THE EFFECT OF DIFFUSION
ON CRITICAL QUASILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We discuss the role of the diffusion coefficient $a(x)$ on the existence of a positive solution for the quasilinear elliptic problem involving critical exponent

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = u^{p^*-1} + \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, $1 < p < n$, $p^* = np/(n-p)$ is the critical exponent from the viewpoint of Sobolev embedding, λ is a real parameter and $a: \overline{\Omega} \to \mathbb{R}$ is a positive continuous function. We prove that if the function $a(x)$ has an interior global minimum point x_0 of order σ, then the range of values λ for which the problem above has a positive solution relies strongly on σ. We discover in particular that the picture changes drastically from $\sigma > p$ to $\sigma \leq p$. Some sharp answers are also provided.

1. Introduction and main theorems

In the 80's decade, Brezis and Nirenberg investigated, in the celebrated paper [9], the existence of a positive solution for the problem

$$(1.1) \qquad \begin{cases} -\Delta u = u^{(n+2)/(n-2)} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, where λ is a real parameter. As it is well known, the existence of such a positive solution relies on the value of the constant λ compared with the first eigenvalue $\lambda_1(-\Delta)$ corresponding to the Laplace operator on Ω under Dirichlet boundary condition. According to one of the main results of [9], the problem (1.1) admits a positive solution if, and only if, $\lambda \in (0, \lambda_1(-\Delta))$ provided that $n \geq 4$ and Ω is star-shaped. Besides, when $n = 3$ and Ω is a ball, a positive solution of (1.1) exists if, and only if, $\lambda \in (\lambda_1(-\Delta)/4, \lambda_1(-\Delta))$. Since then, a lot of attention has been devoted to various questions and extensions related to (1.1). We refer for instance to the Struwe's [36] and Willem's [38] books and references therein for an overview on the so-called Brezis–Nirenberg problem.

A closely related extension which has been widely addressed in the literature is the existence of a positive solution for the problem

(1.2)
$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $n \geq 2$, $1 < p < n$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ represents the p-Laplace operator and $p^* = np/(n-p)$ denotes the critical Sobolev exponent to the embedding of $W_0^{1,p}(\Omega)$ into Lebesgue spaces. Again the existence of a positive solution for (1.2) relies on the location of λ with respect to the first eigenvalue $\lambda_1(-\Delta_p)$ corresponding to the p-Laplace operator on Ω under Dirichlet boundary condition. The above-mentioned result of [9] valid for $n \geq 4$ and star-shaped domains has been extended by Egnell [19], Garcia Azorero and Peral Alonso [21] and Guedda and Veron [25], who have proved that the problem (1.2) admits a positive solution if, and only if, $\lambda \in (0, \lambda_1(-\Delta_p))$ provided that $1 < p \leq \sqrt{n}$. Under this range on p, De Valerioli and Willem [14] has also discussed the existence of solution. The case $p > \sqrt{n}$ has been considered by Egnell [19], who has found a number λ^* so that a positive solution of (1.2) exists for any $\lambda \in (\lambda^*, \lambda_1(-\Delta_p))$. When $\lambda \geq \lambda_1(-\Delta_p)$, results on existence of a nontrivial solution have been established in [11] and [22] for $p = 2$ and in [2] and [15] for $p \neq 2$.

Another important line of investigation consists of the study of existence of positive (or nontrivial) solutions for problems connected to (1.2) in the presence of diffusion such as

(1.3)
$$\begin{cases} -\Delta_p u = \alpha(x)u^{p^*-1} + \beta(x)u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

(1.4)
$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = u^{p^*-1} + \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the diffusion coefficients $\alpha(x)$, $\beta(x)$ and $a(x)$ are positive continuous functions on $\overline{\Omega}$. The problem (1.3) has been focused in various works in the case $p = 2$ as, for instance, [5], [7], [11], [20], [22]–[24], [32], [33], [40] and in the case $p \neq 2$, in [2], [6], [15], [17], [19], [28], among others. Already the problem (1.4) has been investigated in a few works where a comprehensive study has been performed, namely [18], [26] and [31] for $p = 2$ and [19] for $p \neq 2$.

Our main goal is discussing the effect of the order of global minimum points of the diffusion $a(x)$ on the existence of a positive solution for the problem (1.4) when $p \neq 2$.

For simplicity of notation, let $\mathcal{L}_{p,a}u = \mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$. The operator $\mathcal{L}_{p,a}$ is degenerate elliptic for $p > 2$ and singular for $p < 2$. By a positive solution of (1.4) we mean a nontrivial nonnegative weak solution in $W_0^{1,p}(\Omega)$. A positive solution in $C^1(\overline{\Omega})$ will be simply referred as a positive C^1 solution.

Let $\lambda_1(-\mathcal{L}_{p,a})$ be the first eigenvalue corresponding to the problem

$$(1.5) \qquad \begin{cases} -\mathcal{L}_{p,a}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A fairly simple fact that deserves mention due to Egnell [19] is the existence of a number λ^* less than $\lambda_1(-\mathcal{L}_{p,a})$ such that a positive solution of (1.4) exists for any $\lambda \in (\lambda^*, \lambda_1(-\mathcal{L}_{p,a}))$. We are interested in the seeking for optimal values of λ^* or, at least, for good upper and lower bounds of λ^* depending of the order of global minimum points of the function $a(x)$.

We first state two main existence results, the first one to higher order minimum points and the second one dealing with lower order.

THEOREM 1.1. *Assume that $a(x)$ attains its minimum value at $x_0 \in \Omega$ and there exist constants $\sigma > p$ and $C_0 > 0$ such that*

$$(1.6) \qquad a(x) \leq a(x_0) + C_0|x - x_0|^{\sigma}$$

around x_0. If $1 < p \leq \sqrt{n}$, then the problem (1.4) has a positive solution for any $\lambda \in (0, \lambda_1(-\mathcal{L}_{p,a}))$.

