EXISTENCE OF POSITIVE SOLUTIONS FOR A SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEM WITH IMPULSIVE EFFECTS

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Abstract. In this paper, we are mainly concerned with the existence and multiplicity of positive solutions for the following second order periodic boundary value problem involving impulsive effects

$$\begin{cases} -u'' + \rho^2 u = f(t, u), & t \in J', \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), & k = 1, \ldots, m, \\ u(0) - u(2\pi) = 0, \quad u'(0) - u'(2\pi) = 0. \end{cases}$$

Here $J' = J \setminus \{t_1, \ldots, t_m\}$, $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, where $\mathbb{R}^+ = [0, \infty)$, $J = [0, 2\pi]$. The proof of our main results relies on the fixed point theorem on cones. The paper extends some previous results and reports some new results about impulsive differential equations.

1. Introduction

In this paper, we investigate the existence and multiplicity of positive solutions for the following second order periodic boundary value problem involving

impulsive effects

$$(1.1) \quad \begin{cases} -u'' + \rho^2 u = f(t, u), & t \in J', \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), & k = 1, \ldots, m, \\ u(0) - u(2\pi) = 0, \quad u'(0) - u'(2\pi) = 0, \end{cases}$$

where $J' = J \setminus \{t_1, \ldots, t_m\}$, $J = [0, 2\pi]$, $\rho > 0$, $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $t_k (k = 1, \ldots, m)$ are fixed points with $0 < t_1 < \ldots < t_m < 2\pi$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^+)$ and $u'(t_k^-)$ denote the right and left limit of $u'(t)$ at $t = t_k$, respectively.

As one kind of instantaneous mutation phenomena, the impulsive phenomenon exists widespread in practical problems in areas of the modern technology, and its mathematical model is always due to the impulsive differential system. Therefore, the impulsive differential equation, developed in recent years, becomes an important branch of the differential equations. This explains the reason that the last two decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published.

When $I_k \equiv 0$ for $k = 1, \ldots, m$, problem (1.1) reduces to the periodic boundary value problem of ordinary differential equation, which has been studied using various theorems and methods of nonlinear functional analysis, see [1], [3], [4], [7], [8], [10], [11], [16], [17], [21]. In [21], Zhang and Wang studied the following periodic boundary value problem

$$(1.2) \quad \begin{cases} -u'' + \rho^2 u = f(t, u), & t \in J, \\ u(0) - u(2\pi) = 0, \quad u'(0) - u'(2\pi) = 0, \end{cases}$$

and established the existence and multiplicity of positive solutions to the problem (1.2) under the condition that $f: [0, 2\pi] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Carathéodory function.

In recent years, periodic boundary value problems of nonlinear impulsive differential equations have received considerable attention and much literature has been published about the existence of solutions of periodic boundary problems for impulsive differential equations, see, for example, [2], [6], [12]–[15], [18]–[20] In [19], Zhou, Jiang and Tian considered the problem

$$(1.3) \quad \begin{cases} -u'' + \rho^2 u = f(t, u), & t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad -\Delta u'|_{t=t_k} = J_k(u(t_k)), & k = 1, \ldots, m, \\ u(0) - u(2\pi) = 0, \quad u'(0) - u'(2\pi) = 0. \end{cases}$$

The authors adopted a fixed point theorem on a cone to obtain some conditions on f, I_k and J_k, which guarantee the existence of solutions for (1.3).

Motivated by the works mentioned above, in this paper, we study the existence and multiplicity of positive solutions for (1.1). Nevertheless, our methodology and results in this paper are different from those in the papers cited above. We first note the problem (1.1) as a perturbation of the problem (1.2), so we

can construct an integral operator for the corresponding linear periodic boundary value problem without impulsive effects and find out its first eigenvalue and eigenfunction. Then we establish a special cone associated with the Green function of (1.1). Finally, by employing fixed point index theory, combined with a priori estimates of positive solutions, the existence and multiplicity of positive solutions of (1.1) are obtained. It is of interest to note that f, I_k in (1.1) may grow superlinearly and sublinearly. These mean that our methodology and main results here are entirely different from those in [19], [21].

This paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence and multiplicity of positive solutions for (1.1).

