}$$

we have (2.1), and we can easily see (2.2).

For each $\lambda \in (0,1)$, we set $\Sigma_\lambda = \{x \in D : x_1 > \lambda\}$ and we define $h_\lambda\colon \overline{\Sigma_\lambda} \to \overline{D}$ by $h_\lambda(x) = (2\lambda - x_1, x_2)$ for $x = (x_1, x_2) \in \overline{\Sigma_\lambda}$. We note that h_λ satisfies

$$|h_\lambda(x)| < |x| \quad \text{for each } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \cup \mathrm{Int}_{\partial D}(\overline{\Sigma_\lambda} \cap \partial D). \tag{2.3}$$

Here, for a subset E of ∂D, we denote by $\mathrm{Int}_{\partial D} E$, the interior set of E with respect to the relative topology of ∂D. We set $x_\lambda = (2\lambda, 0)$ for $\lambda \in (0,1)$. For the sake of convenience in the arguments in Appendix, we set $\widehat{\lambda} = 1/2$ in this section. We can see

$$x_\lambda \in \begin{cases} \Sigma_\lambda & \text{for each } \lambda \in (0, \widehat{\lambda}), \\ \partial\Sigma_\lambda & \text{for } \lambda = \widehat{\lambda}, \\ \mathbb{R}^2 \setminus \overline{\Sigma_\lambda} & \text{for each } \lambda \in (\widehat{\lambda}, 1) \end{cases} \tag{2.4}$$

and

$$h_\lambda(x_\lambda) = 0 \quad \text{for each } \lambda \in (0, \widehat{\lambda}]. \tag{2.5}$$

For the sake of completeness, we note that $\Sigma_\lambda \setminus \{x_\lambda\} = \Sigma_\lambda$ for each $\lambda \in [\widehat{\lambda}, 1)$ and $\overline{\Sigma_\lambda} \setminus \{x_\lambda\} = \overline{\Sigma_\lambda}$ for each $\lambda \in (\widehat{\lambda}, 1)$.

We define $v_\lambda \in C^2(\Sigma_\lambda \setminus \{x_\lambda\}) \cap C(\overline{\Sigma_\lambda})$ and $c_\lambda \in L^\infty(\Sigma_\lambda)$ by

$$v_\lambda(x) = \widetilde{u}(x) - \widetilde{u}(h_\lambda(x)) \quad \text{for } x \in \overline{\Sigma_\lambda} \tag{2.6}$$

and

$$c_\lambda(x) = \begin{cases} -\dfrac{\widetilde{f}(|x|, \widetilde{u}(x)) - \widetilde{f}(|x|, \widetilde{u}(h_\lambda(x)))}{\widetilde{u}(x) - \widetilde{u}(h_\lambda(x))} & \text{for } x \in \Sigma_\lambda \text{ with } v_\lambda(x) \neq 0, \\ 0 & \text{for } x \in \Sigma_\lambda \text{ with } v_\lambda(x) = 0, \end{cases} \tag{2.7}$$

respectively. By the assumptions of Theorem 1.1, we can see

$$\sup_{r<\lambda<1} \operatorname*{ess\,sup}_{x\in\Sigma_\lambda} |c_\lambda(x)| < \infty \quad \text{for each } r \in (0,1) \tag{2.8}$$

and

$$-\Delta v_\lambda(x) + c_\lambda(x) v_\lambda(x) \leq 0 \quad \text{for } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \setminus \{x_\lambda\}. \tag{2.9}$$

Now, we apply the moving plane argument. Since $\widetilde{u}$ may not be C^2 at the origin, we need additional arguments. We set

$$A_1 = \{\lambda \in [\widehat{\lambda}, 1) : v_\lambda(x) < 0 \text{ for each } x \in \Sigma_\lambda\} \quad \text{and} \quad \mu_1 = \inf_{\lambda \in A_1} \lambda. \tag{2.10}$$

Lemma 2.1. *$A_1 \neq \emptyset$.*

Proof. Let $\lambda \in [\widehat{\lambda}, 1)$ such that λ is sufficiently close to 1. Then we can easily see $v_\lambda(x) \leq 0$ for $x \in \partial\Sigma_\lambda$ and $v_\lambda(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial D \cap \partial\Sigma_\lambda)$ from (2.3). Since $|\Sigma_\lambda| \ll 1$ and (2.9) holds, by the Alexandroff–Bakelman–Pucci inequality [14, Theorem 9.1], which is abbreviated to the ABP inequality below, we have $v_\lambda \leq 0$ on $\overline{\Sigma_\lambda}$. By the strong maximum principle, we have $v_\lambda < 0$ in Σ_λ. Thus we have shown $\lambda \in A_1$, which yields $A_1 \neq \emptyset$. □

