Topological **M**ethods in **N**onlinear **A**nalysis
Volume 43, No. 2, 2014, 323–344

THE EXISTENCE OF NONTRIVIAL CRITICAL POINT FOR A CLASS OF STRONGLY INDEFINITE ASYMPTOTICALLY QUADRATIC FUNCTIONAL WITHOUT COMPACTNESS

Guanggang Liu — Shaoyun Shi* — Yucheng Wei

Abstract. In this paper, we show the existence of nontrivial critical point for a class of strongly indefinite asymptotically quadratic functional without compactness, by using the technique of penalized functionals and an infinite dimensional Morse theory developed by Kryszewski and Szulkin. Two applications are given to Hamiltonian systems and elliptic systems.

1. Introduction

Let E be a real Hilbert space with an inner product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. Consider the functional of the following form

$$\Phi(x) = \frac{1}{2}\langle L_\infty x, x\rangle - \varphi(x), \tag{1.1}$$

where $L_\infty\colon E \to E$ is a bounded linear selfadjoint Fredholm operator of index 0, $\nabla\varphi(0) = 0$, $\nabla\varphi$ is a compact mapping and $\nabla\varphi(x) \to 0$ as $\|x\| \to \infty$. Moreover, we suppose that $0 \in \sigma(L_\infty)$ and $\Phi \in C^2(E,\mathbb{R})$ is a strong indefinite functional,

2010 *Mathematics Subject Classification.* Primary 58E05; Secondary 34C25, 58F05, 35J65.
Key words and phrases. Strongly indefinite functional, filtration, Morse index, critical group, Hamiltonian system, elliptic system.
This work was supported by NSFC grant (11371166) and National 973 project of China (2012CB821200), SRFDP grant (20060183017).

* The corresponding author.

i.e. Φ is unbounded from below and from above on any subspace of finite codimension. Obviously, Φ is an asymptotically quadratic functional with a trivial critical point 0.

The purpose of this paper is to find the nontrivial critical point of Φ. There are two difficulties in treating this problem. On the one hand, since Φ is a strong indefinite functional, it is well known that the Morse index of any critical point of Φ must necessarily be infinite, therefore we can not expect to obtain any useful information from the usual Morse theory. On the other hand, since $0 \in \sigma(L_\infty)$, the global compactness of Φ may be lost.

To overcome the difficulties caused by the strongly indefinite property of functional, some new techniques were developed. In 1997, Kryszewski and Szulkin[12] developed an infinite dimensional Morse theory, which was applied to the asymptotically linear Hamiltonian systems, wave equations and elliptic systems. By developing a method to compute the new cohomology critical groups both at zero and at infinity precisely, A. Szulkin, W. M. Zou[23] and W. M. Zou[25] obtained the existence of (multiple) nontrivial solutions for asymptotically linear Hamiltonian systems, beam equations and noncooperative elliptic systems.

In 1997, Abbondandolo [1] developed another Morse theory for strongly indefinite functionals, which was applied to study the existence of nontrivial periodic solutions for Hamiltonian systems. In [17], a similar result was obtained for an asymptotically linear non-cooperative elliptic system by Abbondandolo's theory.

In [8], a new Morse index theory for strongly indefinite functionals was developed via Galerkin approximation. This method was applied to the asymptotically linear Hamiltonian systems [9], noncooperative elliptic systems [10], wave equation and beam equations [24], respectively.

To overcome the difficulties caused by resonance, some kinds of conditions were imposed to ensure the compactness of functional, for example, Landsman–Lazer type condition [13]. In [15], A. Masiello and L. Pisani considered a bounded resonance problem for semilinear elliptic equations. Since the assumptions that imposed to the nonlinearities can not ensure the compactness of functional, they used the technique of penalized functional. In [21], Su and Liu extended the above result to the nonautonomous case with resonance both at zero and at infinity. Moreover, they considered the multiplicity and sign-changing properties of nontrivial solutions via the classical Morse theory.

In this paper, we consider the general bounded resonance strongly indefinite variational problem with the nonlinearity near infinity similar to that in [15] and [21]. By using the infinite dimensional Morse theory developed by Kryszewski and Szulkin and the technique of penalized functional, we obtain the existence of nontrivial critical point for functional (1.1). As applications, we studied the existence of nontrivial solutions for Hamiltonian system and a class of strongly indefinite elliptic system, respectively.

The paper is organized as follows. In Section 2, we introduce the infinite dimensional Morse theory established by Kryszewski and Szulkin as preliminaries. The main result and its proof will be given in Section 3. In Sections 4 and 5, some applications are given to Hamiltonian system and a class of strongly indefinite elliptic system, respectively.

At the end of the section, we give some notations which will be used.

$d(A,B) := \inf\{\|x-y\| \mid x \in A, y \in B\}$;

$K(\Phi) := \{x \in E \mid \Phi'(x) = 0\}$;

$\Phi^a := \{x \in E \mid \Phi(x) \leq a\}$;

The kernel of operator L is denoted by $N(L)$;

The image of operator L is denoted by $R(L)$;

The space of bounded linear operators from E to F is denoted by $\mathbb{L}(E,F)$;

The Morse index of operator L is denoted by $M^-(L)$;

The nullity of operator L is denoted by $M^0(L)$.

2. Preliminaries

In this section, we introduce the infinite dimensional Morse theory established by Kryszewski and Szulkin. We shall give some necessary definitions and results, for more details, see [12].

Let $(\mathcal{G}_n)_{n=1}^\infty$ be a sequence of abelian groups. The asymptotic group of $(\mathcal{G}_n)_{n=1}^\infty$ is defined by

$$[(\mathcal{G}_n)_{n=1}^\infty] := \prod_{n=1}^\infty \mathcal{G}_n \Big/ \bigoplus_{n=1}^\infty \mathcal{G}_n.$$

Let $\{E_n\}_{n=1}^\infty$ be a filtration of E, i.e. $\{E_n\}_{n=1}^\infty$ is a sequence of closed subspaces of E, $E_n \subseteq E_{n+1}$ and $\overline{\bigcup_{n=1}^\infty E_n} = E$. Denote $\mathcal{E} := \{E_n, d_n\}_{n=1}^\infty$ and the orthogonal projector of E onto E_n by P_n, where $(d_n)_{n=1}^\infty$ is a sequence of nonnegative integers. For any integer q and closed subset (X,A) of E, we define the q-th $\mathcal{E}$-cohomology group of (X,A) with coefficients in a fixed field $\mathcal{F}$ by

$$H^q_{\mathcal{E}}(X,A) := [(H^{q+d_n}(X \cap E_n, A \cap E_n))_{n=1}^\infty].$$

Here $H^*_{\mathcal{E}}$ satisfies all the Eilenberg–Steenrod axioms for cohomology except the dimension axiom.

A functional $\Phi \in C^1(E,\mathbb{R})$ is said to satisfy the $(\mathrm{PS})^*$ condition with respect to $\mathcal{E}$ means that, whenever a sequence $\{y_j\}$ is such that $y_j \in E_{n_j}$ for some n_j, $n_j \to \infty$, there is $M > 0$ such that $|\Phi(y_j)| < M$ for all $j \geq 1$ and $P_{n_j} \nabla \Phi(y_j) \to 0$ as $j \to \infty$, then $\{y_j\}$ has a convergent subsequence.