The conclusion of Theorem 1.1 remains indeed valid if the condition (1.6) is replaced by $a(x) = a(x_0) + o(|x - x_0|^p)$ around x_0, see Section 3 for full details. The importance of working with global minimum is justified by Lemma 2.1.

THEOREM 1.2. *Assume that there exist $x_0 \in \Omega$ and constants $0 < \sigma \leq p$ and $C_0 > 0$ such that*

$$(1.7) \qquad a(x) \geq a(x_0) + C_0|x - x_0|^{\sigma}$$

for all $x \in \Omega$. If $1 < p < n$, then there exists a positive constant λ^* such that the problem (1.4) has a positive solution for any $\lambda \in (\lambda^*, \lambda_1(-\mathcal{L}_{p,a}))$.

Note that the point x_0 in the theorem above is a global minimum of $a(x)$. If further x_0 is unique, it suffices instead of assuming that $a(x) \geq a(x_0) + C_0|x - x_0|^\sigma$ around x_0.

We next present two main nonexistence results, the second one deals with lower order minimum points.

THEOREM 1.3. *Assume that Ω is of C^1 class and star-shaped with respect to $x_0 \in \Omega$, a is of C^1 class on $\overline{\Omega} \setminus \{x_0\}$ and $\nabla a(x) \cdot (x - x_0)$ extends continuously to x_0 and is nonnegative on Ω. If $1 < p < n$, then the problem (1.4) has no positive C^1 solution for any $\lambda \in (-\infty, 0]$.*

It deserves mention that the problem (1.4) has no positive solution whenever $\lambda \geq \lambda_1(-\mathcal{L}_{p,a})$. This assertion follows from a slight adaptation of the proof of Theorem 3.3 of Guedda and Veron [25].

The assumptions in the theorem above imply that x_0 is a global minimum point of $a(x)$, as can easily be checked.

THEOREM 1.4. *Assume that Ω is of C^1 class and star-shaped with respect to $x_0 \in \Omega$, a is of C^1 class on $\overline{\Omega} \setminus \{x_0\}$ and $\nabla a(x) \cdot (x - x_0)$ extends continuously to x_0. Assume further that there exist constants $0 < \sigma \leq p$ and $C_0 > 0$ such that*

$$(1.8) \qquad\qquad \nabla a(x) \cdot (x - x_0) \geq \sigma C_0 |x - x_0|^\sigma$$

for all $x \in \Omega$. If $1 < p < n$, then there exists a positive constant λ_* such that the problem (1.4) has no positive C^1 solution for any $\lambda \in (-\infty, \lambda_*]$.

A quite simple argument shows that the condition (1.8) implies (1.7). So, the conditions assumed in Theorems 1.2 and 1.4 and standard regularity theorems guarantees the existence of constants $0 < \lambda_* \leq \lambda^*$ such that a positive C^1 solution of (1.4) exists for any $\lambda \in (\lambda^*, \lambda_1(-\mathcal{L}_{p,a}))$ and no existence occurs for any $\lambda \in (-\infty, \lambda_*]$. Regarding the case that $n \geq p^2$, Ω is star-shaped with respect to x_0 and $a(x) = 1 + |x - x_0|^\sigma$, we discover that the picture changes considerably according to the cases $\sigma > p$ to $\sigma \leq p$. Precisely, assuming $\Omega \in C^1$ and $\sigma > p$, Theorems 1.1 and 1.3 and the above-mentioned nonexistence result for $\lambda \geq \lambda_1(-\mathcal{L}_{p,a})$ provide a positive C^1 solution for the problem (1.4) if, and only if, $\lambda \in (0, \lambda_1(-\mathcal{L}_{p,a}))$. On the other hand, if $\sigma \leq p$, then a nonexistence range is given by $(-\infty, \lambda_*]$ for some positive number λ_*. Moreover, when $\sigma = p$ and $n = p^2$, we prove in Section 6 that $\lambda_* = \lambda^*$. In order to further exploit this example, we investigate what happens as σ tends to zero. As we shall see later, $\lambda^* = 0$ in the case that $\sigma = 0$. So, in our approach, the classical Brezis–Nirenberg result for p-Laplace operators and $n \geq p^2$ is recovered, which can be framed in the cases $\sigma = 0$ or $\sigma > p$. However, it is interesting to note that

$$\liminf_{\sigma \to 0^+} \lambda^* \geq \lambda_1(-\Delta_p) > 0.$$

In particular, λ^* is not continuous at $\sigma = 0$. All assertions concerning the diffusion term $a(x) = 1 + |x - x_0|^\sigma$ will be proved in Sections 5 and 6.

In conclusion, if Ω is star-shaped with respect to any point (e.g. a ball), then the range of the values λ for which the problem (1.4) has a positive solution relies heavily on the order of interior global minimum points of the diffusion $a(x)$.

An important advantage of Theorems 1.2 and 1.4 is that the constants λ^* and λ_* are constructed explicitly in terms of best constants associated to Euclidean Hardy–Sobolev type inequalities. In particular, we will be able to furnish upper and lower bounds for these constants, see Section 5 for more details.

In order to find solutions to problem (1.1) we can use either minimax methods involving the free energy functional or constrained minimization. We refer for instance to the Willem's book [38] for more details. Here we employ the second strategy. Precisely, consider the functional

$$\Psi_{p,a}(u) = \int_\Omega a(x)|\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx$$

constrained to the set $E_p := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^{p^*} \, dx = 1\}$. Since $\Psi_{p,a}(u) = \Psi_{p,a}(|u|)$ for all $u \in E_p$, the existence of a minimizer for $\Psi_{p,a|E_p}$ leads readily to a positive solution of (1.4).