Lemma 2.2. *$\mu_1 = \widehat{\lambda}$ and $\widehat{\lambda} \in A_1$.*

Proof. We have $v_{\mu_1}(x) \leq 0$ for $x \in \Sigma_{\mu_1}$. Since (2.9) holds with $\lambda = \mu_1$ and $v_{\mu_1}(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1} \cap \partial D)$ from (2.3), by the strong maximum principle, we have $v_{\mu_1}(x) < 0$ for $x \in \Sigma_{\mu_1}$, which yields $\mu_1 \in A_1$. So it is enough to show $\mu_1 = \widehat{\lambda}$. Suppose not, i.e. $\mu_1 > \widehat{\lambda}$. Let G be an open set such that $\overline{G} \subset \Sigma_{\mu_1}$ and $|\Sigma_{\mu_1} \setminus \overline{G}| \ll 1$. We have $\max_{x\in\overline{G}} v_{\mu_1}(x) < 0$. Let $0 < \varepsilon \ll 1$. Then we have $\max_{x\in\overline{G}} v_{\mu_1-\varepsilon}(x) < 0$ and $|\Sigma_{\mu_1-\varepsilon} \setminus \overline{G}| \ll 1$. Since (2.9) holds with $\lambda = \mu_1 - \varepsilon$, $v_{\mu_1-\varepsilon}(x) \leq 0$ for $x \in \partial(\Sigma_{\mu_1-\varepsilon} \setminus \overline{G})$ and $v_{\mu_1-\varepsilon}(x) < 0$ for $x \in (\mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_1-\varepsilon} \cap \partial D)) \cup \partial G$, by the ABP inequality and the strong maximum principle, we have $v_{\mu_1-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_1-\varepsilon}$, which yields $\mu_1 - \varepsilon \in A_1$. This is a contradiction. Thus we have shown $\mu_1 = \widehat{\lambda}$ and $\widehat{\lambda} \in A_1$. □

We set

$$A_2 = \{\lambda \in (0, \widehat{\lambda}) : v_\lambda(x) < 0 \text{ for each } x \in \Sigma_\lambda\} \quad \text{and} \quad \mu_2 = \inf_{\lambda \in A_2} \lambda. \tag{2.11}$$

LEMMA 2.3. $A_2 \neq \emptyset$.

PROOF. We note that $x_{\widehat{\lambda}} = (1,0)$. Let G be an open set such that $\overline{G} \subset \Sigma_{\widehat{\lambda}}$ and $|\Sigma_{\widehat{\lambda}} \setminus G| \ll 1$. From $\widehat{\lambda} \in A_1$ and $\overline{G} \subset \Sigma_{\widehat{\lambda}}$, we have $\max_{x \in \overline{G}} v_{\widehat{\lambda}}(x) < 0$. Let $\lambda \in (0, \widehat{\lambda})$ such that λ is sufficiently close to $\widehat{\lambda}$. We note $|\Sigma_\lambda \setminus \overline{G}| \ll 1$ and x_λ is close to $(1,0)$. We choose a sufficiently small open neighbourhood U of x_λ with $\overline{U} \subset \Sigma_\lambda$, and we set $H = G \cup U$. Then we have $v_\lambda(x) < 0$ for $x \in \overline{H}$, $v_\lambda(x) \leq 0$ for $x \in \partial\Sigma_\lambda \cup \partial H$ and $|\Sigma_\lambda \setminus \overline{H}| \ll 1$. Since (2.9) holds on $\Sigma_\lambda \setminus \overline{H}$, by the ABP inequality and the strong maximum principle, we have $v_\lambda < 0$ on Σ_λ, which yields $\lambda \in A_2$. $\square$

LEMMA 2.4. *Assume that u is of class C^n at the origin. Then there holds*

$$\frac{\partial(\widetilde{u} \circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = 0. \tag{2.12}$$

PROOF. We will show that each j-th partial derivative of u at $(0,0)$ is zero for all $j = 1, \dots, n$. Since u is n-mode with $n \geq 2$ and u is totally differentiable at the origin, we can easily see $\nabla u(0,0) = (0,0)$. Let $j \in \{1, \dots, n\}$. By using mathematical induction, we can see that each $(j-1)$-th partial derivative of u is also n-mode (in a small open ball whose center is the origin), it is totally differentiable at zero and hence each j-th partial differential coefficient of u at the origin is zero. Hence we have shown

$$\frac{\partial^j u}{\partial x_1^k \partial x_2^{j-k}}(0,0) = 0 \quad \text{for } j = 1, \dots, n \text{ and } k = 0, \dots, j. \tag{2.13}$$

Next, we will show

$$\lim_{t \to +0} \frac{u(t^{1/n}\alpha(t), t^{1/n}\beta(t)) - u(0,0)}{t} = 0, \tag{2.14}$$

where $0 < \varepsilon \ll 1$ and $\alpha, \beta \in C([0,\varepsilon), [0,\infty))$ are any bounded functions. By Taylor's theorem and (2.13), for each $0 < t \ll 1$, there exists $\zeta_t \in (0,1)$ such that

$$\begin{aligned} &u(t\alpha(t^n), t\beta(t^n)) \\ &\quad = u(0,0) + \sum_{k=0}^{n-1} \frac{(t\alpha(t^n))^k (t\beta(t^n))^{n-1-k}}{k!(n-1-k)!} \frac{\partial^{n-1} u}{\partial x_1^k \partial x_2^{n-1-k}}(\zeta_t t\alpha(t^n), \zeta_t t\beta(t^n)). \end{aligned}$$