2. Preliminaries

Let $J' = J \setminus \{t_1, \ldots, t_m\}$,

$$\mathrm{PC}(J, \mathbb{R}) := \{x \colon J \mapsto \mathbb{R} :$$
$$x|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), \ x(t_k^-) = x(t_k), \ \exists x(t_k^+), \ k = 1, \ldots, m\}$$

be a Banach space with norm $\|x\|_{\mathrm{PC}} := \sup_{t \in [0, 2\pi]} |x(t)|$. Let $P := \{u \in \mathrm{PC}(J, \mathbb{R}) : u(t) \geq 0, \ t \in J\}$, and then P is a cone in $\mathrm{PC}(J, R)$.

LEMMA 2.1 (see [19]). *Let f and I_k be defined in (1.1). Then the problem (1.1) is equivalent to*

$$(2.1) \qquad u(t) = \int_0^{2\pi} G(t, s) f(s, u(s)) \, ds + \sum_{k=1}^m G(t, t_k) I_k(u(t_k)) =: (Au)(t),$$

where

$$G(t, s) := \frac{1}{2\rho(e^{2\rho\pi} - 1)} \begin{cases} e^{\rho(t-s)} + e^{\rho(2\pi - t + s)}, & 0 \leq s \leq t \leq 2\pi, \\ e^{\rho(s-t)} + e^{\rho(2\pi - s + t)}, & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

Note that if $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, then $A \colon P \to P$ is a completely continuous operator. From (2.1), we easily know that the existence of positive solutions for (1.1) is equivalent to that of positive fixed points of A. Define $\omega := 1/e^{2\rho\pi} > 0$ and $P_0 := \{u \in \mathrm{PC}(J, \mathbb{R}) : u(t) \geq \omega\|u\|, \ t \in J\}$. Clearly, P_0 is also a cone in $\mathrm{PC}(J, \mathbb{R})$.

LEMMA 2.2. *$A(P) \subset P_0$ (in particular $A(P_0) \subset P_0$).*

PROOF. We note that G has the following properties

$$(2.2) \quad \frac{1}{\rho(e^{2\rho\pi} - 1)} < \frac{e^{\rho\pi}}{\rho(e^{2\rho\pi} - 1)} = G(\pi, 0) \leq G(t, s)$$

$$\leq G(0, 2\pi) = \frac{e^{2\rho\pi} + 1}{2\rho(e^{2\rho\pi} - 1)} < \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)}.$$

By the definition of A and (2.2), we have

$$\|Au\| \leq \frac{e^{2\rho\pi} + 1}{2\rho(e^{2\rho\pi} - 1)} \int_0^{2\pi} f(s, u(s)) \, ds + \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \sum_{k=1}^m I_k(u(t_k)).$$

On the other hand, by (2.2) again, we find,

$$
\begin{aligned}
(Au)(t) &\geq \frac{e^{\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \int_0^{2\pi} f(s, u(s)) \, ds + \frac{1}{\rho(e^{2\rho\pi} - 1)} \sum_{k=1}^m I_k(u(t_k)) \\
&= \frac{2e^{\rho\pi}}{e^{2\rho\pi} + 1} \cdot \frac{e^{2\rho\pi} + 1}{2\rho(e^{2\rho\pi} - 1)} \int_0^{2\pi} f(s, u(s)) \, ds \\
&\quad + \frac{1}{e^{2\rho\pi}} \cdot \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \sum_{k=1}^m I_k(u(t_k)) \\
&\geq \frac{1}{e^{2\rho\pi}} \left[\frac{e^{2\rho\pi} + 1}{2\rho(e^{2\rho\pi} - 1)} \int_0^{2\pi} f(s, u(s)) \, ds + \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \sum_{k=1}^m I_k(u(t_k)) \right].
\end{aligned}
$$

As a result, $(Au)(t) \geq \omega \|Au\|$, for all $t \in J$ as claimed. $\qquad \square$

REMARK 2.3. The completely continuous operator A, defined by (2.1), satisfies $A(P) \subset P_0$ and in particular $A(P_0) \subset P_0$. Hence our proof in this paper will be carried out in P_0 rather than in P.

Denote a linear operator by

$$(Lu)(t) := \int_0^{2\pi} G(t, s) u(s) \, ds, \quad t \in J,$$

then it is easy to check that $L \colon C(J) \to C(J)$ is a completely continuous and positive operator. From now on, let $r(L)$ denote the spectral radii of L. By Gelfand's theorem, we have easily the following result.

LEMMA 2.4. L's spectral radii $r(L) > 0$.