For an isolated critical point p, if $\Phi \in C^1(E,\mathbb{R})$ satisfies (PS)* condition, then there is an admissible pair (W,W^-) for Φ and p (see page 3189 of [12]). We define the q-th critical group($q \in \mathbb{Z}$) of Φ at p with respect to $\mathcal{E}$ by

$$C^q_{\mathcal{E}}(\Phi,p) := H^q_{\mathcal{E}}(W,W^-).$$

If $\Phi \in C^1(E,\mathbb{R})$ satisfies (PS)* condition and the critical set $K = K(\Phi)$ is compact, there is also an admissible pair (W,W^-) for Φ and K with respect to $\mathcal{E}$ (see page 3193 of [12]). In particular, if there exist $a < b$ such that $K \subset \operatorname{int}\Phi^{-1}([a,b])$, then $(\Phi^{-1}([a,b]),\Phi^{-1}(a))$ is an admissible pair for Φ and K. We define the q-th critical group of Φ at infinity with respect to $\mathcal{E}$ by

$$C^q_{\mathcal{E}}(\Phi,K) = H^q_{\mathcal{E}}(W,W^-) = H^q_{\mathcal{E}}(\Phi^{-1}([a,b]),\Phi^{-1}(a)).$$

Denote

$$[\mathbb{Z}] := \prod_{n=1}^{\infty}\mathbb{Z}\Big/\bigoplus_{n=1}^{\infty}\mathbb{Z}, \qquad [\mathbb{Z}_+] := \{[(\xi_n)_{n=1}^{\infty}] \in [\mathbb{Z}] : \xi_n \geq 0 \text{ for almost all } n\}.$$

Let (X,B) be a pair of closed subsets of E with the property that for each $q \in \mathbb{Z}$ there is an $n(q)$ such that

$$\dim H^{q+d_n}(X \cap E_n, B \cap E_n) < \infty, \quad \text{for all } n \geq n(q),$$

then $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X,B) := [(\dim H^{q+d_n}(X \cap E_n, B \cap E_n)_{n=1}^{\infty}]$ is a well-defined element of $[\mathbb{Z}_+]$. The sequence $(\dim H^{q+d_n}(X \cap E_n, B \cap E_n)_{n=1}^{\infty}$ will often be constant for almost all n. In such a case we will write $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X,B) = [d]$, d being the constant. We will say that the pair (X,B) is of $\mathcal{E}$-finite type, if $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X,B)$ is well-defined and $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X,B) = [0]$ for almost all $q \in \mathbb{Z}$.

Suppose $\Phi \in C^1(E,\mathbb{R})$ satisfies (PS)* condition, (W,W^-) is an admissible pair for Φ and $K(\Phi) = \{p_1,\ldots p_k\}$. We call p_j is of $\mathcal{E}$-finite type, if some(and therefore every) admissible pair for Φ and p_j is $\mathcal{E}$-finite. If (W,W^-) and all p_j are $\mathcal{E}$-finite, we define

$$M^q_{\mathcal{E}}(W,W^-) := \sum_{j=1}^{k} \dim_{\mathcal{E}} C^q_{\mathcal{E}}(\Phi,p_j), \quad q \in \mathbb{Z},$$
$$\beta^q_{\mathcal{E}}(W,W^-) := \dim_{\mathcal{E}} H^q_{\mathcal{E}}(W,W^-), \quad q \in \mathbb{Z}.$$

Moreover, in such a case we define the Morse and the Poincaré polynomials of (W,W^-) by setting

$$M_{\mathcal{E}}(t,W,W^-) := \sum_{q=-\infty}^{\infty} M^q_{\mathcal{E}}(W,W^-)t^q, \qquad P_{\mathcal{E}}(t,W,W^-) := \sum_{q=-\infty}^{\infty} \beta^q_{\mathcal{E}}(W,W^-)t^q.$$

Here $M_{\mathcal{E}}$ and $P_{\mathcal{E}}$ are not polynomials in the usual sense, some exponents q may be negative, $M_{\mathcal{E}}$ and $P_{\mathcal{E}}$ are elements of $[\mathbb{Z}][t,t^{-1}]$.

Theorem 2.1 ([12]). *Suppose that $\Phi \in C^1(E,\mathbb{R})$ satisfies* (PS)* *condition, (W,W^-) is an admissible pair for Φ and $K(\Phi) := \{p_1,\dots p_k\}$. If all p_j are $\mathcal{E}$-finite, then the pair (W,W^-) is $\mathcal{E}$-finite, and there is a polynomial*

$$Q(t) = \sum_{q=-\infty}^{\infty} a_q t^q$$

such that $a_q \in [\mathbb{Z}_+]$, for all $q \in \mathbb{Z}$, and

$$M_{\mathcal{E}}(t,W,W^-) = P_{\mathcal{E}}(t,W,W^-) + (1+t)Q(t).$$

Let E be a real Hilbert space with a given filtration $\mathcal{E} = \{E_n, d_n\}_{n=1}^{\infty}$.

A mapping $f\colon D \to E$ (D is a closed subset of E) is said to be *A-proper* (with respect to $\mathcal{E}$) if each bounded sequence $\{x_j\} \subset D$ such that $x_j \in E_{n_j} \cap D$ for some n_j, $n_j \to \infty$ and $P_{n_j} f(x_j) \to y \in E$ as $j \to \infty$, has a convergent subsequence.

Proposition 2.2 ([12]). *Assume that $L \in \mathbb{L}(E,E)$ is a self-adjoint Fredholm operator of index* 0 *and $\mathcal{E}$ is a given filtration. $P_n\colon E \to E_n$ and $Q_n\colon R(L) \to R(L)\cap E_n$ are the orthogonal projectors, then*

- (a) *There exists an n_0 such that if $n \ge n_0$, then $P_n|_{N(L)}\colon N(L) \to P_nN(L)$ is a linear isomorphism and $\|z\| \le 2\|P_n z\|$, for all $z \in N(L)$;*
- (b) *$E_n = (R(L)\cap E_n) \oplus P_nN(L)$, and the spaces $R(L)\cap E_n$ and $P_nN(L)$ are orthogonal*;
- (c) *$P_n - Q_n \to 0$ in $\mathbb{L}(R(L),E)$ as $n \to \infty$;*
- (d) *The sequence $\{R(L)\cap E_n\}_{n=1}^{\infty}$ is a filtration of $R(L)$. More precisely, for each $x \in R(L), Q_n x \to x$ as $n \to \infty$.*

Proposition 2.3 ([12]). *Let $L \in \mathbb{L}(E,E)$ be a self-adjoint operator. Then the following conditions are equivalent*:

- (a) *L is A-proper*;
- (b) *L is a Fredholm operator of index 0 and there exist $c > 0$, $n_0 \ge 1$ such that if $n \ge n_0$, then $\|P_nLx\| \ge c\|x\|$ for all $x \in R(L)\cap E_n$.*

Let $L \in \mathbb{L}(E,E)$ be a self-adjoint Fredholm operator, we define the $\mathcal{E}$-Morse index $M_{\mathcal{E}}^-(L)$ of L by

$$M_{\mathcal{E}}^-(L) := \lim_{n\to\infty}(M^-(Q_nL|_{R(L)\cap E_n}) - d_n),$$

where $Q_n\colon R(L) \to R(L)\cap E_n$ is the orthogonal projector.