Using (2.13) again, we obtain

$$\begin{aligned} \lim_{t \to +0} \frac{u(t\alpha(t^n), t\beta(t^n)) - u(0,0)}{t^n} &= \lim_{t \to +0} \sum_{k=0}^{n-1} \frac{(\alpha(t^n))^k (\beta(t^n))^{n-1-k}}{k!(n-1-k)!} \\ &\quad \cdot \frac{\dfrac{\partial^{n-1} u}{\partial x_1^k \partial x_2^{n-1-k}}(\zeta_t t\alpha(t^n), \zeta_t t\beta(t^n)) - \dfrac{\partial^{n-1} u}{\partial x_1^k \partial x_2^{n-1-k}}(0,0)}{t} = 0, \end{aligned}$$

which yields (2.14). From the definitions of $\widetilde{u}$ and h_{μ_2} and (2.14), we have

$$\lim_{t\to+0}\frac{\widetilde{u}(h_{\mu_2}(x_{\mu_2}+t(1,0)))-\widetilde{u}(h_{\mu_2}(x_{\mu_2}))}{t} = \lim_{t\to+0}\frac{u(t^{1/n}\cos\pi/n, t^{1/n}\sin\pi/n)-u(0,0)}{t}=0,$$

$$\lim_{t\to+0}\frac{\widetilde{u}(h_{\mu_2}(x_{\mu_2}-t(1,0)))-\widetilde{u}(h_{\mu_2}(x_{\mu_2}))}{t} = \lim_{t\to+0}\frac{u(t^{\frac{1}{n}},0)-u(0,0)}{t}=0.$$

Hence we have shown (2.12). □

Lemma 2.5. $\mu_2 = 0$.

Proof. Suppose $\mu_2 \neq 0$. Then we have $\mu_2 \in (0,\widehat{\lambda})$ by Lemma 2.3, and we can see $v_{\mu_2} \le 0$ on $\overline{\Sigma_{\mu_2}}$. We will show $v_{\mu_2} < 0$ on $\Sigma_{\mu_2}\setminus\{x_{\mu_2}\}$. We have $v_{\mu_2}(x) < 0$ for $x \in \mathrm{Int}_{\partial D}(\partial\Sigma_{\mu_2}\cap\partial D)$ from (2.3). By (2.9) with $\lambda = \mu_2$ and the strong maximum principle, we have $v_{\mu_2} < 0$ on $\Sigma_{\mu_2}\setminus\{x_{\mu_2}\}$.

Next, we will show $v_{\mu_2}(x_{\mu_2}) < 0$. In the case $u(0) > u(x)$ for $x \in \overline{D}\setminus\{0\}$, we can easily see $v_{\mu_2}(x_{\mu_2}) < 0$. So we consider the case that u is of class C^n at the origin. Suppose $v_{\mu_2}(x_{\mu_2}) < 0$ does not hold, i.e. $v_{\mu_2}(x_{\mu_2}) = 0$. Let $\nu_1 = (-1,0)$ and $\nu_2 = (1,0)$. From (2.12), we have

$$\begin{aligned}\frac{\partial v_{\mu_2}}{\partial\nu_1}(x_{\mu_2}) &= -\frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}) + \frac{\partial(\widetilde{u}\circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = -\frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}),\\ \frac{\partial v_{\mu_2}}{\partial\nu_2}(x_{\mu_2}) &= \frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}) - \frac{\partial(\widetilde{u}\circ h_{\mu_2})}{\partial x_1}(x_{\mu_2}) = \frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}).\end{aligned}$$

By Hopf's lemma, we obtain

$$-\frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}) < 0 \quad\text{and}\quad \frac{\partial\widetilde{u}}{\partial x_1}(x_{\mu_2}) < 0,$$

which is a contradiction. So we have shown $v_{\mu_2}(x_{\mu_2}) < 0$. Thus we have $v_{\mu_2} < 0$ on Σ_{μ_2}, which shows $\mu_2 \in A_2$.

We choose an open set G such that $\overline{G} \subset \Sigma_{\mu_2}$ and $|\Sigma_{\mu_2}\setminus\overline{G}| \ll 1$. We have $\max_{\overline{G}} v_{\mu_2} < 0$. Let $0 < \varepsilon \ll 1$. Then we have $|\Sigma_{\mu_2-\varepsilon}\setminus\overline{G}| \ll 1$ and $\max_{\overline{G}} v_{\mu_2-\varepsilon} < 0$. Since (2.9) holds with $\lambda = \mu_2-\varepsilon$, by the ABP inequality and the strong maximum principle, we have $v_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon}\setminus\overline{G}$. Hence we have shown $v_{\mu_2-\varepsilon}(x) < 0$ for $x \in \Sigma_{\mu_2-\varepsilon}$. Thus we have $\mu_2-\varepsilon \in A_2$, which is a contradiction. Therefore we obtain $\mu_2 = 0$. □

Proof of Theorem 1.1. By the lemmas above, we can infer that $\widetilde{u}$ is radially symmetric and $\widetilde{u}_r(x) < 0$ for $r = |x| \in (0,1)$. From the definition of $\widetilde{u}$, we can find u is also radially symmetric and $u_r(x) < 0$ for $r = |x| \in (0,1)$. □

Proof of Theorem 1.3. Since the arguments for the proof of Theorem 1.1 almost similarly work for that of Theorem 1.3, we prove it briefly.