PROOF. By the definition of L, we have

$$\|L\| = \max_{t \in J} \int_0^{2\pi} G(t, s) \, ds \geq \max_{t \in J} \int_0^{2\pi} G(\pi, 0) \, ds = 2\pi G(\pi, 0) > 0.$$

Similarly,

$$
\begin{aligned}
\|L^2\| &= \max_{t \in J} \int_0^{2\pi} \int_0^{2\pi} G(t, s) G(s, \tau) \, d\tau \, ds \\
&\geq \max_{t \in J} \int_0^{2\pi} \int_0^{2\pi} G(\pi, 0) G(\pi, 0) \, d\tau \, ds = [2\pi G(\pi, 0)]^2.
\end{aligned}
$$

Inductively, we have $\|L^n\| \geq [2\pi G(\pi, 0)]^n$, $n = 1, 2, \dots$ Therefore, Gelfand's Theorem gives

$$r(L) = \inf_{n \geq 1} \sqrt[n]{\|L^n\|} \geq 2\pi G(\pi, 0) > 0. \qquad \square$$

By Lemma 2.4 and the Krein–Rutman Theorem [9], there exist two functions $\varphi \in P \setminus \{0\}$ and $\psi \in P \setminus \{0\}$ with $\psi(x) \geq 0$ for which

$$\int_0^{2\pi} G(t,s)\varphi(s)\,ds = r(L)\varphi(t),$$

(2.3)
$$\int_0^{2\pi} G(t,s)\psi(t)\,dt = r(L)\psi(s),$$

$$\int_0^{2\pi} \psi(t)\,dt = 1.$$

We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

LEMMA 2.5 (see [5]). *If* $A : \overline{B}_\rho \cap P \to P$ *is a completely continuous operator. If there exists* $u_0 \in P \setminus \{0\}$ *such that* $u - Au \neq \lambda u_0$, *for all* $\lambda \geq 0$, $u \in \partial B_\rho \cap P$, *then* $i(A, B_\rho \cap P, P) = 0$.

LEMMA 2.6 (see [5]). *If* $0 \in B_\rho$ *and* $A : \overline{B}_\rho \cap P \to P$ *is a completely continuous operator. If* $u \neq \lambda Au$, *for all* $u \in \partial B_\rho \cap P$, $0 \leq \lambda \leq 1$, *then* $i(A, B_\rho \cap P, P) = 1$.

3. Main results

Let $\lambda_1 := 1/r(L)$. We now list our hypotheses.

(H1) There exist $c > 0$ and $a_1 \geq 0$, $a_2 \geq 0$ satisfying $a_1 + \omega a_2 \sum_{k=1}^m \psi(t_k) > \lambda_1$, such that

(3.1) $f(t,u) \geq a_1 u - c$, $\quad I_k(u) \geq a_2 u - c$, $\quad$ for all $t \in [0, 2\pi]$, $u \geq 0$.

(H2) There exist $r > 0$ and $b_1 \geq 0$, $b_2 \geq 0$ satisfying

$$b_1^2 + b_2^2 \neq 0, \qquad b_1 + \omega^{-1} b_2 \sum_{k=1}^m \psi(t_k) < \lambda_1,$$

such that

(3.2) $f(t,u) \leq b_1 u$, $\quad I_k(u) \leq b_2 u$, $\quad$ for all $t \in [0, 2\pi]$, $0 < u < r$.

(H3) There exist $r > 0$ and $a_3 \geq 0$, $a_4 \geq 0$ satisfying $a_3 + \omega a_4 \sum_{k=1}^m \psi(t_k) > \lambda_1$, such that

(3.3) $f(t,u) \geq a_3 u$, $\quad I_k(u) \geq a_4 u$, $\quad$ for all $t \in [0, 2\pi]$, $0 < u < r$.

(H4) There exist $c > 0$ and $b_3 \geq 0$, $b_4 \geq 0$ satisfying

$$b_3^2 + b_4^2 \neq 0, \qquad b_3 + \omega^{-1} b_4 \sum_{k=1}^m \psi(t_k) < \lambda_1,$$

such that

(3.4) $f(t,u) \leq b_3 u + c$, $\quad I_k(u) \leq b_4 u + c$, $\quad$ for all $t \in [0, 2\pi]$, $u \geq 0$.

