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NONCONVEX RETRACTS AND COMPUTATION FOR FIXED POINT INDEX IN CONES

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Abstract. In this paper we construct two retracts in a cone by nonnegative functionals of convex and concave types, and an example is given to illustrate that the retracts are nonconvex. Then the nonconvex retracts are used to compute the fixed point index for the completely continuous operator in the domains $D_1 \cap D_2$ and $D_1 \cup D_2$, where D_1 and D_2 are bounded open sets in the cone. The computation for fixed point index can be applied to the existence and the more precise location of positive fixed points.

1. Introduction

Let E be a real Banach space with the zero element denoted by θ. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions: (i) $\lambda x \in P$ for $x \in P$ and $\lambda \geq 0$; (ii) $\pm x \in P$ implies $x = \theta$. For the theory and properties of cone and fixed point index in Banach spaces we refer to [13], [14]. A nonnegative functional $\gamma\colon P \to [0, +\infty)$ is said to be convex or concave if $\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$ and $\gamma(tx + (1-t)y) \geq t\gamma(x) + (1-t)\gamma(y)$ for all $x, y \in P$ and $t \in [0,1]$, respectively. γ is bounded if its range of bounded set in E is bounded. For $D \subset P$, $\overline{D}$ and ∂D are respectively the closure and boundary of D in P. The open ball centered at θ with the radius

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$R > 0$ is denoted by $B_R = \{x \in E \mid \|x\| < R\}$ and $[x]$ stands for $x/\|x\|$ for $x \in E \setminus \{\theta\}$. Throughout this paper, the notations

$$\begin{aligned} D_1 &= \{x \in P \mid \alpha(x) < R_1\}, & D_2 &= \{x \in P \mid \beta(x) < R_2\}, \\ D_1' &= \{x \in P \mid \alpha(x) > R_1\}, & D_2' &= \{x \in P \mid \beta(x) > R_2\} \end{aligned}$$

are always used for the functionals $\alpha, \beta\colon P \to [0,+\infty)$ and the constants $R_1, R_2 > 0$.

Computation for fixed point index in cones and topological degrees about nonlinear completely continuous operators plays a very important role in the fixed point theory, see the references [1]–[10], [12]–[23] listed in this paper and others. Krasnosel'skiĭ is the first author who considered fixed point index results about cone compression/expansion (see [15]). In [1], [3] there are the following results of computation for fixed point index.

Theorem 1.1. *Let P be a cone in E, $\alpha, \beta\colon P \to [0,+\infty)$ be continuous functionals, D_1 and D_2 be nonempty bounded subsets of P.*

(a) *Suppose that $A\colon \overline{D}_1 \to P$ is a completely continuous operator with*

$$\inf_{x \in \partial D_1} \|Ax\| > 0 \quad \text{and} \quad A(\partial D_1) \subset \overline{D}_1'.$$

If $\alpha(\lambda x) \leq \lambda \alpha(x)$ for $x \in \partial D_1$, $\lambda \in (0,1]$, then the fixed point index $i(A, D_1, P) = 0$.

(b) *Suppose that $A\colon \overline{D}_2 \to P$ is a completely continuous operator with $A(\partial D_2) \subset \overline{D}_2$. If $\beta(\theta) = 0$ and $\beta(\mu x) \geq \mu \beta(x)$ for $x \in \partial D_2$, $\mu \geq 1$, then the fixed point index $i(A, D_2, P) = 1$.*

Theorem 1.2. *Let P be a cone in E, $\alpha\colon P \to [0,+\infty)$ be a continuous convex functional and $\beta\colon P \to [0,+\infty)$ be a continuous concave functional, D_1 and D_2 be nonempty bounded subsets of P with $D_1 \cap D_2' \neq \emptyset$. Suppose that $A\colon P \to P$ is a completely continuous operator.*

(a) *If $Ax \in D_1$ for $x \in (\partial D_1) \cap \overline{D}_2'$ and $x \in (\partial D_1) \cap \{x \in P \mid Ax \in D_2\}$, then the fixed point index $i(A, D_1, P) = 1$.*

(b) *If $Ax \in D_2'$ for $x \in (\partial D_2) \cap \overline{D}_1$ and $x \in (\partial D_2) \cap \{x \in P \mid Ax \in D_1'\}$, then the fixed point index $i(A, D_2, P) = 0$.*

Through these computations for fixed point index, the fixed point theorems of expansion-compression type in cones were deduced and applied to the existence and the location of positive solutions for differential and difference equations. In this paper we shall discuss the fixed point index in the domains $D_1 \cap D_2$ and $D_1 \cup D_2$ which can be applied to the existence and the more precise location of positive fixed points, and thus positive solutions for differential and difference equations under different conditions from those in previous work.