We define $\widetilde{u} \in C^2(D \setminus \{0\}) \cap C(\overline{D} \setminus \{0\})$ as before. For $\lambda \in (0,1)$, we define $v_\lambda \in C^2(\Sigma_\lambda \setminus \{x_\lambda\}) \cap C(\overline{\Sigma_\lambda} \setminus \{x_\lambda\})$ by

$$v_\lambda(x) = \widetilde{u}(x) - \widetilde{u}(h_\lambda(x)) \quad \text{for } x \in \overline{\Sigma_\lambda} \setminus \{x_\lambda\},$$

and we define c_λ by (2.7). We note that the definition of v_λ is almost same as (2.6). For the sake of simplicity, we consider $v_\lambda(x_\lambda) = -\infty$ in the case $x_\lambda \in \overline{\Sigma_\lambda}$ with $\lambda \in (0,1)$. We note c_λ may not belong to $L^\infty(\Sigma_\lambda)$. However, we have

$$\sup_{r<\lambda<1} \operatorname{ess\,sup}\{|c_\lambda(x)| : x \in \Sigma_\lambda, v_\lambda(x) \geq -1\} < \infty \quad \text{for each } r \in (0,1).$$

We can easily see that (2.9) holds.

We define A_1 and μ_1 by (2.10). Since $x_\lambda \notin \Sigma_\lambda$ for $\lambda \in [\widehat{\lambda}, 1)$, we can show $\mu_1 = \widehat{\lambda}$ by similar lines as those in the proofs of Lemmas 2.1 and 2.2. We define A_2 and μ_2 by (2.11). By a similar proof of Lemma 2.3, we can show $A_2 \neq \emptyset$. We will show $\mu_2 = 0$. Suppose not, i.e. $\mu_2 > 0$. Since $v_{\mu_2} \leq 0$ on $\overline{\Sigma_{\mu_2}}$, $v_{\mu_2} \leq -1$ in a small neighbourhood U of x_{μ_2} with $\overline{U} \subset \Sigma_{\mu_2}$, and (2.9) holds for all $x \in \Sigma_{\mu_2} \setminus \overline{U}$, we have $v_{\mu_2} < 0$ in Σ_{μ_2} by the strong maximum principle. Thus we have $\mu_2 \in A_2$. Let G be an open set such that $\overline{G} \subset \Sigma_{\mu_2}$ and $|\Sigma_{\mu_2} \setminus \overline{G}| \ll 1$. We have $\max_{x \in \overline{G}} v_{\mu_2}(x) < 0$. Let $\lambda \in (0, \mu_2)$ such that λ is sufficiently close to μ_2. Let $V = \{x \in \Sigma_\lambda : v_\lambda(x) < -1\}$ and set $H = G \cup V$. Then we have $x_\lambda \in H$, $\overline{H} \subset \Sigma_\lambda \cup \operatorname{Int}_{\partial D}(\partial \Sigma_\lambda \cap \partial D)$, $\max_{\overline{H}} v_\lambda < 0$, $|\Sigma_\lambda \setminus \overline{H}| \ll 1$, and (2.9) holds for all $x \in \Sigma_\lambda \setminus \overline{H}$. By the ABP inequality and the strong maximum principle, we have $v_\lambda(x) < 0$ for all $x \in \Sigma_\lambda$, which is a contradiction. Thus we have shown $\mu_2 = 0$, and we can infer that u is radially symmetric and $u_r(x) < 0$ for $r = |x| \in (0,1)$.□

3. Proof of Theorem 1.4

We will show (a). We note that each positive solution of (1.4) is of class C^{k_α}, where

$$k_\alpha = \begin{cases} \lfloor \alpha \rfloor + 2 & \text{if } \alpha \in (0,\infty) \setminus \mathbb{N}, \\ \alpha + 1 & \text{if } \alpha \text{ is an odd natural number}, \\ \infty & \text{if } \alpha \text{ is an even natural number}. \end{cases}$$

Let u be an n-mode, positive solution of (1.4) with $n \geq 1 + \lceil \alpha/2 \rceil$. We choose $\widehat{n} \in \mathbb{N}$ such as $(\alpha+2)\widehat{n} \leq 2n < 2(\alpha+2)\widehat{n}$. We also choose $m \in \mathbb{N}$ such that

$$m((\alpha+2)\widehat{n} - n) \geq n\widehat{n}, \quad m/\widehat{n} \in \mathbb{N} \quad \text{and} \quad m(\alpha+2) > 2n.$$

We set $\widehat{m} = m/\widehat{n}$. We note that $\widehat{m} \geq 2$. We set $\overline{u}(r,\theta) = u(r^{m/n}, (m/n)\theta)$. We note that $\overline{u}$ is m-mode and hence it is $\widehat{m}$-mode. From $u(r,\theta) = \overline{u}(r^{n/m}, (n/m)\theta)$, we can easily see that

$$\Delta\overline{u}+\left(\frac{m}{n}\right)^2|x|^{(m(\alpha+2)-2n)/n}\,\overline{u}^{p-1}=0\quad\text{in } D\setminus\{0\}.$$

We will show

$$\begin{cases}\Delta\overline{u}+\left(\dfrac{m}{n}\right)^2|x|^{(m(\alpha+2)-2n)/n}\,\overline{u}^{p-1}=0 & \text{in } D,\\ \overline{u}=0 & \text{on } \partial D.\end{cases}\tag{3.1}$$