Proposition 2.4 ([12]). *Suppose $\widetilde{L} \in \mathbb{L}(E,E)$ is a self-adjoint Fredholm operator of index* 0 *such that $\widetilde{L}(E_n) \subset E_n$ for almost all n and $B \in \mathbb{L}(E,E)$ is* *a self-adjoint compact operator. Then $\widetilde{L} - B$ is A-proper. If $M^-(\widetilde{L}|_{E_n}) = d_n + k$ for almost all n and some $k \in \mathbb{Z}$, then $M^-_{\mathcal{E}}(L)$ is well-defined and finite.*

Denote the dimension of the space $N(L)$ by $M^0(L)$. As usual $P_n\colon E \to E_n$ is the orthogonal projector from E to E_n.

Remark 2.5 ([12]). If $N(L) \subset E_{n_0}$ for some n_0, then when $n \geq n_0$,

$$M^-_{\mathcal{E}}(L) = \lim_{n\to\infty}(M^-(Q_nL|_{R(L)\cap E_n}) - d_n) = \lim_{n\to\infty}(M^-(P_nL|_{E_n}) - d_n).$$

We may make use of $\mathcal{E}$-Morse index of L to compute the critical group $C^*_{\mathcal{E}}(\Phi, p)$ at isolated critical point p.

Theorem 2.6 ([12]). *Suppose that $\Phi \in C^1(E, \mathbb{R})$, p is an isolated critical point of Φ, and*

$$\Phi(x) = \Phi(p) + \frac{1}{2}\langle L(x-p), x-p\rangle - \psi(x),$$

where L is an invertible A-proper operator and $\nabla\psi(x) = o(\|x-p\|)$ as $x \to p$. If $M^-_{\mathcal{E}}(L)$ is well-defined and finite, then

$$C^q_{\mathcal{E}}(\Phi, p) = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L), \\ [0], & q \neq M^-_{\mathcal{E}}(L). \end{cases}$$

3. The main result

Let $E = E^+ \oplus E^- \oplus N(L_\infty)$ be the decomposition corresponding to the positive, negative and zero part of the spectrum of L_∞, $L_0 = \Phi''(0)$, and $\varphi_0(x) = \Phi(x) - \langle L_0x, x\rangle/2$.

We make the following assumptions:

- (H1) There is a $\widetilde{L} \in \mathbb{L}(E, E)$ such that:
  - (1) $\widetilde{L}$ is a linear self-adjoint Fredholm operator of index 0 in E;
  - (2) $L_\infty = \widetilde{L} - B_\infty$, $L_0 = \widetilde{L} - B_0$, where B_∞ and B_0 are compact linear self-adjoint operators in E;
  - (3) There exists $\{E_n, d_n\}_{n=1}^\infty$ with $E_n \subset E_{n+1} \subset E$, $\overline{\bigcup_{n=1}^\infty E_n} = E$, d_n and k are nonnegative integer, such that $\widetilde{L}(E_n) \subset E_n$ and $M^-(\widetilde{L}|_{E_n}) = d_n + k$;
- (H2) $\nabla\varphi, \nabla\varphi_0$ are compact mappings, and $\nabla\varphi_0(x) = o(\|x\|)$ as $\|x\| \to 0$;
- (H3) L_0 is nondegenerate, i.e. $M^0(L_0) = 0$;
- (H4) There exists a constant $C > 0$, such that $\|\nabla\varphi(x)\| < C$ for all $x \in E$;
- (H5) For a sequence $\{x_j\}$, let $x_j = x_j^- + x_j^0 + x_j^+$, with $x_j^\pm \in E^\pm$ and $x_j^0 \in N(L_\infty)$, if $x_j^+ + x_j^-$ are bounded and $\|x_j^0\| \to \infty$ as $j \to \infty$, then $\lim_{j\to\infty}\|\varphi''(x_j)\| = 0$.

The following is our main result.

Theorem 3.1. *Suppose* (H1)–(H5) *hold. If* $M^-_{\mathcal{E}}(L_0) < M^-_{\mathcal{E}}(L_\infty) - 1$ *or* $M^-_{\mathcal{E}}(L_0) > M^-_{\mathcal{E}}(L_\infty) + M^0(L_\infty) + 1$, *then* $\Phi(x)$ *has at least one nontrivial critical point.*

To prove Theorem 3.1, we need some lemmas. Denote $\mathcal{E}' = \{E'_n, d_n\}_{n=1}^\infty$, where $E'_n := (R(L_\infty) \cap E_n) \oplus N(L_\infty)$.

Let $P'_n: E \to E'_n$ and $Q_n: R(L_\infty) \to R(L_\infty) \cap E_n$ be the orthogonal projectors.

Lemma 3.2. *Suppose* (H1) *holds, then* L_∞ *and* L_0 *are A-proper with respect to* $\mathcal{E}'$.

Proof. Since $L_\infty = \widetilde{L} - B_\infty$, $\widetilde{L} \in \mathbb{L}(E,E)$ is a self-adjoint Fredholm operator of index 0 and $B_\infty \in \mathbb{L}(E,E)$ is a self-adjoint compact operator, by Proposition 2.4, L_∞ is A-proper with respect to $\mathcal{E}$.

Let $\{x_j\}$ be a bounded sequence such that $x_j \in E'_{n_j}$ for some n_j, $n_j \to \infty$ and $P'_{n_j} L_\infty \to y \in E$ as $j \to \infty$.

Suppose $x_j = \overline{x}_j + x^0_j \in (R(L_\infty) \cap E_{n_j}) \oplus N(L_\infty) = E'_{n_j}$, let $\widehat{x}_j = \overline{x}_j + P_{n_j} x^0_j$. By (b) of Proposition 2.2, $E_{n_j} = (R(L_\infty) \cap E_{n_j}) \oplus P_n N(L_\infty)$, then $\widehat{x}_j \in E_{n_j}$ and $\|\widehat{x}_j - x_j\| \to 0$ as $j \to \infty$. Since $\{x_j\}$ is bounded, $\{\widehat{x}_j\}$ is also bounded.