First we construct two nonconvex retracts in a cone by nonnegative functionals of convex and concave types, and then the retracts are used to compute the fixed point index for the completely continuous operator in the domains $D_1 \cap D_2$ and $D_1 \cup D_2$. A subset $X \subset E$ is called a retract of E if there exists a continuous mapping $r: E \to X$, a retraction, satisfying $r(x) = x$, $x \in X$. By a theorem due to Dugundji [11], every nonempty closed convex subset of E is a retract of E. An example is given to illustrate that the retracts obtained in this paper are nonconvex.

2. Nonconvex retracts

Theorem 2.1. *Let P be a cone in E, $\alpha: P \to [0, +\infty)$ be a continuous convex functional and $\beta: P \to [0, +\infty)$ be a bounded continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0$, $\beta(x) > 0$ for $x \neq \theta$, both $\{x \in P \mid \alpha(x) \leq R\}$ and $\{x \in P \mid \beta(x) \leq R\}$ be bounded for all $R > 0$. If*

$$\beta(\mu x) > \beta(x), \quad \text{for all } \mu > 1,\ x \in P \setminus \{\theta\}, \tag{2.1}$$

then $\overline{D}_1 \cap \overline{D}_2$ is a retract of E.

Proof. Since both α and β are continuous with $\alpha(\theta) = \beta(\theta) = 0$, there exist nonzero elements in $\overline{D}_1 \cap \overline{D}_2$, i.e. $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$.

(a) Since $\overline{D}_1$ is a closed convex set, there exists a retraction $g_1: E \to \overline{D}_1$.

(b) If $\overline{D}_1 \cap \overline{D}_2' = \emptyset$, then $\overline{D}_1 \subset \overline{D}_2$ and $\overline{D}_1 \cap \overline{D}_2 = \overline{D}_1$ is a retract of E. Afterwards we may suppose that $\overline{D}_1 \cap \overline{D}_2' \neq \emptyset$. Since $\overline{D}_1$ is bounded, there exists a constant $R_1' > 0$ such that $\|x\| \leq R_1'$, for all $x \in \overline{D}_1 \cap \overline{D}_2'$. Because $\beta(x)$ is a bounded functional, we have a constant $M > R_2$ such that $\beta(x) \leq M$, for all $x \in \overline{D}_1 \cap \overline{D}_2'$. Due to the boundedness of $\{x \in P \mid \beta(x) \leq M + 1\}$, we know that there exists a constant $R_2' > R_1'$ such that $\beta(x) > M + 1$ for $x \in P \cap \partial B_{R_2'}$. Owing to $\theta \notin \overline{D}_1 \cap \overline{D}_2'$, we can define

$$g_2(x) = \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}(x - R_2'[x]), \quad \text{for all } x \in \overline{D}_1 \cap \overline{D}_2'.$$

Obviously, g_2 is continuous.

(c) For $x \in \overline{D}_1 \cap \overline{D}_2'$, define

$$g_3(x) = \begin{cases} g_2(x) + R_2'[x], & \|g_2(x)\| \leq R_2'; \\ \theta, & \|g_2(x)\| > R_2'. \end{cases}$$

Thus $g_3: \overline{D}_1 \cap \overline{D}_2' \to P$ is continuous. In fact, if $\|g_2(x)\| \leq R_2'$, i.e.

$$\left\| \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}(x - R_2'[x]) \right\| = \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}(R_2' - \|x\|) \leq R_2', \tag{2.2}$$

then
$$g_3(x) = \left(\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}(\|x\| - R_2') + R_2'\right)[x] \in P,$$
which implies that $g_3\colon \overline{D}_1 \cap \overline{D}_2' \to P$. When $\|g_2(x)\| = R_2'$, by (2.2) we have $g_3(x) = \theta$ and hence $g_3(x)$ is continuous.

(d) Define
$$g_4(x) = \begin{cases} g_3(x), & x \in \overline{D}_1 \cap \overline{D}_2'; \\ x, & x \in \overline{D}_1 \cap \overline{D}_2. \end{cases}$$
For $x \in \{x \in P \mid \beta(x) = R_2,\ \alpha(x) \le R_1\}$, we have that $g_2(x) = x - R_2'[x]$ and $\|g_2(x)\| = R_2' - \|x\| < R_2'$. Therefore $g_3(x) = x$ and $g_4\colon D_1 \to P$ is well defined and continuous.