Let $\varphi\in C_0^\infty(D)$ and let $0<\varepsilon<1$. We have

$$\begin{aligned}0&=\int_{\{x:\varepsilon<|x|<1\}}\left(\Delta\overline{u}+\left(\frac{m}{n}\right)^2|x|^{(m(\alpha+2)-2n)/n}\,\overline{u}^{p-1}\right)\varphi\,dx\\
&=\int_{\{x:\varepsilon<|x|<1\}}\Delta\overline{u}\,\varphi\,dx+\left(\frac{m}{n}\right)^2\int_{\{x:\varepsilon<|x|<1\}}|x|^{(m(\alpha+2)-2n)/n}\,\overline{u}^{p-1}\varphi\,dx\\
&=-\int_{|x|=\varepsilon}\frac{\partial\overline{u}}{\partial r}\,\varphi\,d\sigma-\int_{\{x:\varepsilon<|x|<1\}}\nabla\overline{u}\,\nabla\varphi\,dx\\
&\quad+\left(\frac{m}{n}\right)^2\int_{\{x:\varepsilon<|x|<1\}}|x|^{(m(\alpha+2)-2n)/n}\,\overline{u}^{p-1}\varphi\,dx.\end{aligned}$$

Noting $u\in C^1(\overline{D})$, and letting $\varepsilon\to 0$, we can find that $\overline{u}$ is a weak solution of (3.1). By the elliptic regularity, $\overline{u}$ is a strong solution of (3.1). Since we have

$$\frac{m(\alpha+2)-2n}{n}+1\geq\widehat{m}$$

from $m((\alpha+2)\widehat{n}-n)\geq n\widehat{n}$, we can infer $\overline{u}\in C^{\widehat{m}}(\overline{D})$. From $(\alpha+2)\widehat{n}\leq 2n$, we can easily see

$$r\mapsto r^{2-2\widehat{m}+(m(\alpha+2)-2n)/n}$$

is nonincreasing. Noting $\overline{u}$ is $\widehat{m}$-mode and applying Theorem 1, we obtain that $\overline{u}$ is radially symmetric, which implies that so is u. Hence we have shown (a).

We will show (b). We set $H_n=\{u\in H_0^1(D): u \text{ is } n\text{-mode}\}$ for each $n\in\mathbb{N}$ and $H_\infty=\{u\in H_0^1(D): u\text{ is radially symmetric}\}$. For each $\alpha\geq 0$ and $p>2$, we set

$$S_{\alpha,p,n}=\inf_{u\in H_n\setminus\{0\}}R_{\alpha,p}(u)\quad\text{for each } n\in\mathbb{N}\cup\{\infty\}.$$

For the sake of completeness, we note that $H_1=H_0^1(D)$ and

$$S_{0,p,1}=\inf_{u\in H_0^1(D)\setminus\{0\}}R_{0,p}(u).$$

Let $\alpha,p\in(2,\infty)$. Setting $v_u(|x|)=u(|x|^{2/(\alpha+2)})$ for each $u\in H_\infty$, we can find

$$S_{\alpha,p,\infty}=\inf_{u\in H_\infty\setminus\{0\}}\left(\frac{\alpha+2}{2}\right)^{1+2/p}R_{\alpha,p}(v_u)\geq S_{0,p,1}\left(\frac{\alpha+2}{2}\right)^{1+2/p};\tag{3.2}$$

see [32, Theorem 4.1] and its proof. Next, let φ be any element of $C_0^\infty(D)$. Since we can consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ by the zero extension, we can define $\varphi_\alpha \in C_0^\infty(D)$ by $\varphi_\alpha((x_1, x_2)) = \varphi(\alpha(x_1 - (1 - 1/\alpha)), \alpha x_2)$ for $(x_1, x_2) \in D$. We set $D_1 = D$ and

$$D_n = \{(r\cos\theta, r\sin\theta) : 0 < r < 1, -\pi/n < \theta < \pi/n\} \quad \text{for } n \in \mathbb{N} \setminus \{1\}.$$

Following the arguments in [32, Theorem 4.2], [11, Proposition 2] and [21, Lemma 1.3], we will show

$$S_{\alpha,p,n} \le S_{0,p,1} n^{1-2/p} \alpha^{4/p} \left(\frac{\alpha}{\alpha - 2}\right)^{2\alpha/p} \tag{3.3}$$

for each $n \in \mathbb{N}$ with $\operatorname{supp} \varphi_\alpha \subset D_n$. Let $n \in \mathbb{N}$ with $\operatorname{supp} \varphi_\alpha \subset D_n$. For the sake of completeness, we note that $\alpha \in (2, \infty)$ and $n \le \lceil \alpha/2 \rceil$ is a sufficient condition for $\operatorname{supp} \varphi_\alpha \subset D_n$. We define $P_n \colon D \to D$ by

$$P_n(r\cos\theta, r\sin\theta) = (r\cos(\theta + 2\pi/n), r\sin(\theta + 2\pi/n))$$

for $(r, \theta) \in [0, 1) \times \mathbb{R}$. We set $\widetilde{\varphi}(x) = \varphi_\alpha(x) + \varphi_\alpha(P_n(x)) + \ldots + \varphi_\alpha(P_n^{n-1}(x))$ for $x \in D$. Since we have

$$\int_D |\nabla \varphi_\alpha|^2 \, dx = \int_D |\nabla \varphi|^2 \, dx \quad \text{and} \quad \int_D |x|^\alpha |\varphi_\alpha|^p \, dx \ge \alpha^{-2} \left(1 - \frac{2}{\alpha}\right)^\alpha \int_D |\varphi|^p \, dx,$$

we obtain

$$S_{\alpha,p,n} \le R_{\alpha,p}(\widetilde{\varphi}) \le \frac{n \displaystyle\int_D |\nabla \varphi|^2 \, dx}{\left(n\alpha^{-2} \left(1 - \dfrac{2}{\alpha}\right)^\alpha \displaystyle\int_D |\varphi|^p \, dx \right)^{2/p}}.$$

Since $\varphi \in C_0^\infty(D)$ is arbitrary, we have shown (3.3). From (3.2) and (3.3), we can see that $n \le n_\alpha$ is a sufficient condition for $S_{\alpha,p,n} < S_{\alpha,p,\infty}$.