(H5) There is a $\delta > 0$ such that $0 \le u \le \delta$ and $0 \le t \le 2\pi$ implies

$$f(t,u) \le \eta\delta, \qquad I_k(u) \le \eta_k\delta,$$

where $\eta, \eta_k \ge 0$ satisfy

$$\eta + \sum_{k=1}^{m} \eta_k > 0, \qquad 2\pi\eta + \sum_{k=1}^{m} \eta_k < \frac{\rho(e^{2\rho\pi} - 1)}{e^{2\rho\pi}}.$$

(H6) There is a $\delta > 0$ such that $\omega\delta \le u \le \delta$ and $0 \le t \le 2\pi$ implies

$$f(t,u) \ge \xi\delta, \qquad I_k(u) \ge \xi_k\delta,$$

where $\xi, \xi_k \ge 0$ satisfy

$$\xi + \sum_{k=1}^{m} \xi_k > 0, \quad 2\pi\xi + \sum_{k=1}^{m} \xi_k > \rho(e^{2\rho\pi} - 1).$$

THEOREM 3.1. *Suppose that* (H1), (H2) *are satisfied. Then* (1.1) *has at least one positive solution.*

PROOF. Let $\mathcal{M}_1 = \{u \in P : u = Au + \lambda\varphi, \lambda \ge 0\}$, where $\varphi \in P$ is determined by (2.3). We shall prove that $\mathcal{M}_1$ is bounded. Indeed, if $u \in \mathcal{M}_1$, then we have $u \ge Au$ by definition. This leads to

$$(3.5) \qquad u(t) \ge \int_0^{2\pi} G(t,s)f(s,u(s))\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)).$$

Multiply by $\psi(t)$ on both sides of the above and integrate over $[0,1]$ and use (2.3) to obtain

$$(3.6) \quad \int_0^{2\pi} u(t)\psi(t)\,dt$$

$$\ge \int_0^{2\pi} \psi(t)\left[\int_0^{2\pi} G(t,s)f(s,u(s))\,ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k))\right]dt$$

$$= \lambda_1^{-1}\int_0^{2\pi} f(t,u(t))\psi(t)\,dt + \lambda_1^{-1}\sum_{k=1}^{m} \psi(t_k)I_k(u(t_k)).$$

Consequently,

$$(3.7) \qquad \lambda_1\int_0^{2\pi} u(t)\psi(t)\,dt \ge \int_0^{2\pi} f(t,u(t))\psi(t)\,dt + \sum_{k=1}^{m} \psi(t_k)I_k(u(t_k)).$$

Combining this and (3.1), we find

$$(3.8) \quad \lambda_1\int_0^{2\pi} u(t)\psi(t)\,dt \ge \int_0^{2\pi} \psi(t)(a_1u(t) - c)\,dt + \sum_{k=1}^{m} \psi(t_k)(a_2u(t_k) - c).$$

From (3.8) we obtain

$$(3.9) \qquad (\lambda_1 - a_1) \int_0^{2\pi} u(t)\psi(t)\,dt + c_1 \geq a_2 \sum_{k=1}^m u(t_k)\psi(t_k),$$

where $c_1 = c\left(1 + \sum_{k=1}^m \psi(t_k)\right)$. By Lemma 2.2, we have

$$(3.10) \qquad (\lambda_1 - a_1)\|u\| + c_1 \geq \omega a_2 \|u\| \sum_{k=1}^m \psi(t_k).$$

Therefore,

$$\|u\| \leq \frac{c_1}{a_1 + \omega a_2 \sum_{k=1}^m \psi(t_k) - \lambda_1} =: \mathcal{N}_1, \quad \text{for all } u \in \mathcal{M}_1,$$

which establishes the boundedness of $\mathcal{M}_1$, as required. Taking $R > \mathcal{N}_1$, we obtain

$$u \neq Au + \lambda\varphi, \quad \text{for all } u \in \partial B_R \cap P, \ \lambda \geq 0.$$