It is clear that $P'_{n_j} x = Q_{n_j} x$ for any $x \in R(L_\infty)$. By (c) of Proposition 2.2, $P_{n_j} - Q_{n_j} \to 0$ in $\mathbb{L}(R(L_\infty), E)$, we have

$$P_{n_j} L_\infty \overline{x}_j - P'_{n_j} L_\infty \overline{x}_j = P_{n_j} L_\infty \overline{x}_j - Q_{n_j} L_\infty \overline{x}_j \to 0 \quad \text{as } j \to \infty.$$

Furthermore, by $\dim N(L_\infty) < \infty$, we have $(P_{n_j} - I)|_{N(L_\infty)} \to 0$ in $\mathbb{L}(N(L_\infty), E)$ as $j \to \infty$, so

$$P_{n_j} L_\infty P_{n_j} x^0_j = P_{n_j} L_\infty (P_{n_j} - I) x^0_j \to 0 \quad \text{as } j \to \infty.$$

Hence

$$\begin{aligned} P_{n_j} L_\infty \widehat{x}_j - P'_{n_j} L_\infty x_j &= P_{n_j} L_\infty (\overline{x}_j + P_{n_j} x^0_j) - P'_{n_j} L_\infty (\overline{x}_j + x^0_j) \\ &= P_{n_j} L_\infty \overline{x}_j - P'_{n_j} L_\infty \overline{x}_j + P_{n_j} L_\infty P_{n_j} x^0_j \to 0 \end{aligned} \tag{3.1}$$

as $j \to \infty$. It follows from (3.1) and $P'_{n_j} L_\infty x_j \to y$ that $P_{n_j} L_\infty \widehat{x}_j \to y$. Since L_∞ is A-proper with respect to $\mathcal{E}$, $\{\widehat{x}_j\}$ has a convergent subsequence. Hence $\{x_j\}$ also has a convergent subsequence, and L_∞ is A-proper with respect to $\mathcal{E}'$.

Similarly, we can prove L_0 is A-proper with respect to $\mathcal{E}'$. □

Remark 3.3. Since L_∞ is A-proper with respect to $\mathcal{E}'$, by Proposition 2.3, there exist $n_0 > 0$ and $C > 0$ such that for $n \geq n_0$ and $x \in R(L_\infty) \cap E_n$, $\|P'_n L_\infty x\| \geq C\|x\|$, thus $N(P'_n L_\infty|_{E'_n}) \cap (R(L_\infty) \cap E_n) = \{0\}$. Note that $E'_n = (R(L_\infty) \cap E_n) \oplus N(L_\infty)$, so for $n \geq n_0$, $N(P'_n L_\infty|_{E'_n}) = N(L_\infty)$. Let $E'_n = E'^+_n \oplus E'^-_n \oplus N(L_\infty)$ be the decomposition corresponding to the positive, negative and zero part of the spectrum of $P'_n L_\infty|_{E'_n}$. By (b) of Proposition 2.3, there exist n_0 and a constant $C_1 > 0$ such that $\langle P'_n L_\infty x, x\rangle \geq C_1\|x\|^2$ for $n \geq n_0$ and $x \in E'^+_n$, $\langle P'_n L_\infty x, x\rangle \leq -C_1\|x\|^2$ for $n \geq n_0$ and $x \in E'^-_n$.

Let $\Phi_\rho(x) = \Phi(x) + \chi_\rho(\|x^0\|^2)$, where $\chi_\rho(t) \in C^2(\mathbb{R},\mathbb{R})$, $\rho > 0$ and

$$\chi_\rho(t) = \begin{cases} 0, & t \leq \rho, \\ (t-\rho)^4, & t > \rho. \end{cases} \tag{3.2}$$

Lemma 3.4. *Suppose* (H1), (H2) *and* (H4) *hold, then for every $\rho > 0$, Φ_ρ satisfies* (PS)* *condition with respect to $\mathcal{E}'$, and for n large enough $\Phi_\rho|_{E'_n}$ satisfies* (PS) *condition.*

Proof. Let $\{x_j\}$ be a sequence with $x_j \in E'_{n_j}$ for some n_j, $n_j \to \infty$ and $P'_{n_j}\nabla\Phi_\rho(x_j) \to 0$ as $j \to \infty$.

First we show the boundedness of $\{x_j\}$. Let $x_j = x_j^+ + x_j^- + x_j^0 \in E'^+_{n_j} \oplus E'^-_{n_j} \oplus N(L_\infty)$. Since $P'_{n_j}\nabla\Phi_\rho(x_j) \to 0$ as $j \to \infty$, by Remark 3.3 and (H4), we have

$$\begin{aligned} o(\|x_j^+\|) &= \langle P'_{n_j}\nabla\Phi_\rho(x_j), x_j^+\rangle \\ &= \langle P'_{n_j}L_\infty(x_j), x_j^+\rangle - \langle P'_{n_j}\nabla\varphi(x_j), x_j^+\rangle \geq C_1\|x_j^+\|^2 - C_0\|x_j^+\| \end{aligned}$$

as $j \to \infty$, and

$$\begin{aligned} o(\|x_j^-\|) &= \langle P'_{n_j}\nabla\Phi_\rho(x_j), x_j^-\rangle \\ &= \langle P'_{n_j}L_\infty(x_j), x_j^-\rangle - \langle P'_{n_j}\nabla\varphi(x_j), x_j^-\rangle \leq -C_1\|x_j^-\|^2 + C_0\|x_j^-\| \end{aligned}$$

as $j \to \infty$. Hence $\{x_j^+\}$ and $\{x_j^-\}$ are bounded, and thus $\{x_j^+ + x_j^-\}$ is bounded. Moreover, by (H4),

$$\begin{aligned} o(\|x_j^0\|) = \langle P'_{n_j}\nabla\Phi_\rho(x_j), x_j^0\rangle &= -\langle P'_{n_j}\nabla\varphi(x_j), x_j^0\rangle + 2\chi'_\rho(\|x_j^0\|^2)\|x_j^0\|^2 \\ &\geq -C_0\|x_j^0\| + 2\chi'_\rho(\|x_j^0\|^2)\|x_j^0\|^2. \end{aligned} \tag{3.3}$$

By the definition of χ_ρ and (3.3), $\{x_j^0\}$ is bounded. Hence $\{x_j\}$ is bounded.

Now we show that $\{x_j\}$ has a convergent subsequence. Since $\{x_j\}$ is bounded, $\nabla\varphi$ is compact and $\dim N(L_\infty) < \infty$, there exists a subsequence of $\{x_j\}$ (for simplicity still denote by $\{x_j\}$) and $y \in E$ such that $P'_{n_j}\nabla\varphi(x_j) - 2\chi'_\rho(\|x_j^0\|^2)x_j^0 \to y$. Since

$$P'_{n_j}\nabla\Phi_\rho(x_j) = P'_{n_j}L_\infty(x_j) - P'_{n_j}\nabla\varphi(x_j) + 2\chi'_\rho(\|x_j^0\|^2)x_j^0 \to 0,$$

we have $P'_{n_j}L_\infty(x_j) \to y$. By Lemma 3.2, L_∞ is A-proper with respect to $\mathcal{E}'$, so $\{x_j\}$ has a convergent subsequence. This means that Φ_ρ satisfies (PS)* condition with respect to $\mathcal{E}'$.