Following [21, Lemma 1.5], we will show that if $n > 1$ and $S_{\alpha,p,n} < S_{\alpha,p,\infty}$ then $S_{\alpha,p,1} < \ldots < S_{\alpha,p,n}$. Let $n > 1$ and $S_{\alpha,p,n} < S_{\alpha,p,\infty}$. We can choose $u \in H_n \setminus \{0\}$ such that $R_{\alpha,p}(u) = S_{\alpha,p,n}$ and $u \ge 0$. We note that $u \notin H_\infty$ and u is a positive solution of (1.4). Let $m \in \{1, \ldots, n-1\}$. We define $v \in H_m$ by

$$v(r\cos\theta, r\sin\theta) = u(r\cos(m\theta/n), r\sin(m\theta/n))$$

for $(r, \theta) \in [0, 1) \times \mathbb{R}$. Since we can see

$$\int_D |x|^\alpha |v|^p \, dx = \int_D |x|^\alpha |u|^p \, dx$$

and

$$\int_D |\nabla v|^2 \, dx = \int_0^{2\pi} \int_0^1 \left(\left| \frac{\partial u}{\partial r} \right|^2 + \frac{m^2}{n^2 r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 \right) r \, dr \, d\theta < \int_D |\nabla u|^2 \, dx,$$

we have

$$S_{\alpha,p,m} \le R_{\alpha,p}(v) < R_{\alpha,p}(u) = S_{\alpha,p,n}.$$

By a similar argument, we can conclude that $S_{\alpha,p,1} < \ldots < S_{\alpha,p,n}$. Hence we can infer that if $S_{\alpha,p,n} < S_{\alpha,p,\infty}$, then for each $m = 1, \ldots, n$, there exists a nonradial positive solution $u_m \in H_m$ of (1.4) satisfying $R_{\alpha,p}(u_m) = S_{\alpha,p,m}$.

We set the number in (1.6) as $\eta(\alpha,p)$. For a fixed $\alpha \in (2,\infty)$, we have $\eta(\alpha,p) \to 1+\alpha/2$ as $p \to \infty$, which yields (i). For a fixed $p \in (2,\infty)$, we have $\eta(\alpha,p) \to \infty$ as $\alpha \to \infty$, which yields (ii). Hence, we can finish the proof of Theorem 1.4. □

Appendix. Proof of Theorem 1.7

Let $a \in [-1,1] \setminus \{0\}$. We consider that (D,g) is a Riemannian manifold, where the metric tensor g is defined by

$$\frac{4|a||dx|^2}{(1+a|x|^2)^2}. \tag{A.1}$$

For each $\lambda \in (0,1)$, let $T_\lambda \subset D$ be the geodesic which intersects x_1-axis orthogonally at $(\lambda, 0)$, i.e.

$$T_\lambda = \left\{ x \in D : |x - e_\lambda| = \frac{1+a\lambda^2}{2|a|\lambda} \right\}, \tag{A.2}$$

where

$$e_\lambda = \left(-\frac{1-a\lambda^2}{2a\lambda}, 0 \right).$$

For each $\lambda \in (0,1)$, we define $\Sigma_\lambda \subset D$ by

$$\Sigma_\lambda = \begin{cases} \left\{ x \in D : |x-e_\lambda| > \dfrac{1+a\lambda^2}{2a\lambda} \right\} & \text{if } a \in (0,1], \\ \left\{ x \in D : |x-e_\lambda| < \dfrac{1+a\lambda^2}{2|a|\lambda} \right\} & \text{if } a \in [-1,0). \end{cases}$$

For each $\lambda \in (0,1)$ and $x \in \Sigma_\lambda$, there is the reflection $h_\lambda(x)$ of x with respect to T_λ in (D,g) and it is given by

$$h_\lambda(x) = e_\lambda + \left(\frac{1+a\lambda^2}{2a\lambda} \right)^2 \frac{x-e_\lambda}{|x-e_\lambda|^2} = e_\lambda + \left(|e_\lambda|^2 + \frac{1}{a} \right) \frac{x-e_\lambda}{|x-e_\lambda|^2}. \tag{A.3}$$

Considering Σ_λ as a subset of the Euclidean space $\mathbb{R}^2$, we can see that h_λ is uniquely continuously extended to $\overline{\Sigma_\lambda}$. We also denote it by h_λ. Then h_λ satisfies (A.3) for all $x \in \overline{\Sigma_\lambda}$. In the case of $a \ne -1$, (2.3) holds. In the case of $a = -1$, it holds that

$$|h_\lambda(x)| < |x| \quad \text{for each } \lambda \in (0,1) \text{ and } x \in \Sigma_\lambda \tag{A.4}$$

and $|h_\lambda(x)| = 1$ for each $\lambda \in (0,1)$ and $x \in \overline{\Sigma_\lambda} \cap \partial D$. For the proofs of (A.2), (A.3), (2.3) and (A.4), see [25], [26], [31]. We note that

$$h_\lambda(x_1, 0) = \left(\frac{x_1(a\lambda^2 - 1) + 2\lambda}{2a\lambda x_1 + 1 - a\lambda^2}, 0\right) \tag{A.5}$$

for $(x_1, 0) \in \overline{\Sigma_\lambda}$, $\lambda \in (0,1)$ and $a \in [-1,1] \setminus \{0\}$.