(e) In the following we shall show that $\beta(g_3(x)) \le R_2$, $\alpha(g_3(x)) \le R_1$ for $x \in \overline{D}_1 \cap \overline{D}_2'$, that is, $g_4\colon D_1 \to \overline{D}_1 \cap \overline{D}_2$.

Actually when $\|g_2(x)\| > R_2'$, we have that $g_3(x) = \theta$, and hence $\alpha(g_3(x)) = 0 \le R_1$, $\beta(g_3(x)) = 0 \le R_2$.

When $\|g_2(x)\| \le R_2'$, it follows from $\beta(x) \ge R_2$ that
$$\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)} \ge 1,$$
and
$$\begin{aligned} g_3(x) &= \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}(x - R_2'[x]) + R_2'[x], \\ x &= \frac{\beta(R_2'[x]) - \beta(x)}{\beta(R_2'[x]) - R_2} g_3(x) + \left(1 - \frac{\beta(R_2'[x]) - \beta(x)}{\beta(R_2'[x]) - R_2}\right) R_2'[x]. \end{aligned}$$
By the concavity of β, we have
$$\beta(x) \ge \frac{\beta(R_2'[x]) - \beta(x)}{\beta(R_2'[x]) - R_2}\beta(g_3(x)) + \left(1 - \frac{\beta(R_2'[x]) - \beta(x)}{\beta(R_2'[x]) - R_2}\right)\beta(R_2'[x]),$$

$$\beta(g_3(x)) \le \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\beta(x) - \left(\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)} - 1\right)\beta(R_2'[x]) = R_2. \tag{2.3}$$
On the other hand, when $\|g_2(x)\| \le R_2'$, if
$$\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\left(1 - R_2'\frac{1}{\|x\|}\right) + R_2'\frac{1}{\|x\|} > 1,$$
from (2.1) we have
$$\beta(g_3(x)) = \beta\left(\left(\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\left(1 - R_2'\frac{1}{\|x\|}\right) + R_2'\frac{1}{\|x\|}\right)x\right) > \beta(x).$$
Since $\beta(x) \ge R_2$, it follows that $\beta(g_3(x)) > R_2$, which contradicts (2.3), and thus
$$0 \le \frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\left(1 - R_2'\frac{1}{\|x\|}\right) + R_2'\frac{1}{\|x\|} \le 1.$$

By the convexity of α,

$$\begin{aligned}\alpha(g_3(x)) &= \alpha\bigg(\bigg(\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\bigg(1 - R_2'\frac{1}{\|x\|}\bigg) + R_2'\frac{1}{\|x\|}\bigg)x\bigg)\\ &\leq \bigg(\frac{\beta(R_2'[x]) - R_2}{\beta(R_2'[x]) - \beta(x)}\bigg(1 - R_2'\frac{1}{\|x\|}\bigg) + R_2'\frac{1}{\|x\|}\bigg)\alpha(x) \leq \alpha(x) \leq R_1.\end{aligned}$$

(f) Let $g(x) = g_4(g_1(x))$, for all $x \in E$, then $g: E \to \overline{D}_1 \cap \overline{D}_2$ is a retraction.□

Theorem 2.2. *Let P be a cone in E, $\alpha: P \to [0, +\infty)$ be a continuous functional and $\beta: P \to [0, +\infty)$ be a continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0$, $\beta(x) > 0$ for $x \neq \theta$. If $D_1' \cap D_2' \neq \emptyset$, (2.1) holds and*

$$\alpha(\lambda x) \leq \lambda\alpha(x), \quad \textit{for all } \lambda \in [0,1],\ x \in P, \tag{2.4}$$

then $\overline{D}_1' \cap \overline{D}_2'$ is a retract of E.

Proof. (a) Since $\overline{D}_2'$ is a closed convex set, there exists a retraction

$$g_1: E \to \overline{D}_2'.$$

(b) If $\overline{D}_1 \cap \overline{D}_2' = \emptyset$, then $\overline{D}_1' \supset \overline{D}_2'$ and $\overline{D}_1' \cap \overline{D}_2' = \overline{D}_2'$ is a retract of E. Afterwards we may suppose that $\overline{D}_1 \cap \overline{D}_2' \neq \emptyset$. Owing to $\theta \notin \overline{D}_1 \cap \overline{D}_2'$, we can define

$$g_2(x) = \begin{cases} \dfrac{R_1}{\alpha(x)}x, & x \in \overline{D}_1 \cap \overline{D}_2'; \\ x, & x \in \overline{D}_1' \cap \overline{D}_2'. \end{cases}$$

Because $g_2(x) = x$ if $\alpha(x) = R_1$ and $\beta(x) \geq R_2$, we have that $g_2(x)$ is continuous on $\overline{D}_2'$.