By a similar argument we can show that for n large enough, any (PS) sequence $\{x_j\}(x_j \in E'_n)$ of $\Phi_\rho|_{E'_n}$ is bounded. Since E'_n is finite dimensional, $\{x_j\}$ has a convergent subsequence. Therefore for n large enough $\Phi_\rho|_{E'_n}$ satisfies (PS) condition. □

Remark 3.5. Note that in Lemma 3.4 one needs not to assume $\{\Phi_\rho(x_j)\}$ to be bounded. Hence the critical point set of Φ_ρ is compact for every $\rho > 0$.

Lemma 3.6. *Suppose* (H1), (H2) *and* (H4) *hold, then* $M^-_{\mathcal{E}'}(L_\infty) = M^-_{\mathcal{E}}(L_\infty)$.

Proof. Since $N(L_\infty) \subset E'_n$, according to Remark 2.5, we have

$$M^-_{\mathcal{E}'}(L_\infty) = \lim_{n\to\infty}(M^-(P'_n L_\infty|_{E'_n}) - d_n).$$

On the other hand,

$$E'_n := (R(L_\infty) \cap E_n) \oplus N(L_\infty) \quad \text{and} \quad P'_n L_\infty|_{R(L_\infty)\cap E_n} = Q_n L_\infty|_{R(L_\infty)\cap E_n},$$

so

$$\begin{aligned} M^-_{\mathcal{E}'}(L_\infty) &= \lim_{n\to\infty}(M^-(P'_n L_\infty|_{E'_n}) - d_n) \\ &= \lim_{n\to\infty}(M^-(Q_n L_\infty|_{R(L_\infty)\cap E_n}) - d_n) = M^-_{\mathcal{E}}(L_\infty). \end{aligned} \qquad \square$$

Lemma 3.7. *Suppose* (H1) *holds. If* $\Phi''_\rho(x_0)$ *is invertible, then*

$$M^-_{\mathcal{E}'}(\Phi''_\rho(x_0)) = M^-_{\mathcal{E}}(\Phi''_\rho(x_0)).$$

Proof. Denote $L = \Phi''_\rho(x_0)$. Since L is invertible, according to Remark 2.5, we have

$$M^-_{\mathcal{E}'}(L) = \lim_{n\to\infty}(M^-(P'_n L|_{E'_n}) - d_n), \qquad M^-_{\mathcal{E}}(L) = \lim_{n\to\infty}(M^-(P_n L|_{E_n}) - d_n).$$

Since $\dim N(L_\infty) < \infty$, there exists $\varepsilon_n \to 0$ such that

$$\|P_n z - z\| \le \varepsilon_n \|z\|, \quad \text{for all } z \in N(L_\infty). \tag{3.4}$$

By Proposition 2.2, there exists a $n_0 > 0$ such that $P_n\colon N(L_\infty) \to P_n N(L_\infty)$ is a linear isomorphism when $n \ge n_0$. Note that

$$E'_n = (R(L_\infty) \cap E_n) \oplus N(L_\infty), E_n = (R(L_\infty) \cap E_n) \oplus P_n N(L_\infty).$$

When $n \ge n_0$, for $x = \overline{x} + x^0 \in (R(L) \cap E_n) \oplus N(L_\infty) = E'_n$, we define $H_n\colon E'_n \to E_n$ by $H_n(x) = \overline{x} + P_n x^0$, then H_n is a linear isomorphism.

Since L is invertible, according to Lemma 3.2 and Proposition 2.3, there exists a $n_1 > n_0$ such that $P'_n L|_{E'_n}$ is also invertible when $n \ge n_1$. Hence, for $n \ge n_1$, $N(P'_n L|_{E'_n}) = 0$, and let $E'_n = E'^+_n \oplus E'^-_n$ be the decomposition corresponding to the positive and the negative part of the spectrum of $P'_n L|_{E'_n}$. For $n \ge n_1$ there exists a constant $C_2 > 0$ such that $\langle Lx, x\rangle = \langle P'_n Lx, x\rangle \ge C_2\|x\|^2$ for $x \in E'^+_n$ and $\langle Lx, x\rangle = \langle P'_n Lx, x\rangle \le -C_2\|x\|^2$ for $x \in E'^-_n$.

For $x \in E_n'^- \setminus \{0\}$, let $x = \overline{x} + x^0 \in (R(L_\infty) \cap E_n) \oplus N(L_\infty)$, then by (3.4), for $n \geq n_1$ large enough

$$
\begin{aligned}
(3.5) \qquad \langle P_n L H_n x, H_n x\rangle &= \langle L(\overline{x} + P_n x^0), (\overline{x} + P_n x^0)\rangle \\
&= \langle L(x - x^0 + P_n x^0), (x - x^0 + P_n x^0)\rangle \\
&= \langle Lx, x\rangle + 2\langle Lx, (P_n x^0 - x^0)\rangle + \langle L(P_n x^0 - x^0), (P_n x^0 - x^0)\rangle \\
&\leq -C_2\|x\|^2 + 3\varepsilon_n\|L\|\|x\|^2 < 0.
\end{aligned}
$$

Since $H_n : E_n' \to E_n$ is an isomorphism, then $E_n = H_n E_n'^+ \oplus H_n E_n'^-$. Thus by (3.5), we have $\langle P_n L x, x\rangle < 0$ for $x \in H_n E_n'^-$, this implies that

$$M^-(P_n L|_{E_n}) \geq M^-(P_n' L|_{E_n'}).$$

Similarly we can prove that $\langle P_n L x, x\rangle > 0$ for $x \in H_n E_n'^+$, and thus

$$M^-(P_n L|_{E_n}) \leq M^-(P_n' L|_{E_n'}).$$

Hence $M^-(P_n' L|_{E_n'}) = M^-(P_n L|_{E_n})$. Therefore,

$$M^-_{\mathcal{E}'}(L) = \lim_{n\to\infty}(M^-(P_n' L|_{E_n'}) - d_n) = \lim_{n\to\infty}(M^-(P_n L|_{E_n}) - d_n) = M^-_{\mathcal{E}}(L). \quad \square$$

According to Theorem 2.6, (H2), (H3), Lemma 3.2 and Lemma 3.7, we have

$$
(3.6) \qquad C^q_{\mathcal{E}'}(\Phi_\rho, 0) = C^q_{\mathcal{E}'}(\Phi, 0) = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L_0), \\ [0], & q \neq M^-_{\mathcal{E}}(L_0). \end{cases}
$$

Lemma 3.8. *Suppose* (H1), (H2) *and* (H4) *hold, then for any $\rho > 0$, the critical group for Φ_ρ at infinity is*

$$
C^q_{\mathcal{E}'}(\Phi_\rho, K(\Phi_\rho)) = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L_\infty), \\ [0], & q \neq M^-_{\mathcal{E}}(L_\infty). \end{cases}
$$

Proof. By Remark 3.5, $K(\Phi_\rho)$ is compact, so we can choose $b > 0$ such that $K(\Phi_\rho) \subset \operatorname{int} \Phi_\rho^{-1}([-b, b])$. By the definition of $C^q_{\mathcal{E}'}(\Phi_\rho, K(\Phi_\rho))$, we need to compute $H^q_{\mathcal{E}'}(\Phi_\rho^{-1}([-b, b]), \Phi_\rho^{-1}(-b))$.