As we shall see in Section 2, the condition (1.6) assumed in Theorem 1.1 combined with standard estimates on bubbles around x_0 yields

$$\Psi_{p,a}(u_0) < m_a K(n,p)^{-p}$$

on some function $u_0 \in E_p$, where m_a denotes the minimum value of $a(x)$ on $\overline{\Omega}$ and $K(n,p)$ is the best constant to the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$. Once this is done, the existence of a positive solution is classically deduced from the preceding estimate. We present a complete proof for convenience of the reader. A different proof can also be deduced by using the Ekeland variational principle and a compactness result recently obtained by Mercuri and Willem in [30].

The proof of Theorem 1.2 involves the validity on $C_0^\infty(\Omega)$ of the sharp Sobolev inequality

$$(1.9) \qquad \left(\int_\Omega |u|^{p^*} \, dx \right)^{p/p^*} \le K(n,p)^p \int_\Omega |\nabla u|^p \, dx$$

and of the sharp Hardy–Sobolev inequality

$$(1.10) \qquad \int_\Omega |u|^p \, dx \le K_0(n,p,\sigma,\Omega,x_0)^p \int_\Omega |x - x_0|^\sigma |\nabla u|^p \, dx$$

provided that $0 < \sigma \le p$. Inequality (1.10) is a special case of the famous Caffarelli–Konh–Nirenberg inequality presented in [10]. When $\sigma = p$, the value of $K_0(n,p,\sigma,\Omega,x_0)$ does not depend on Ω and x_0, namely $K_0(n,p,\sigma,\Omega,x_0) = p/n$, as can be seen in [27] for $p = 2$ and in [1] for $p \ne 2$. Thanks to (1.9) and (1.10)

and to the condition (1.7), the positive constant λ^* will be constructed as the supremum of values $\lambda > 0$ for which the inequality

$$\left(\int_\Omega |u|^{p^*} dx \right)^{p/p^*} + \lambda m_a^{-1} K(n,p)^p \int_\Omega |u|^p \, dx \leq m_a^{-1} K(n,p)^p \int_\Omega a(x) |\nabla u|^p \, dx$$

holds for all $u \in C_0^\infty(\Omega)$. Section 3 will be devoted to the detailed proof of Theorems 1.1 and 1.2.

Pohožaev type integral identities are powerful tools in proving nonexistence of solutions for quasilinear elliptic boundary problems. Related to Theorem 1.3, we use one of them, namely a version of the Pucci–Serrin identity [34] for C^2 solutions refined by Degiovanni, Musesti and Squassina in [15] for C^1 solutions. The same identity is useful in the proof of Theorem 1.4 together with the sharp Hardy–Sobolev inequality (1.10) which plays a key role. The proof of Theorems 1.3 and 1.4 will be presented in Section 4. In Section 5, upper and lower bounds will be obtained for λ^* and λ_* considering some particularly interesting examples of diffusion coefficients $a(x)$. In Section 6, it will be outlined some open problems which are naturally motivated by our results. Part of the tools required in proofs will be provided in Section 2.

2. Preliminary tools

Our aim in this section is to provide two suitable lemmas for the proof of the main theorems.

As mentioned in the introduction, nontrivial solutions of (1.4) can be found as critical points of the functional

$$\Psi_{p,a}(u) = \int_\Omega a(x) |\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx$$

constrained to the set $E_p := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^{p^*} dx = 1\}$. Such a critical point is called a least energy solution of (1.4) and

$$c_{p,a} := \inf_{u \in E_p} \Psi_{p,a}(u)$$

is called the least energy level of $\Psi_{p,a}$ on E_p.

The conclusion of Theorems 1.1 and 1.2 relies on a rather standard lemma whose proof will be outlined for reader's convenience. The proof given below follows the ideas of an elegant and simple proof of minimizers existence presented by Barstch and Willem in [4] and communicated to us by M. Willem.

Given a positive continuous function $a(x)$ on $\overline{\Omega}$, recall that m_a denotes its minimum value on $\overline{\Omega}$ and $K(n,p)$ denotes the best constant to the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$.

LEMMA 2.1. *If $0 < c_{p,a} < m_a K(n,p)^{-p}$, then the constrained functional $\Psi_{p,a}|_{E_p}$ attains its infimum.*

PROOF. Let $(u_j) \subset E_p$ be such that

$$\Psi_{p,a}(u_j) = \int_\Omega a(x)|\nabla u_j|^p \, dx - \lambda \int_\Omega |u_j|^p \, dx \to c_{p,a}$$

as $j \to +\infty$. We can assume that $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_j \to u$ in $L^p(\Omega)$ and $u_j \to u$ almost everywhere. on Ω. By the sharp Sobolev inequality,

$$c_{p,a} = \lim_{j \to +\infty} \left(\int_\Omega a(x)|\nabla u_j|^p \, dx - \lambda \int_\Omega |u_j|^p \, dx \right) \geq m_a K(n,p)^{-p} - \lambda \int_\Omega |u|^p \, dx.$$