(c) For $x \in \overline{D}_1 \cap \overline{D}_2'$, $g_2(x) = R_1 x/\alpha(x)$, i.e. $x = \alpha(x)g_2(x)/R_1$. It follows from $\alpha(x) \leq R_1$ that $\alpha(x)/R_1 \leq 1$, and thus by (2.4),

$$\alpha(x) = \alpha\bigg(\frac{\alpha(x)}{R_1}g_2(x)\bigg) \leq \frac{\alpha(x)}{R_1}\alpha(g_2(x)),$$

that is, $\alpha(g_2(x)) \geq R_1$. We have from (2.1) that

$$\beta(g_2(x)) = \beta\bigg(\frac{R_1}{\alpha(x)}x\bigg) \geq \beta(x) \geq R_2,$$

and then $g_2: \overline{D}_2' \to \overline{D}_1' \cap \overline{D}_2'$.

(d) Let $g(x) = g_2(g_1(x))$, for all $x \in E$, then $g: E \to \overline{D}_1' \cap \overline{D}_2'$ is a retraction.□

3. Computation for the fixed point index

In this section we shall use the retracts obtained above to compute the fixed point index for nonlinear completely continuous operators. The next theorem follows the idea from [19].

Theorem 3.1. *Let P be a cone in E and Ω be a bounded open set in P, $A:\overline{\Omega} \to P$ be completely continuous with $Ax \neq x$, for all $x \in \partial\Omega$. Suppose that $D \subset P$ is a retract of E satisfying $A(\partial\Omega) \subset D$.*

(a) *If $D \subset \overline{\Omega}$, then the fixed point index $i(A, \Omega, P) = 1$;*
(b) *If $D \cap \Omega = \emptyset$, then the fixed point index $i(A, \Omega, P) = 0$.*

Proof. Let $g: E \to D$ be a retraction.

(a) Take R sufficiently large such that $\overline{\Omega} \subset P_R = \{x \in P \mid \|x\| < R\}$. By the extension theorem, $A|_{\partial\Omega}$ has a completely continuous extension $A_1: \overline{P}_R \to P$ with $A_1x = Ax$ for $x \in \partial\Omega$. Denote $A_2 = gA_1$, then $A_2: \overline{P}_R \to D$ is completely continuous and $A_2x = Ax$ for $x \in \partial\Omega$ since $A(\partial\Omega) \subset D$. It follows from $D \subset \overline{\Omega} \subset P_R$ and the homotopy invariance of fixed point index that

$$i(A_2, P_R, P) = i(\theta, P_R, P) = 1. \tag{3.1}$$

Notice that $A_2: \overline{P}_R \to D \subset \overline{\Omega}$ and $Ax \neq x$ for $x \in \partial\Omega$, then A_2 has no fixed point in $\overline{P}_R \setminus \Omega$ and hence

$$i(A_2, P_R, P) = i(A_2, \Omega, P) = i(A, \Omega, P). \tag{3.2}$$

Therefore, from (3.1) and (3.2) it follows that $i(A, \Omega, P) = 1$.

(b) $A|_{\partial\Omega}$ has a completely continuous extension $A_3: \overline{\Omega} \to P$ with $A_3x = Ax$ for $x \in \partial\Omega$. Denote $A_4 = gA_3$, thus $A_4: \overline{\Omega} \to D$ is completely continuous and $A_4x = Ax$ for $x \in \partial\Omega$ since $A(\partial\Omega) \subset D$. If $i(A, \Omega, P) \neq 0$, then $i(A_4, \Omega, P) \neq 0$ and A_4 has a fixed point in $D \cap \Omega$, which contradicts $D \cap \Omega = \emptyset$. □

Theorem 3.2. *Let P be a cone in E, $\alpha: P \to [0, +\infty)$ be a continuous convex functional and $\beta: P \to [0, +\infty)$ be a bounded continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0$, $\beta(x) > 0$ for $x \neq \theta$, both $\{x \in P \mid \alpha(x) \leq R\}$ and $\{x \in P \mid \beta(x) \leq R\}$ be bounded for all $R > 0$, $A: P \to P$ be completely continuous. Suppose that* (2.1) *holds.*