Consider the operator $P_n' L_\infty|_{E_n'} : E_n' \to E_n'$. By Remark 3.3, for $x \in E_n'$, we can set $x = x^+ + x^- + x^0 \in E_n'^+ \oplus E_n'^- \oplus N(L_\infty)$. By (H4), for $n \geq n_0$ we have

$$\langle P_n' \nabla\Phi_\rho(x), x^-\rangle = \langle P_n' L_\infty x, x^-\rangle - \langle P_n' \nabla\varphi(x), x^-\rangle \leq -C_1\|x^-\|^2 + C\|x^-\|.$$

Hence there is a $R > 0$ large enough such that

$$
(3.7) \qquad \langle P_n' \nabla\Phi_\rho(x), x^-\rangle < -a, \quad x \in E_n' \setminus (E_n' \cap U_R),
$$

where $a > 0$ is a constant, $U_R := \{x \in E \mid \|x^-\| \leq R\}$. This implies that $\Phi_\rho|_{E_n'}$ has no critical point in $E_n' \setminus (E_n' \cap U_R)$, and the negative gradient vector field of $\Phi_\rho|_{E_n'}$ points outward of $E_n' \cap U_R$ on $\partial(E_n' \cap U_R)$.

By Remark 3.3 and (H4), for any $x \in E'_n \cap U_R$, we have

$$\begin{aligned}(3.8)\quad \Phi_\rho(x) &= \frac{1}{2}\langle L_\infty x^+, x^+\rangle + \frac{1}{2}\langle L_\infty x^-, x^-\rangle - \varphi(x) + \chi_\rho(\|x^0\|^2)\\ &\geq \frac{1}{2}C_1\|x^+\|^2 - \frac{1}{2}\|L\|R^2 - C(\|x^+\| + R + \|x^0\|) + \chi_\rho(\|x^0\|^2),\end{aligned}$$

By the definition of χ_ρ and (3.8), we can choose $b > 0$ sufficiently large such that $\Phi_\rho(x) > -b$ for $n \geq n_0$ and $x \in E'_n \cap U_R$, therefore $\Phi_\rho^{-b} \cap E'_n \subset E'_n \setminus (E'_n \cap U_R)$. By Lemma 3.4, we can take n_0 large enough such that $\Phi_\rho|_{E'_n}$ satisfies (PS) condition for $n \geq n_0$. Note that $\Phi_\rho|_{E'_n}$ has no critical points in $E'_n \setminus (E'_n \cap U_R)$ and the negative gradient vector field of $\Phi_\rho|_{E'_n}$ points outward of $E'_n \cap U_R$ on $\partial(E'_n \cap U_R)$, we can construct a deformation mapping $\gamma_1 \colon E'_n \setminus (E'_n \cap U_R) \to \Phi_\rho^{-b} \cap E'_n$ by the flow generated by $-P'_n \nabla \Phi_\rho|_{E'_n}$.

We claim that one can choose b and n_0 large enough such that for $n \geq n_0$, $K(\Phi_\rho|_{E'_n}) \subset \Phi_\rho^b \cap E'_n$. Indeed, if this is not true, there is a sequence $\{x_j\}$ such that $x_j \in E'_{n_j}$ for some n_j, $P'_{n_j}\nabla\Phi_\rho(x_j) = 0$, $n_j \to \infty$ and $\Phi_\rho(x_j) \to \infty$ as $j \to \infty$. Since in the proof of Lemma 3.4 we do not need to assume $\{\Phi_\rho(x_j)\}$ to be bounded, by Lemma 3.4 there exists $x_0 \in E$ such that $\{x_j\}$ has a subsequence (still denoted by $\{x_j\}$) which converges to x_0, thus $\Phi_\rho(x_j) \to \Phi_\rho(x_0)$, therefore we get a contradiction. On the other hand, $\Phi_\rho|_{E'_n}$ satisfies (PS) condition, by deformation lemma, there is a deformation retract $\gamma_2 \colon E'_n \to \Phi_\rho^b \cap E'_n$.

Therefore, for $n \geq n_0$, we have

$$H^q(\Phi_\rho^b \cap E'_n, \Phi_\rho^{-b} \cap E'_n) \cong H^q(E'_n, E'_n \setminus (E'_n \cap U_R)) = \begin{cases} \mathcal{F}, & q = M^-(P'_n L_\infty|_{E'_n}),\\ 0, & q \neq M^-(P'_n L_\infty|_{E'_n}).\end{cases}$$

By Lemma 3.6 and $M^-_{\mathcal{E}'}(L_\infty) = \lim_{n\to\infty}(M^-(P'_n L_\infty|_{E'_n}) - d_n)$, there exists $n_1 > n_0$ such that $M^-_{\mathcal{E}}(L_\infty) = M^-_{\mathcal{E}'}(L_\infty) = M^-(P'_n L_\infty|_{E'_n}) - d_n$ for any $n \geq n_1$. Hence we have

$$H^q(\Phi_\rho^b \cap E'_n, \Phi_\rho^{-b} \cap E'_n) = \begin{cases} \mathcal{F}, & q = M^-_{\mathcal{E}}(L_\infty) + d_n,\\ 0, & q \neq M^-_{\mathcal{E}}(L_\infty) + d_n,\end{cases}$$

and

$$H^q_{\mathcal{E}'}(\Phi_\rho^b, \Phi_\rho^{-b}) \cong [(H^{q+d_n}(\Phi_\rho^b \cap E'_n, \Phi_\rho^{-b} \cap E'_n))_{n=1}^\infty] = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L_\infty),\\ [0], & q \neq M^-_{\mathcal{E}}(L_\infty).\end{cases}$$

By excision and the definition of $C^q_{\mathcal{E}'}(\Phi_\rho, K(\Phi_\rho))$,

$$C^q_{\mathcal{E}'}(\Phi_\rho, K(\Phi_\rho)) \cong H^q_{\mathcal{E}'}(\Phi_\rho^{-1}([-b,b]), \Phi_\rho^{-1}(-b)) \cong H^q_{\mathcal{E}'}(\Phi_\rho^b, \Phi_\rho^{-b}). \qquad \square$$

Lemma 3.9. *There exists a constant $M > 0$ independent of ρ such that for any $x \in K(\Phi_\rho)$ with $x = x^+ + x^- + x^0 \in E^+ \oplus E^- \oplus N(L_\infty)$, we have $\|x^+ + x^-\| < M$.*