Thus, since $c_{p,a} < m_a K(n,p)^{-p}$, it follows that $u \neq 0$. We now show that $\|u\|_{L^{p^*}(\Omega)} = 1$. As in [4], consider, for $h > 0$, the auxiliary functions

$$T_h(s) = \min(\max(s, -h), h) \quad \text{and} \quad R_h(s) = s - T_h(s).$$

As can be easily checked, $T_h(u), R_h(u) \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega a(x)|\nabla u|^p \, dx = \int_\Omega a(x)|\nabla T_h(u)|^p \, dx + \int_\Omega a(x)|\nabla R_h(u)|^p \, dx$$

for any $u \in W_0^{1,p}(\Omega)$. Consequently,

$$
\begin{aligned}
c_{p,a} &= \lim_{j \to +\infty} \left(\int_\Omega a(x)|\nabla u_j|^p \, dx - \lambda \int_\Omega |u_j|^p \, dx \right) \\
&= \lim_{j \to +\infty} \left(\int_\Omega a(x)|\nabla T_h(u_j)|^p \, dx + \int_\Omega a(x)|\nabla R_h(u_j)|^p \, dx \right) - \lambda \int_\Omega |u|^p \, dx \\
&= \lim_{j \to +\infty} \Bigg(\int_\Omega a(x)|\nabla T_h(u_j)|^p \, dx \\
&\quad - \lambda \int_\Omega |T_h(u_j)|^p \, dx + \int_\Omega a(x)|\nabla T_h(u_j)|^p \, dx - \lambda \int_\Omega |R_h(u_j)|^p \, dx \Bigg) \\
&\quad + \lambda \int_\Omega |T_h(u)|^p \, dx + \lambda \int_\Omega |R_h(u)|^p \, dx - \lambda \int_\Omega |u|^p \, dx \\
&\geq c_{p,a} \lim_{j \to +\infty} \left[\left(\int_\Omega |T_h(u_j)|^{p^*} \, dx \right)^{p/p^*} + \left(\int_\Omega |R_h(u_j)|^{p^*} \, dx \right)^{p/p^*} \right] \\
&\quad + \lambda \int_\Omega |T_h(u)|^p \, dx + \lambda \int_\Omega |R_h(u)|^p \, dx - \lambda \int_\Omega |u|^p \, dx.
\end{aligned}
$$

Thanks to Lemma 3.1 of [4], one gets

$$
\begin{aligned}
c_{p,a} \geq c_{p,a} &\left[\left(\int_\Omega |T_h(u)|^{p^*} \, dx \right)^{p/p^*} + \left(1 + \int_\Omega |R_h(u)|^{p^*} \, dx - \int_\Omega |u|^{p^*} \, dx \right)^{p/p^*} \right] \\
&+ \lambda \int_\Omega |T_h(u)|^p \, dx + \lambda \int_\Omega |R_h(u)|^p \, dx - \lambda \int_\Omega |u|^p \, dx.
\end{aligned}
$$

Using that $c_{p,a} > 0$, when $h \to +\infty$, one obtains

$$1 \geq \left(\int_\Omega |u|^{p^*} dx \right)^{p/p^*} + \left(1 - \int_\Omega |u|^{p^*} dx \right)^{p/p^*},$$

so that $\|u\|_{L^*(\Omega)} = 1$. By lower semi-continuity, it then follows that u is a minimizer for $c_{p,a}$. $\qquad\square$

As previously already quoted, nonexistence results for elliptic problems are often established through integral identities satisfied by any solution. Particularly, the proof of Theorems 1.3 and 1.4 relies on the following integral identity:

LEMMA 2.2 ([15], [34]). *Let* $1 < p < n$. *Assume that* Ω *is of* C^1 *class, a is of* C^1 *class on* $\overline{\Omega} \setminus \{x_0\}$ *and* $b(x) = \nabla a(x) \cdot (x - x_0)$ *extends continuously to* x_0. *Let u be a* C^1 *solution of* (1.4). *Then u satisfies the identity*

$$(2.1) \qquad \frac{1}{p} \int_{\partial\Omega} a(x) |\nabla u|^p (x - x_0) \cdot \nu \, ds = \lambda \int_\Omega |u|^p dx - \frac{1}{p} \int_\Omega b(x) |\nabla u|^p dx,$$

where ν denotes the exterior normal field to $\partial\Omega$ *and ds stands for the element of area corresponding to* $\partial\Omega$.

3. Proof of Theorems 1.1 and 1.2

The proof of Theorems 1.1 and 1.2 involves Lemma 2.1, bubbles estimates around x_0 and the valid of the sharp Hardy–Sobolev inequality (1.10) on $C_0^\infty(\Omega)$.

PROOF OF THEOREM 1.1. Although we could provide an alternative proof by using some ideas due to Egnell [18], [19], we choose considering another functional on E defined in the introduction, namely $\Psi_{p,a}$, in a similar way to the one of Garcia Azorero and Peral Alonso [21]. As explained in detail in the introduction, the key point of this proof is Lemma 2.1. First, without loss of generality, assume that $x_0 = 0$. By the assumption (1.6), there exists $\delta > 0$ such that $B_\delta(0) \subset \Omega$ and

$$a(x) \leq a(0) + C_0 |x|^\sigma$$

for all $x \in B_\delta(0)$. Using this condition, we have

$$c_{p,a} \leq \inf_{u \in W_0^{1,p}(B_\delta(0)) \setminus \{0\}} \frac{\displaystyle\int_{B_\delta(0)} a(x) |\nabla u|^p dx - \int_{B_\delta(0)} \lambda |u|^p dx}{\left(\displaystyle\int_{B_\delta(0)} |u|^{p^*} dx \right)^{p/p^*}}$$

$$\leq \inf_{u \in W_0^{1,p}(B_\delta(0)) \setminus \{0\}} \frac{\displaystyle\int_{B_\delta(0)} a(0) |\nabla u|^p dx + C_0 \int_{B_\delta(0)} |x|^\sigma |\nabla u|^p dx - \lambda \int_{B_\delta(0)} |u|^p dx}{\left(\displaystyle\int_{B_\delta(0)} |u|^{p^*} dx \right)^{p/p^*}}.$$

In order to prove that $c_{p,a} < m_a K(n,p)^{-p}$, consider now a cutoff function $\eta \in C_0^\infty(B_\delta(0))$ with $\eta = 1$ around 0 and extremal functions for $K(n,p)$ on $\mathcal{D}(\mathbb{R}^n)$ given to one-parameter $\varepsilon > 0$ by

$$v_\varepsilon(x) = c_{n,p}\varepsilon^{(n-p)/p(p-1)}(\varepsilon^{p/(p-1)} + |x|^{p/(p-1)})^{(p-n)/p},$$

where $c_{n,p}$ is normalized so that $||v_\varepsilon||_{p^*} = 1$, see [3] or [37].