Now, let n, a, f and u be as in Theorem 1.7. For the sake of simplicity, we assume $a \neq 0$. We define $\widetilde{u} \in C^2(D \setminus \{0\}) \cap C(\overline{D})$ and $\widetilde{f} \in C((0,1) \times (0,\infty), \mathbb{R})$ as before. We can see that $\widetilde{u}$ and $\widetilde{f}$ satisfy (2.1) and for each $t > 0$,

$$\begin{cases} r \mapsto (1 + ar^2)^2 \widetilde{f}(r,t) & \text{is nonincreasing} \\ & \text{in the case of } a \in (-1,1] \setminus \{0\}, \\ r \mapsto (1 + ar^2)^2 \widetilde{f}(r,t) & \text{is decreasing in the case of } a = -1. \end{cases} \tag{A.6}$$

We set

$$\widehat{\lambda} = \frac{1}{\sqrt{1+a} + 1} \quad \text{and} \quad x_\lambda = \left(\frac{2\lambda}{1 - a\lambda^2}, 0\right) \quad \text{for } \lambda \in (0,1).$$

Noting (A.5), we can see that in the case of $a \in (-1,1] \setminus \{0\}$, (2.4) and (2.5) hold. In the case of $a = -1$, we can see that

$$x_\lambda \in \Sigma_\lambda \text{ and } h_\lambda(x_\lambda) = 0 \text{ for each } \lambda \in (0,1).$$

We respectively define $v_\lambda \in C^2(\Sigma_\lambda \setminus \{x_\lambda\}) \cap C(\overline{\Sigma_\lambda})$ and $c_\lambda \in L^\infty(\Sigma_\lambda)$ by (2.6) and (2.7), and we can show (2.8). We recall that for $a \in [-1,1] \setminus \{0\}$, the Laplace–Beltrami operator on (D,g) at $x \in D$ is given by

$$\Delta_{(g,x)} = \frac{(1 + a|x|^2)^2}{4|a|} \Delta.$$

LEMMA A.1. *The inequality* (2.9) *holds.*

PROOF. Let $\lambda \in (0,1)$ and $x \in \Sigma_\lambda \setminus \{x_\lambda\}$. First, we will show that $h_\lambda \colon (\Sigma_\lambda, g) \to (h_\lambda(\Sigma_\lambda), g)$ is Riemannian isometric. We set $h_\lambda(x) = (h_{\lambda,1}(x), h_{\lambda,2}(x))$ for $x \in \Sigma_\lambda$. From (A.3), we have

$$\left(\frac{\partial h_{\lambda,1}}{\partial x_p}(x)\right)^2 + \left(\frac{\partial h_{\lambda,2}}{\partial x_p}(x)\right)^2 = \left(|e_\lambda|^2 + \frac{1}{a}\right)^2 \frac{1}{|x - e_\lambda|^4} \quad \text{for } p = 1,2$$

and

$$\frac{\partial h_{\lambda,1}}{\partial x_1}(x) \frac{\partial h_{\lambda,1}}{\partial x_2}(x) + \frac{\partial h_{\lambda,2}}{\partial x_1}(x) \frac{\partial h_{\lambda,2}}{\partial x_2}(x) = 0.$$

Noting $2(x - e_\lambda) \cdot e_\lambda = |x|^2 - |e_\lambda|^2 - |x - e_\lambda|^2$, we can show

$$(1 + a|h_\lambda(x)|^2)|x - e_\lambda|^2 = (1 + a|x|^2)\left(|e_\lambda|^2 + \frac{1}{a}\right),$$

where $(x-e_\lambda)\cdot e_\lambda$ is the standard inner product of $(x-e_\lambda)$ and e_λ. Using these equalities and (A.1), we can easily see

$$g_{pq}(x)=\sum_{i,j=1}^{2} g_{ij}(h_\lambda(x))\frac{\partial h_{\lambda,i}}{\partial x_p}(x)\frac{\partial h_{\lambda,j}}{\partial x_q}(x)\quad\text{for } p,q=1,2.$$

Thus we have shown $h_\lambda\colon(\Sigma_\lambda,g)\to(h_\lambda(\Sigma_\lambda),g)$ is Riemannian isometric.