Now Lemma 2.5 yields

$$(3.11) \qquad i(A, B_R \cap P, P) = 0.$$

Let $\mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda A u, \ \lambda \in [0,1]\}$. We shall prove $\mathcal{M}_2 = \{0\}$. Indeed, if $u \in \mathcal{M}_2$, we have

$$u(t) \leq (Au)(t) = \int_0^{2\pi} G(t,s)f(s,u(s))\,ds + \sum_{k=1}^m G(t,t_k)I_k(u(t_k)),$$

for all $u \in \overline{B}_r \cap P$. Similar to (3.6), multiply by $\psi(t)$ on both sides of the above and integrate over $[0,1]$ and use (2.3) to obtain

$$(3.12) \qquad \lambda_1 \int_0^{2\pi} u(t)\psi(t)\,dt \leq \int_0^{2\pi} \psi(t)f(t,u(t))\,dt + \sum_{k=1}^m \psi(t_k)I_k(u(t_k)),$$

for all $u \in \overline{B}_r \cap P$. This, together with (3.2), implies

$$(3.13) \qquad \lambda_1 \int_0^{2\pi} u(t)\psi(t)\,dt \leq b_1 \int_0^{2\pi} u(t)\psi(t)\,dt + b_2 \sum_{k=1}^m u(t_k)\psi(t_k).$$

Lemma 2.2 leads to

$$\omega(\lambda_1 - b_1)\|u\| \leq (\lambda_1 - b_1)\int_0^{2\pi} u(t)\psi(t)\,dt \leq b_2 \sum_{k=1}^m u(t_k)\psi(t_k) \leq b_2\|u\|\sum_{k=1}^m \psi(t_k),$$

which contradicts the condition $b_1 + \omega^{-1}b_2 \sum_{k=1}^m \psi(t_k) < \lambda_1$. This implies $\mathcal{M}_2 = \{0\}$ and thus $u \neq \lambda A u$ for all $u \in \partial B_r \cap P$ and $\lambda \in [0,1]$. Now Lemma 2.6 yields

$$(3.14) \qquad i(A, B_r \cap P, P) = 1.$$

Combining this with (3.11) gives $i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1$. Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. $\square$

THEOREM 3.2. *Suppose that* (H3), (H4) *are satisfied. Then* (1.1) *has at least one positive solution.*

PROOF. Let $\mathcal{M}_3 := \{u \in \overline{B}_r \cap P : u = Au + \lambda\varphi, \lambda \geq 0\}$. We claim $\mathcal{M}_3 \subset \{0\}$. Indeed, if $u \in \mathcal{M}_3$, then we have $u \geq Au$ by definition. By (3.3) and (3.7), we have

$$(3.15) \qquad \lambda_1 \int_0^{2\pi} u(t)\psi(t)\,dt \geq a_3 \int_0^{2\pi} u(t)\psi(t)\,dt + a_4 \sum_{k=1}^m u(t_k)\psi(t_k).$$

Lemma 2.2 and (3.15) yield

$$(\lambda_1 - a_3)\|u\| \geq (\lambda_1 - a_3) \int_0^{2\pi} u(t)\psi(t)\,dt \geq a_4 \sum_{k=1}^m u(t_k)\psi(t_k) \geq \omega a_4 \|u\| \sum_{k=1}^m \psi(t_k),$$

which contradicts the property $a_3 + \omega a_4 \sum_{k=1}^m \psi(t_k) > \lambda_1$. This implies $\|u\| \equiv 0$, for all $u \in \mathcal{M}_3$. Hence $\mathcal{M}_3 \subset \{0\}$, as claimed. As a result, we have $u - Au \neq \lambda\psi$ for all $u \in \partial B_r \cap P$ and $\lambda \geq 0$. Now Lemma 2.5 gives

$$(3.16) \qquad\qquad i(A, B_r \cap P, P) = 0.$$

Let $\mathcal{M}_4 := \{u \in P : u = \lambda Au, \lambda \in [0,1]\}$. We assert $\mathcal{M}_4$ is bounded. Indeed, if $u \in \mathcal{M}_4$, then $u \leq Au$, which can be written in the form

$$u(t) \leq \int_0^{2\pi} G(t,s)f(s,u(s))\,ds + \sum_{k=1}^m G(t,t_k)I_k(u(t_k)).$$

By (3.4) and (3.12), we obtain

$$\lambda_1 \int_0^{2\pi} u(t)\psi(t)\,dt \leq \int_0^{2\pi} \psi(t)(b_3 u(t) + c)\,dt + \sum_{k=1}^m \psi(t_k)(b_4 u(t_k) + c).$$