According to [21], $w_\varepsilon = \eta v_\varepsilon \in W_0^{1,p}(B_\delta(0))$ and satisfies

$$
(3.1) \quad
\begin{aligned}
||\nabla w_\varepsilon||_p^p &= K(n,2)^{-p} + O(\varepsilon^{p+(n-p^2)/(p-1)}), \\
||w_\varepsilon||_{p^*}^p &= 1 + O(\varepsilon^n), \\
||w_\varepsilon||_p^p &= \begin{cases} a_{n,p}\varepsilon^p + O(\varepsilon^{p+(n-p^2)/(p-1)}) & \text{if } n > p^2, \\ a_{n,p}\varepsilon^p|\log\varepsilon| + O(\varepsilon^p) & \text{if } n = p^2, \end{cases}
\end{aligned}
$$

where

$$a_{n,p} = c_{n,p}^p \omega_{n-1} \frac{(p-1)\Gamma\left(\dfrac{n-p^2}{p}\right)\Gamma\left(\dfrac{np-n}{p}\right)}{p\Gamma(n-p)} \quad \text{for } n > p^2,$$

$$a_{n,p} = c_{n,p}^p \omega_{n-1} \quad \text{for } n = p^2.$$

Here ω_{n-1} stands for the area of the round unit sphere S^{n-1}.

On the other hand, when $\sigma > p$, straightforward computations yield

$$\int_{B_\delta(0)} |x|^\sigma |\nabla w_\varepsilon|^p \, dx = \begin{cases} O(\varepsilon^\sigma) & \text{if } \sigma < \dfrac{n-p}{p-1}, \\ O(\varepsilon^\sigma|\log\varepsilon|) & \text{if } \sigma = \dfrac{n-p}{p-1}, \\ O(\varepsilon^{(n-p)/(p-1)}) & \text{if } \sigma > \dfrac{n-p}{p-1}. \end{cases}$$

In particular,

$$(3.2) \quad \int_{B_\delta(0)} |x|^\sigma |\nabla w_\varepsilon|^p \, dx = \begin{cases} o(\varepsilon^p) & \text{if } n > p^2, \\ O(\varepsilon^p) & \text{if } n = p^2. \end{cases}$$

In conclusion, a combination between (3.1) and (3.2) produces, for $n \geq p^2$ and $\lambda > 0$,

$$c_{p,a} \leq \frac{K(n,p)^{-p} - a_{n,p}\lambda\varepsilon^p + o(\varepsilon^p)}{m_a^{-1}(1 + O(\varepsilon^n))} < m_a K(n,p)^{-p}$$

provided $\varepsilon > 0$ is small enough.

For $n = p^2$ and $\lambda > 0$, again we have

$$c_{p,a} \leq \frac{K(n,p)^{-p} - a_{n,p}\lambda\varepsilon^p|\log\varepsilon| + O(\varepsilon^p)}{m_a^{-1}(1 + O(\varepsilon^n))} < m_a K(n,p)^{-p}$$

for $\varepsilon > 0$ small too.

By Lemma 2.1, the constrained functional $\Psi_{p,a}|_{E_p}$ admits a minimizer u which can be taken nonnegative. In particular, u satisfies for some Lagrange multiplier $\mu \in \mathbb{R}$,

$$\begin{cases} -\mathcal{L}_{p,a}u = \mu u^{p^*-1} + \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u \in E_p$, we have $\mu = \Psi_{p,a}(u)$. Using now that $\lambda < \lambda_1(-\mathcal{L}_{p,a})$, one obtains $\mu > 0$. Finally, one easily checks that $\mu^{1/(p^*-p)}u$ is a positive solution of (1.4).

If the condition $a(x) = a(x_0) + o(|x - x_0|^p)$ is assumed around x_0, then (3.2) is replaced by

$$\int_\Omega |x - x_0|^p |\nabla w_\varepsilon|^p \, dx = \begin{cases} b\varepsilon^p + o(\varepsilon^p) & \text{if } n > p^2, \\ b\varepsilon^p |\log \varepsilon| + O(\varepsilon^p) & \text{if } n = p^2, \end{cases}$$

for constants $b > 0$ small enough. The rest of the proof then follows. $\qquad\square$

PROOF OF THEOREM 1.2. The building strategy for λ^* changes when $0 < \sigma \le p$ since estimates with bubbles has no effect. In this case, the validity of (1.10) is essential. Indeed, by (1.7) and (1.10) we can write

$$m_a^{-1} K(n,p)^p \int_\Omega a(x) |\nabla u|^p \, dx$$

$$\ge K(n,p)^p \int_\Omega |\nabla u|^p \, dx + C_0 m_a^{-1} K(n,p)^p \int_\Omega |x - x_0|^\sigma |\nabla u|^p \, dx$$

$$\ge \left(\int_\Omega |u|^{p^*} \, dx \right)^{p/p^*} + C_0 K_0(n,p,\sigma,\Omega,x_0)^{-p} m_a^{-1} K(n,p)^p \int_\Omega |u|^p \, dx.$$

Consider the set Θ of values $\lambda > 0$ for which the inequality

$$m_a^{-1} K(n,p)^p \int_\Omega a(x) |\nabla u|^p \, dx \ge \left(\int_\Omega |u|^{p^*} \, dx \right)^{p/p^*} + \lambda m_a^{-1} K(n,p)^p \int_\Omega |u|^p \, dx$$

holds for all $u \in C_0^\infty(\Omega)$.