(a) *If $A(\partial(D_1 \cap D_2)) \subset \overline{D}_1 \cap \overline{D}_2$ with $Ax \neq x$, for all $x \in \partial(D_1 \cap D_2)$, then $i(A, D_1 \cap D_2, P) = 1$;*
(b) *If $A(\partial(D_1 \cap D_2)) \subset \overline{D}_1' \cap \overline{D}_2'$ with $Ax \neq x$, for all $x \in \partial(D_1 \cap D_2)$, then $i(A, D_1 \cap D_2, P) = 0$;*
(c) *If $A(\partial(D_1 \cup D_2)) \subset \overline{D}_1 \cap \overline{D}_2$ with $Ax \neq x$, for all $x \in \partial(D_1 \cup D_2)$, then $i(A, D_1 \cup D_2, P) = 1$;*
(d) *If $A(\partial(D_1 \cup D_2)) \subset \overline{D}_1' \cap \overline{D}_2'$ with $Ax \neq x$, for all $x \in \partial(D_1 \cup D_2)$, then $i(A, D_1 \cup D_2, P) = 0$.*

Proof. It is clear that $\overline{D}_1' \cap \overline{D}_2' \neq \emptyset$ since both $\{x \in P \mid \alpha(x) \leq R\}$ and $\{x \in P \mid \beta(x) \leq R\}$ are bounded for all $R > 0$. By the convexity of α and $\alpha(\theta) = 0$ we know that (2.4) is satisfied. From Theorems 2.1 and 2.2 it follows that $\overline{D}_1 \cap \overline{D}_2$ and $\overline{D}_1' \cap \overline{D}_2'$ are retracts of E, and then Theorem 3.1 tells us that the conclusions of this theorem holds. □

COROLLARY 3.3. *Under the same conditions as those in Theorem* 3.2*, if* $(\overline{D}_1 \cup \overline{D}_2) \setminus (D_1 \cap D_2) \neq \emptyset$ *and*

(a) $A(\partial(D_1 \cap D_2)) \subset \overline{D}_1 \cap \overline{D}_2$, $A(\partial(D_1 \cup D_2)) \subset \overline{D}_1' \cap \overline{D}_2'$; *or*
(b) $A(\partial(D_1 \cap D_2)) \subset \overline{D}_1' \cap \overline{D}_2'$, $A(\partial(D_1 \cup D_2)) \subset \overline{D}_1 \cap \overline{D}_2$,

then A has a fixed point in $(\overline{D}_1 \cup \overline{D}_2) \setminus (D_1 \cap D_2)$.

4. An example

In this section we will give an example to illustrate that the retracts obtained are nonconvex.

Let $E = C[0,1]$ be Banach space with the norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$, for all $x \in C[0,1]$,

$$P = \left\{ x \in C[0,1] \,\middle|\, x(t) \geq 0, \text{ for all } t \in [0,1], \min_{t \in [1/3,2/3]} x(t) \geq \frac{1}{9}\|x\| \right\}.$$

Obviously, P is a cone in E. Define

$$\alpha(x) = \max_{t \in [1/3,2/3]} x(t), \ \beta(x) = \min_{t \in [1/3,2/3]} x(t)$$

for $x \in P$, it is clear that $\alpha: P \to [0,+\infty)$ is a continuous convex functional and $\beta: P \to [0,+\infty)$ is a bounded continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0$, $\beta(x) > 0$ for $x \neq \theta$, both $\{x \in P \mid \alpha(x) \leq R\}$ and $\{x \in P \mid \beta(x) \leq R\}$ are bounded for all $R > 0$. Moreover, (2.1) and (2.4) are satisfied.

Let $R_1 = 7/9$ and $R_2 = 5/18$. If we take $x_1(t) = 5t/6$ and $x_2(t) = 5(1-t)/6$ for $t \in [0,1]$, then $x_1, x_2 \in \overline{D}_1 \cap \overline{D}_2$. However $\alpha((x_1+x_2)/2) = 5/12 > R_2$, thus $\overline{D}_1 \cap \overline{D}_2$ is not convex. If we take $x_3(t) = 7t/6$ and $x_4(t) = 7(1-t)/6$ for $t \in [0,1]$, then $x_3, x_4 \in \overline{D}_1' \cap \overline{D}_2'$. However $\alpha((x_3+x_4)/2) = 7/12 < R_1$, thus $\overline{D}_1' \cap \overline{D}_2'$ is not convex. Besides, if we take $x_5(t) = t$ and $x_6(t) = 2(t-1)^2$ for $t \in [0,1]$, then it is easy to see that $x_5 \in \overline{D}_1 \setminus D_2$ and $x_6 \in \overline{D}_2 \setminus D_1$, i.e. $(\overline{D}_1 \cup \overline{D}_2) \setminus (D_1 \cap D_2) \neq \emptyset$.

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