From Lemma 2.2, we get

$$\omega(\lambda_1 - b_3)\|u\| \leq (\lambda_1 - b_3) \int_0^{2\pi} u(t)\psi(t)\,dt$$

$$\leq b_4 \sum_{k=1}^m u(t_k)\psi(t_k) + c_1 \leq b_4 \|u\| \sum_{k=1}^m \psi(t_k) + c_1,$$

where $c_1 = c\left(1 + \sum_{k=1}^m \psi(t_k)\right)$. Consequently, we see

$$\|u\| \leq \frac{\omega^{-1}c_1}{\lambda_1 - \left(b_3 + \omega^{-1}b_4 \sum_{k=1}^m \psi(t_k)\right)} =: \mathcal{N}_2.$$

Now the boundedness of $\mathcal{M}_4$, as asserted. Taking $R > \mathcal{N}_2$, we have $u \neq \lambda Au$ for all $u \in \partial B_R \cap P$ and $\lambda \in [0,1]$. Now Lemma 2.6 yields

$$(3.17) \qquad i(A, B_R \cap P, P) = 1.$$

Combining this with (3.16) gives $i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1$. Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. $\qquad \square$

THEOREM 3.3. *Suppose that* (H1), (H3) *and* (H5) *are satisfied. Then* (1.1) *has at least two positive solutions.*

PROOF. By (H5) and (2.2), we have

$$\|Au(t)\| \leq \int_0^{2\pi} \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \eta\delta \, ds + \sum_{k=1}^m \frac{e^{2\rho\pi}}{\rho(e^{2\rho\pi} - 1)} \eta_k \delta$$

$$= \frac{e^{2\rho\pi}\delta}{\rho(e^{2\rho\pi} - 1)} \left[2\pi\eta + \sum_{k=1}^m \eta_k \right] < \delta.$$

Consequently, $\|Au\| < \|u\|$, for all $u \in \partial B_\delta \cap P$, and thus $u \neq \lambda Au$ for all $u \in \partial B_\delta \cap P$ and $\lambda \in [0,1]$. Now Lemma 2.6 yields

$$(3.18) \qquad i(A, B_\delta \cap P, P) = 1.$$

On the other hand, in view of (H1) and (H3), we may choose $R > \delta$ and $r \in (0, \delta)$ such that (3.11) and (3.16) hold (see the proofs of Theorems 3.1 and 3.2). Together with (3.11), (3.16) and (3.18), we have

$$i(A, (B_R \setminus \overline{B}_\delta) \cap P, P) = 0 - 1 = -1, \qquad i(A, (B_\delta \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Therefore, A has at least two fixed points, one on $(B_R \setminus \overline{B}_\delta) \cap P$ and the other on $(B_\delta \setminus \overline{B}_r) \cap P$. Equivalently, (1.1) has at least two positive solutions. $\qquad \square$

THEOREM 3.4. *Suppose that* (H2), (H4) *and* (H6) *are satisfied. Then* (1.1) *has at least two positive solutions.*

PROOF. By Lemma 2.2, red for all $u \in \partial B_\delta \cap P$ and $t \in [0, 2\pi]$, we obtain $\omega\delta \leq u(t) \leq \delta$. Therefore, from (H6) and (2.2), we have

$$(3.19) \qquad \|Au(t)\| \geq \int_0^{2\pi} \frac{1}{\rho(e^{2\rho\pi} - 1)} \xi\delta \, ds + \sum_{k=1}^m \frac{1}{\rho(e^{2\rho\pi} - 1)} \xi_k \delta$$

$$= \frac{\delta}{\rho(e^{2\rho\pi} - 1)} \left[2\pi\xi + \sum_{k=1}^m \xi_k \right] > \delta.$$

Consequently, we find $u - Au \neq \lambda\varphi$, for all $u \in \partial B_\delta \cap P$ and $\lambda \geq 0$. We obtain by Lemma 2.5

$$(3.20) \qquad i(A, B_\delta \cap P, P) = 0.$$

On the other hand, in view of (H2) and (H4), we may choose $R > \delta$ and $r \in (0, \delta)$ such that (3.14) and (3.17) hold (see the proofs of Theorem 3.1 and 3.2). We obtain by (3.14), (3.17) and (3.20)

$$i(A, (B_R \setminus \overline{B}_\delta) \cap P, P) = 1 - 0 = 1, \qquad i(A, (B_\delta \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Hence A has at least two fixed points, one on $(B_R \setminus \overline{B}_\delta) \cap P$ and the other on $(B_\delta \setminus \overline{B}_r) \cap P$, and thus (1.1) has at least two positive solutions. $\square$

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