Proof. For $x \in K(\Phi_\rho)$ with $x = x^+ + x^- + x^0 \in E^+ \oplus E^- \oplus N(L_\infty)$, by (H4), one has

$$\begin{aligned} 0 &= \langle \nabla\Phi_\rho(x), x^+\rangle = \langle L_\infty x, x^+\rangle - \langle \nabla\varphi(x), x^+\rangle \geq C\|x^+\|^2 - C_0\|x^+\|, \\ 0 &= \langle \nabla\Phi_\rho(x), x^-\rangle = \langle L_\infty x, x^-\rangle - \langle \nabla\varphi(x), x^-\rangle \leq -C\|x^-\|^2 + C_0\|x^-\|, \end{aligned}$$

where C_0 and C are constants independent of ρ. So there exist constants $M_1, M_2 > 0$ independent of $\rho > 0$ such that for $x \in K(\Phi_\rho)$, we have $\|x^+\| < M_1$, $\|x^-\| < M_2$. Let $M = M_1 + M_2$, we have $\|x^+ + x^-\| < M$. □

Proof of Theorem 3.1. For $x \in E$, let $x = x^+ + x^- + x^0 \in E^+ \oplus E^- \oplus N(L_\infty)$. If Φ has no nontrivial critical point, then Φ_ρ has no nontrivial critical point in set $E_\rho := \{x \in E \mid \|x^0\|^2 \leq \rho\}$. Since $M^-_{\mathcal{E}}(L_0) \neq M^-_{\mathcal{E}}(L_\infty)$, the critical set $K(\Phi_\rho) \setminus \{0\}$ is not empty. According to Remark 3.5, $K(\Phi_\rho) \setminus \{0\}$ is compact, so $d := d(K(\Phi_\rho), E_\rho) > 0$. Let $N_\delta(A) := \{x \in E \mid d(x, A) < \delta\}$. By the Marino–Prodi perturbation technique [20], for any $\varepsilon > 0$, $0 < \tau < \min\{1, d/3\}$, there exists a C^2 functional I_ρ such that

(a) $\|\Phi_\rho - I_\rho\|_{C^2} < \varepsilon$;
(b) $\Phi_\rho(x) = I_\rho(x)$, $x \in E \setminus N_{2\tau}(K(\Phi_\rho))$;
(c) $\Phi''_\rho(x) = I''_\rho(x)$, $x \in N_\tau(K(\Phi_\rho))$, $K(I_\rho) \subset N_\tau(K(\Phi_\rho))$, and the critical points of I_ρ are all nondegenerate.

By (b), (3.6) and Lemma 3.8 we have

$$C^q_{\mathcal{E}'}(I_\rho, K(I_\rho)) = C^q_{\mathcal{E}'}(\Phi_\rho, K(\Phi_\rho)) = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L_\infty), \\ [0], & q \neq M^-_{\mathcal{E}}(L_\infty), \end{cases} \tag{3.9}$$

$$C^q_{\mathcal{E}'}(I_\rho, 0) = C^q_{\mathcal{E}'}(\Phi_\rho, 0) = \begin{cases} [\mathcal{F}], & q = M^-_{\mathcal{E}}(L_0), \\ [0], & q \neq M^-_{\mathcal{E}}(L_0). \end{cases} \tag{3.10}$$

We claim that: there exists a constant $\beta > 0$ independent of ρ such that if $x \in K(I_\rho)$ satisfies $M^-_{\mathcal{E}}(I''_\rho(x)) < M^-_{\mathcal{E}}(L_\infty)$ or $M^-_{\mathcal{E}}(I''_\rho(x)) > M^-_{\mathcal{E}}(L_\infty) + M^0(L_\infty)$, then $\|x^0\| \leq \beta$.

If the claim is not true, then there exists a sequence $\{x_i\}$ such that $x_i \in K(I_{\rho_i})$, $\rho_i > 0$, $M^-_{\mathcal{E}}(I''_{\rho_i}(x_i)) < M^-_{\mathcal{E}}(L_\infty)$ or $M^-_{\mathcal{E}}(I''_{\rho_i}(x_i)) > M^-_{\mathcal{E}}(L_\infty) + M^0(L_\infty)$, and $\|x_i^0\| \to \infty$ as $i \to \infty$

Since $K(I_{\rho_i}) \subset N_\tau(K(\Phi_{\rho_i}))$, there exists $x'_i \in K(\Phi_{\rho_i})$ such that $\|x_i - x'_i\| < \tau < 1$. By Lemma 3.9, $\|x'^+_i + x'^-_i\| < M$. Hence $\|x^+_i + x^-_i\| < M + 1$. On the other hand $\|x^0_i\| \to \infty$ as $i \to \infty$, so by (H5), $|\varphi''(x_i)\| \to 0$ as $i \to \infty$. Hence there exists $i_0 > 0$ such that for any $i \geq i_0$, we have $\|\varphi''(x_i)\| < C_1/2$, where C_1 is given in Remark 3.3.

For $n \geq n_0$, $y^- \in E_n'^- \setminus \{0\}$, $y^+ \in E_n'^+ \setminus \{0\}$, by Remark 3.3 and (c), we get

$$\begin{aligned}
\langle P_n' I_\rho''(x_i)y^-, y^-\rangle &= \langle P_n'\Phi_\rho''(x_i)y^-, y^-\rangle \\
&= \langle P_n' L_\infty y^-, y^-\rangle - \langle P_n'\varphi''(x_i)y^-, y^-\rangle \\
&\leq -C_1\|y^-\|^2 + \|\varphi''(x_i)\|\|y^-\|^2 < 0, \\
\langle P_n' I_\rho''(x_i)y^+, y^+\rangle &= \langle P_n'\Phi_\rho''(x_i)y^+, y^+\rangle \\
&= \langle P_n' L_\infty y^+, y^+\rangle - \langle P_n'\varphi''(x_i)y^+, y^+\rangle \\
&\geq C_1\|y^+\|^2 - \|\varphi''(x_i)\|\|y^+\|^2 > 0.
\end{aligned}$$

Hence for $n \geq n_0$ and $i \geq i_0$,

$$M^-(P_n' I_\rho''(x_i)|_{E_n'}) \geq M^-(P_n' L_\infty|_{E_n'}), \tag{3.11}$$

$$M^-(P_n' I_\rho''(x_i)|_{E_n'}) \leq M^-(P_n' L_\infty|_{E_n'}) + M^0(L_\infty). \tag{3.12}$$

Since the critical points x_i are all nondegenerate, by Lemma 3.7, (3.11), (3.12) and (c) for $i \geq i_0$, we have

$$\begin{aligned}
M_{\mathcal{E}}^-(I_\rho''(x_i)) &= M_{\mathcal{E}'}^-(I_\rho''(x_i)) = \lim_{n\to\infty}(M^-(P_n' I_\rho''(x_i)|_{E_n'}) - d_n) \\
&\geq \lim_{n\to\infty}(M^-(P_n' L_\infty|_{E_n'}) - d_n) = M_{\mathcal{E}'}^-(L_\infty) = M_{\mathcal{E}}^-(L_\infty), \\
M_{\mathcal{E}}^-(I_\rho''(x_i)) &= M_{\mathcal{E}'}^-(I_\rho''(x_i)) = \lim_{n\to\infty}(M^-(P_n' I_\rho''(x_i)|_{E_n'}) - d_n) \\
&\leq \lim_{n\to\infty}(M^-(P_n' L_\infty|_{E_n'}) - d_n) + M^0(L_\infty) \\
&= M_{\mathcal{E}'}^-(L_\infty) + M^0(L_\infty) = M_{\mathcal{E}}^-(L_\infty) + M^0(L_\infty).
\end{aligned}$$

This leads to a contradiction. The claim is proved.