Remark that the set Θ is non-empty, since $C_0 K_0(n,p,\sigma,\Omega,x_0)^{-p}$ belongs to Θ. Let $\lambda^* := \sup \Theta$. In particular, $\lambda^* \ge C_0 K_0(n,p,\sigma,\Omega,x_0)^{-p}$. Note also that λ^* is finite and Θ is a closed set, since the inequality

$$(3.3) \quad m_a^{-1} K(n,p)^p \int_\Omega a(x) |\nabla u|^p \, dx$$

$$\ge \left(\int_\Omega |u|^{p^*} \, dx \right)^{p/p^*} + \lambda^* m_a^{-1} K(n,p)^p \int_\Omega |u|^p \, dx.$$

holds for all $u \in C_0^\infty(\Omega)$.

It now suffices to prove that $c_{p,a} < m_a K(n,p)^{-p}$ for any $\lambda \in (\lambda^*, +\infty)$ and $\lambda^* < \lambda_1(-\mathcal{L}_{p,a})$. The first assertion follows immediately from the definition

of λ^*. In fact, since $\lambda \notin \Theta$ for any $\lambda > \lambda^*$, there exists $u_0 \in C_0^\infty(\Omega)$ such that

$$(3.4) \quad m_a^{-1} K(n,p)^p \int_\Omega a(x)|\nabla u_0|^p \, dx$$
$$< \left(\int_\Omega |u_0|^{p^*} \, dx \right)^{p/p^*} + \lambda m_a^{-1} K(n,p)^p \int_\Omega |u_0|^p \, dx,$$

or equivalently, $c_{p,a} < m_a K(n,p)^{-p}$. For the second one, we choose an eigenfunction φ_1 corresponding to the first eigenvalue $\lambda_1(-\mathcal{L}_{p,a})$ of the operator $-\mathcal{L}_{p,a}$ as a test function in (3.3). So, we are led to

$$\lambda_1(-\mathcal{L}_{p,a}) \int_\Omega |\varphi_1|^p \, dx = \int_\Omega a(x)|\nabla \varphi_1|^p \, dx > \lambda^* \int_\Omega |\varphi_1|^p \, dx,$$

so that the assertion follows. Proceeding now exactly as in the previous proof, one derives a positive solution of (1.4) for any $\lambda \in (\lambda^*, \lambda_1(-\mathcal{L}_{p,a}))$. $\square$

4. Proof of Theorems 1.3 and 1.4

The proof of Theorems 1.3 and 1.4 is based on the Pohožaev type integral identity (2.1) given in Lemma 2.2.

PROOF OF THEOREM 1.3. Assume, by contradiction, that (1.4) admits a positive C^1 solution u and that, without loss of generality, $x_0 = 0$. By the Hopf lemma due to Pucci, Serrin and Zhou (Theorem 1' of [35]), the gradient of u never vanishes on $\partial\Omega$. Since Ω is star-shaped with respect to the origin, $x \cdot \nu$ is nonnegative and non-identically zero on $\partial\Omega$. So, thanks to the positivity of the function a, the left-hand side of (2.1) is positive too. But, by assumption, we have $\nabla a(x) \cdot x \geq 0$ for all $x \in \Omega$ and, therefore, the equation (2.1) would be violated if $\lambda \leq 0$. $\square$

PROOF OF THEOREM 1.4. Arguing as in the preceding proof and using the assumption $\nabla a(x) \cdot x \geq \sigma C_0 |x|^\sigma$ for all $x \in \Omega$, the identity (2.1) yields

$$\lambda \int_\Omega |u|^p \, dx > \frac{\sigma}{p} C_0 \int_\Omega |x|^\sigma |\nabla u|^p \, dx.$$

Using now that $0 < \sigma \leq p$ and evoking the sharp Hardy-Sobolev inequality (1.10), one gets

$$\lambda \int_\Omega |u|^p \, dx > \frac{\sigma}{p} C_0 K_0(n,p,\sigma,\Omega,0)^{-p} \int_\Omega |u|^p \, dx,$$

so that $\lambda > \lambda_* =: \frac{\sigma}{p} C_0 K_0(n,p,\sigma,\Omega,0)^{-p}$. Consequently, for star-shaped domains about the origin, the problem (1.4) has no positive C^1 solution for any $\lambda \in (-\infty, \lambda_*]$. $\square$

5. Bounds for λ^* and λ_*

This section deals mainly with upper and lower bounds for λ^* and λ_* for some interesting examples of non-uniformly elliptic operators $\mathcal{L}_{p,a}$ of the type $\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$.