Let $x\in\Sigma_\lambda$ and set $y=h_\lambda(x)$. Noting $\widetilde{u}\in C^2(D\setminus\{0\})$, we have

$$\Delta_{(g,y)}\widetilde{u}(y)=\Delta_{(g,x)}(\widetilde{u}(h_\lambda(x))).$$

Then by (2.3), (A.4), (A.6), we have

$$\begin{aligned}
0&=\Delta_{(g,y)}(\widetilde{u}(y))+\frac{(1+a|y|^2)^2}{4|a|}\widetilde{f}(|y|,\widetilde{u}(y))\\
&\quad-\Delta_{(g,x)}\widetilde{u}(x)-\frac{(1+a|x|^2)^2}{4|a|}\widetilde{f}(|x|,\widetilde{u}(x))\\
&=\Delta_{(g,x)}(\widetilde{u}(h_\lambda(x)))+\frac{(1+a|h_\lambda(x)|^2)^2}{4|a|}\widetilde{f}(|h_\lambda(x)|,\widetilde{u}(h_\lambda(x)))\\
&\quad-\Delta_{(g,x)}\widetilde{u}(x)-\frac{(1+a|x|^2)^2}{4|a|}\widetilde{f}(|x|,\widetilde{u}(x))\\
&\geq-\Delta_{(g,x)}v_\lambda(x)+\frac{(1+a|x|^2)^2}{4|a|}\widetilde{f}(|x|,\widetilde{u}(h_\lambda(x)))-\frac{(1+a|x|^2)^2}{4|a|}\widetilde{f}(|x|,\widetilde{u}(x))\\
&=\frac{(1+a|x|^2)^2}{4|a|}(-\Delta v_\lambda(x)+c_\lambda(x)v_\lambda(x)).
\end{aligned}$$

Hence, we obtain (2.9). □

Proof of Theorem 1.7. As in the case of $a=0$, we apply the moving sphere (plane) argument. First, we consider the case $a\in(-1,1]\setminus\{0\}$. Since the arguments in Section 2 work except the proof for (2.12), we give a proof for it only. Putting $r\exp(i\theta)=(r\cos\theta,r\sin\theta)$, we have

$$\begin{aligned}
&\lim_{t\to+0}\frac{\widetilde{u}(h_{\mu_2}(x_{\mu_2}+t(1,0)))-\widetilde{u}(h_{\mu_2}(x_{\mu_2}))}{t}\\
&=\lim_{t\to+0}\frac{1}{t}\bigg[u\bigg(\bigg(\frac{t(1-a\mu_2^2)^2}{2a\mu_2(2\mu_2-(1-a\mu_2^2)t)+(1-a\mu_2^2)^2}\bigg)^{1/n}\exp\Big(\frac{i\pi}{n}\Big)\bigg)-u(0,0)\bigg]\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{t\to+0}\frac{\widetilde{u}(h_{\mu_2}(x_{\mu_2}-t(1,0)))-\widetilde{u}(h_{\mu_2}(x_{\mu_2}))}{t}\\
&=\lim_{t\to+0}\frac{1}{t}\bigg[u\bigg(\bigg(\frac{t(1-a\mu_2^2)^2}{2a\mu_2(2\mu_2-(1-a\mu_2^2)t)+(1-a\mu_2^2)^2}\bigg)^{1/n},0\bigg)-u(0,0)\bigg]=0,
\end{aligned}$$

which yield (2.12). We note that these calculations are also valid for $a=-1$.

Next, we consider the case $a = -1$. Since we do not need to define A_1 and μ_1 in this case, we start the moving sphere arguments in Section 2 by setting A_2 and μ_2. We can easily see $A_2 \neq \emptyset$. Since the arguments in Section 2 also work except the proof for the first paragraph of Lemma 2.5, we give a proof for it only. That is, assuming $\mu_2 \in (0, \widehat{\lambda})$, we will show $v_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. By (A.4), (2.9) with $\lambda = \mu_2$ and the strong maximum principle, we have $v_{\mu_2} = 0$ on Σ_{μ_2} or $v_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. If $v_{\mu_2} = 0$ on Σ_{μ_2}, from the calculation in the proof of Lemma (A.1), we have

$$(1 - |x|^2)^2 \widetilde{f}(|x|, \widetilde{u}(x)) = (1 - |h_{\mu_2}(x)|^2)^2 \widetilde{f}(|h_{\mu_2}(x)|, \widetilde{u}(x))$$

for $x \in \Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$, which contradicts (A.6). Thus we have shown $v_{\mu_2} < 0$ on $\Sigma_{\mu_2} \setminus \{x_{\mu_2}\}$. $\square$

PROOF OF COROLLARY 1.9. Let $m \in \mathbb{N}$ with $m > n$. Since

$$\begin{aligned}
\frac{d}{dr}\left(\frac{r^{-m} + ar^m}{r^{-n} + ar^n}\right) &= \frac{(m-n)(a^2 r^{m+n} - r^{-m-n}) + (m+n)(ar^{m-n} - ar^{-m+n})}{r(r^{-n} + ar^n)^2} \\
&\equiv \frac{\gamma(r)}{r(r^{-n} + ar^n)^2}, \\
\frac{d}{dr}\gamma(r) &= \frac{m^2 - n^2}{r}(r^{-m} + ar^m)(r^{-n} + ar^n),
\end{aligned}$$

and $\gamma(1) \le 0$, the function $r \mapsto (1+ar^{2m})^2 r^{2-2m}/((1+ar^{2n})^2 r^{2-2n})$ is decreasing in $(0,1)$. From $f \ge 0$, we can find that for each $m \in \mathbb{N}$ with $m \ge n$, (a) in Theorem 1.7 holds for m instead of n. Now, assume that u is m-mode with $n \le m \le k$. Then from Theorem 1.7, we can see that u is radially symmetric. $\square$

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