We first focus on the constant λ^* provided in Theorem 1.2. Let $x_0 \in \Omega$ be a global minimum point of $a(x)$ and assume that the function $a(x)$ satisfies (1.7), namely there exist constants $0 < \sigma \le p$ and $C_0 > 0$ such that

$$a(x) \ge a(x_0) + C_0|x - x_0|^\sigma$$

for all $x \in \Omega$. The construction of λ^* produces

(5.1) $$\lambda^* \ge C_0 K_0(n, p, \sigma, \Omega, x_0)^{-p}.$$

We recall that when $\sigma = p$, the exact value of the best constant $K_0(n, p, \sigma, \Omega, x_0)$ associated to the Hardy-Sobolev inequality (1.10) is given by

(5.2) $$K_0(n, p, p, \Omega, x_0) = \frac{p}{n}.$$

Unfortunately, the value of this optimal constant is not known for $0 < \sigma < p$, although some rough bounds can easily be deduced. Since $\lambda^* < \lambda_1(-\mathcal{L}_{p,a})$, upper bounds for λ^* can be obtained explicitly in terms of Ω. Nevertheless, when $\sigma = p$ and $a(x)$ also satisfies

(5.3) $$a(x) \le a(x_0) + D_0|x - x_0|^p$$

around x_0 for some constant $D_0 > 0$, then estimates on bubbles concentrated at x_0 readily lead to an upper bound for λ^* depending only on the dimension n. In fact, for the bubbles w_ε defined in the proof of Theorem 1.1, we have

(5.4) $$\int_\Omega a(x_0)|\nabla w_\varepsilon|^p \, dx = m_a K(n,p)^{-p} + O(\varepsilon^{p+(n-p^2)/(p-1)})$$

and

(5.5) $$\int_\Omega |x - x_0|^p |\nabla w_\varepsilon|^p \, dx = \begin{cases} b_{n,p}\varepsilon^p + o(\varepsilon^p) & \text{if } n > p^2, \\ b_{n,p}\varepsilon^p|\log \varepsilon| + O(\varepsilon^p) & \text{if } n = p^2, \end{cases}$$

with

$$b_{n,p} = c_{n,p}^p \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \frac{(p-1)\Gamma\left(\dfrac{n-p^2}{p}\right)\Gamma\left(\dfrac{np+p^2-n}{p}\right)}{p\Gamma(n)} \qquad \text{for } n > p^2,$$

$$b_{n,p} = p^p c_{n,p}^p \omega_{n-1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n = p^2.$$

Thus, choosing $u = w_\varepsilon$ in (3.3), using the inequality (5.3) and after the bubbles estimates (3.1), (5.4) and (5.5), we deduce that

$$(5.6) \qquad \lambda^* \leq D_0 \frac{b_{n,p}}{a_{n,p}}.$$

By contrast, note that the constant λ_* provided in Theorem 1.4 appears in an explicit way. Precisely, assume that Ω is a star-shaped domain with respect to $x_0 \in \Omega$ and the function $a(x)$ satisfies (1.8), precisely there exist constants $0 < \sigma \leq p$ and $C_0 > 0$ such that

$$\nabla a(x) \cdot (x - x_0) \geq \sigma C_0 |x - x_0|^\sigma$$

for all $x \in \Omega$. By definition,

$$\lambda_* := \frac{\sigma}{p} C_0 K_0(n, p, \sigma, \Omega, x_0)^{-p}.$$

On the other hand, the condition (1.8) implies (1.7), so that

$$(5.7) \qquad \lambda_* := \frac{\sigma}{p} C_0 K_0(n, p, \sigma, \Omega, x_0)^{-p} \leq \frac{\sigma}{p} \lambda^*.$$

Before going further, it can easily be checked from Lemma 2.2 and (5.7) that $\lambda_* = 0$ for $\sigma = 0$ and

$$\lim_{\sigma \to 0^+} \lambda_* = 0$$

for any star-shaped bounded C^1 domain Ω.

We now derive the value of λ^* for $\sigma = 0$ and its asymptotic behavior for σ near zero. Namely, $\lambda^* = 0$ for $\sigma = 0$ and, however,

$$\liminf_{\sigma \to 0^+} \lambda^* \geq \lambda_1(-\Delta_p) > 0.$$

When $\sigma = 0$, the term of diffusion $a(x) = 1 + |x - x_0|^\sigma$ becomes the constant function $a(x) = 2$, so that $m_a = 2$. Thus, the Euclidean sharp Sobolev inequality (1.9) can be rewritten as

$$\left(\int_\Omega |u|^{p^*} dx \right)^{p/p^*} \leq m_a^{-1} K(n, p)^p \int_\Omega a(x) |\nabla u|^p dx.$$

In particular, we deduce that $\lambda^* \geq 0$. The reverse inequality follows from a direct combination between the estimates of bubbles (3.1) and the inequality (3.3). Finally, from (5.1), we have

$$\liminf_{\sigma \to 0^+} \lambda^* \geq \liminf_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p}.$$

It remains then to show that $\lim_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p} = \lambda_1(-\Delta_p)$. But the inequality

$$\int_\Omega |u|^p \, dx \leq K_0(n, p, \sigma, \Omega, x_0)^p \int_\Omega |x - x_0|^\sigma |\nabla u|^p \, dx$$

$$\leq K_0(n, p, \sigma, \Omega, x_0)^p (\text{diam}\, \Omega)^\sigma \int_\Omega |\nabla u|^p \, dx$$

implies $\limsup_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p} \leq \lambda_1(-\Delta_p)$.

In order to achieve the desired limit, we evoke a result of [39] which asserts the existence, for each $0 < \sigma < p$, of a weak solution $\varphi_\sigma \in W^{1,p}_{0,\sigma}(\Omega)$ of the problem

$$\begin{cases} -\text{div}(|x - x_0|^\sigma |\nabla u|^{p-2} \nabla u) = K_0(n, p, \sigma, \Omega, x_0)^{-p} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $||\varphi_\sigma||_{L^p(\Omega)} = 1$, where $W^{1,p}_{0,\sigma}(\Omega)$ denotes the completion of $C^1_0(\Omega)$ under the norm

$$||u||_{W^{1,p}_{0,\sigma}} := \left(\int_\Omega |x - x_0|^\sigma |\nabla u|^p \, dx \right)^{1/p}.$$

According to the proof of Theorem 1.4 of[12], namely pages 673 and 674, up to a subsequence, $\nabla \varphi_\sigma$ converges to $\nabla \varphi$ almost everywhere, where $\varphi \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\})$. Probably, an alternative proof of this convergence can be given by using the ideas of [14]. For a fixed number $0 < \gamma < p$, there exist constants $c_0, c_1 > 0$, independent of σ, such that

$$\int_\Omega |x - x_0|^\gamma |\nabla \varphi_\sigma|^p \, dx \leq c_0 \int_\Omega |x - x_0|^\sigma |\nabla \varphi_\sigma|^p \, dx$$

$$= c_0 K_0(n, p, \sigma, \Omega, x_0)^{-p} \int_\Omega \varphi_\sigma^p \, dx = K_0(n, p, \sigma, \Omega, x_0)^{-p} \leq c_1$$

for all $0 < \sigma < \gamma$. In particular, the sequence (φ_σ) is bounded in $W^{1,p}_{0,\gamma}(\Omega)$ and then, by a natural extension of Lemma 2.1 of [13] to $p \neq 2$, namely Theorem 2.1 of [39], φ_σ converges, modulo a subsequence, to φ in $L^p(\Omega)$, so that $||\varphi||_{L^p(\Omega)} = 1$. Finally, Fatou's lemma yields

$$\int_\Omega |\nabla \varphi|^p \, dx \leq \liminf_{\sigma \to 0^+} \int_\Omega |x - x_0|^\sigma |\nabla \varphi_\sigma|^p \, dx$$

$$= \liminf_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p} \int_\Omega |\varphi_\sigma|^p \, dx = \liminf_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p},$$

so that

$$\liminf_{\sigma \to 0^+} K_0(n, p, \sigma, \Omega, x_0)^{-p} \geq \lambda_1(-\Delta_p).$$

6. Open related problems

Motivated by the discussion done in the previous section on bounds of λ^* and λ_*, it naturally arises some interesting issues. First of all, under the condition (1.8), we saw that

$$\lambda_* := \frac{\sigma}{p} C_0 K_0(n, p, \sigma, \Omega, x_0)^{-p} \leq \frac{\sigma}{p} \lambda^*.$$

In particular, this inequality produces a gap whenever $0 < \sigma < p$. Besides, when $\sigma = p$, one has $\lambda_* =: (n^p/p^p)C_0 \leq \lambda^*$. So, two interesting questions arises under the condition (1.8):

(a) what happens when $\lambda \in (\lambda_*, \lambda^*]$ in the case $0 < \sigma < p$;

(b) whether or not λ_* is equal to λ^* in the case $\sigma = p$.

The question (b) is affirmative for $n = p^2$ under the condition (1.8) with equality or, equivalently,

$$a(x) = a(x_0) + C_0|x - x_0|^p \quad \text{in } \Omega.$$

Precisely, when $n = p^2$, one has $b_{n,p}/a_{n,p} = p^p$ and, thanks to (5.6), the equality $\lambda_* = \lambda^*$ follows. This fact was established by Egnell [18] in the case $p = 2$. On the other hand, when $n > p^2$, one easily checks that

$$\lambda_* := C_0 \frac{n^p}{p^p} < C_0 \frac{b_{n,p}}{a_{n,p}}.$$

Surely it would be interesting to understand (a) and (b) in the apparently simpler situation $a(x) = 1 + |x - x_0|^\sigma$ with $x_0 \in \Omega$. In this example, the number λ^* corresponds to the optimal constant λ associated to the weighted Sobolev inequality

$$\left(\int_\Omega |u|^{p^*} dx \right)^{p/p^*} + \lambda K(n, p)^p \int_\Omega |u|^p \, dx \leq K(n, p)^p \int_\Omega (1 + |x - x_0|^\sigma)|\nabla u|^p \, dx.$$

Summarizing, we have $\lambda_* < \lambda^*$ whenever $0 < \sigma < p$, $\lambda_* = (n/p)^p \leq \lambda^*$ when $\sigma = p$ and, by last, $\lambda_* = \lambda^*$ when $\sigma = p$ and $n = p^2$.

Perhaps the question (b) can be answered from a view point of optimal constants which consists in knowing the dependence of λ^* on Ω and x_0 and, better yet, in determining its exact value. Still on the example $a(x) = 1 + |x - x_0|^\sigma$, a related question is to understand what occurs if x_0 does not belong to Ω. This is the case when its minimum value is achieved on the boundary of Ω.

Another particularly interesting example corresponding to $\sigma = p$ is

$$a(x) = a(x_0) + ((x - x_0)^t B(x - x_0))^{p/2} \quad \text{on } \Omega,$$

where B is a positive definite symmetric matrix. In this case,

$$\lambda_* = K_0(n, p, B, \Omega, x_0)^{-p} \leq \lambda^*,$$

where $K_0(n,p,B,\Omega,x_0)^p$ is the best constant on $C_0^\infty(\Omega)$ to

$$\int_\Omega |u|^p\,dx \le K_0(n,p,B,\Omega,x_0)^p \int_\Omega ((x-x_0)^t B(x-x_0))^{p/2}|\nabla u|^p\,dx,$$

and no answer to the question (b) is known even in dimension $n=p^2$.

Finally, concerning with operators $\mathcal{L}_{p,a}$, it would be important to know what happens if $a(x)$ has a behavior around x_0 distinct of those described in (1.6), (1.7) and (1.8). In particular, it may occur that the function $a(x)$ behaves differently around x_0 on each one direction of $\mathbb{R}^n$.

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