Take $\rho = \beta^2 + 1$, then I_ρ has no nontrivial critical point in the set E_ρ, so for $x \in K(I_\rho) \setminus \{0\}$, we have $\|x^0\| > \beta$. By the above claim, for any $x \in K(I_\rho) \setminus \{0\}$,

$$M_{\mathcal{E}}^-(L_\infty) \leq M_{\mathcal{E}}^-(I_\rho''(x)) \leq M_{\mathcal{E}}^-(L_\infty) + M^0(L_\infty). \tag{3.13}$$

Since the critical set $K(I_\rho)$ of I_ρ is compact, and the elements of $K(I_\rho)$ are all nondegenerate, $K(I_\rho)$ is a finite set. Assume that $K(I_\rho) \setminus \{0\} = \{x_1, \dots, x_n\}$, and denote $m_{\mathcal{E}}^-(x_i) = M_{\mathcal{E}}^-(I_\rho''(x_i))$. From (3.9), (3.10), Theorems 2.6 and 2.1 we know that

$$t^{M_{\mathcal{E}}^-(L_0)} + \sum_{i=1}^n t^{m_{\mathcal{E}}^-(x_i)} = t^{M_{\mathcal{E}}^-(L_\infty)} + (1+t)Q(t). \tag{3.14}$$

Since the left-hand side of (3.14) contains the exponent $M_{\mathcal{E}}^-(L_0)$ and $M_{\mathcal{E}}^-(L_0) \neq M_{\varepsilon}^-(L_\infty)$. Therefore $Q(t)$ must have a nonzero term with exponent $M_{\mathcal{E}}^-(L_0)$ or $M_{\mathcal{E}}^-(L_0)-1$, and it follows that there is a nonzero term with exponent $M_{\mathcal{E}}^-(L_0)+1$ or $M_{\mathcal{E}}^-(L_0)-1$ on the left-hand side. Hence there exists $x_i \in K(I_\rho)$ such that $m_{\mathcal{E}}^-(x_i) = M_{\varepsilon}^-(L_0) - 1$ or $m_{\mathcal{E}}^-(x_i) = M_{\mathcal{E}}^-(L_0) + 1$. So according to the condition $M_{\mathcal{E}}^-(L_0) < M_{\mathcal{E}}^-(L_\infty) - 1$ or $M_{\mathcal{E}}^-(L_0) > M_{\mathcal{E}}^-(L_\infty) + M^0(L_\infty) + 1$, we have $m^-_{\mathcal{E}}(x_i) < M^-_{\mathcal{E}}(L_\infty)$ or $m^-_{\mathcal{E}}(x_i) > M^-_{\mathcal{E}}(L_\infty) + M^0(L_\infty)$. This contradicts to (3.13), so Φ has at least one nontrivial critical point. □

4. Application to Hamiltonian systems

Consider the following Hamiltonian systems

$$\dot{z} = JH_z(z,t), \tag{4.1}$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard symplectic matrix.

First we introduce the following assumptions:

- (A1) $H \in C^2(\mathbb{R}^{2N} \times \mathbb{R})$ is 2π-periodic in t;
- (A2) $H(z,t) = \frac{1}{2}A(t)z \cdot z + G(z,t)$, where $A(t)$ is a symmetric $2N \times 2N$ matrix with 2π-periodic entries, and $|G_z(z,t)| \le C$ for some $C > 0$ and all $(z,t) \in \mathbb{R}^{2N} \times \mathbb{R}$;
- (A3) $H(z,t) = \frac{1}{2}A_0(t)z \cdot z + G_0(z,t)$,where $A_0(t)$ is a symmetric $2N \times 2N$ matrix with 2π-periodic entries, and $(G_0)_z(z,t) = o(|z|)$ uniformly in t as $|z| \to 0$;
- (A4) The equation $\dot{z} = JA_0(t)z$ has no nontrivial 2π-periodic solution;
- (A5) $\|G_{zz}(z,t)\| \to 0$ uniformly in t as $|z| \to \infty$.

Let

$$E = \bigg\{ z(t) \;\bigg|\; z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt,\ a_0, a_k, b_k \in \mathbb{R}^{2N}, \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty \bigg\}.$$

Then E is a Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ defined by

$$\langle z, z' \rangle = 2\pi a_0 \cdot a_0' + \pi \sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k').$$

Let

$$\Phi = \frac{1}{2}\int_0^{2\pi} (-J\dot{z} \cdot z)dt - \int_0^{2\pi} H(z,t)\,dt.$$

By (A2) and (A5), $\|H_{zz}(z,t)\| \le C'(1 + |z|^s)$ for some $C' > 0, s \in (0,\infty)$ and all $(z,t) \in \mathbb{R}^{2N} \times \mathbb{R}$, then it is known [18] that $\Phi \in C^2(E,\mathbb{R})$ and $z(t)$ is a 2π-periodic solution of (4.1) if and only if it is a critical point of Φ.

Let $\widetilde{L}$, B_∞ and B_0 be the linear operators from E to E defined by

$$\langle \widetilde{L}z, z' \rangle := \int_0^{2\pi} (-J\dot{z} \cdot z')\,dt, \qquad \text{forall } z, z' \in E,$$

$$\langle B_\infty z, z' \rangle := \int_0^{2\pi} (A(t)z \cdot z')\,dt, \qquad \text{for all } z, z' \in E,$$

$$\langle B_0 z, z' \rangle := \int_0^{2\pi} (A_0(t)z \cdot z')\, dt, \quad \text{for all } z, z' \in E.$$

Then $\widetilde{L}$ is a self-adjoint Fredholm operator of index 0, B_∞ and B_0 are compact self-adjoint operators.

Let

$$\varphi(z) = \int_0^{2\pi} G(z,t)\, dt, \varphi_0(z) = \int_0^{2\pi} G_0(z,t)\, dt.$$

Then $\nabla\varphi$ and $\nabla\varphi_0$ are compact mappings, satisfy $\nabla\varphi(x) = o(\|x\|)$ as $\|x\| \to \infty$ and $\nabla\varphi_0(x) = o(\|x\|)$ as $\|x\| \to 0$.

Denote $L_\infty = \widetilde{L} - B_\infty, L_0 = \widetilde{L} - B_0$, we can rewrite Φ by

$$\Phi = \frac{1}{2}\langle L_\infty z, z\rangle - \varphi(z) = \frac{1}{2}\langle L_0 z, z\rangle - \varphi_0(z).$$

Let

$$E_n := \left\{ z \in E \;\middle|\; z(t) = a_0 + \sum_{k=1}^{n} a_k \cos kt + b_k \sin kt, a_0, a_k, b_k \in \mathbb{R}^{2N} \right\},$$

then $\{E_n\}_{n=1}^\infty$ is a filtration of E. Set $d_n := N(2n+1)$ and $\mathcal{E} := \{E_n, d_n\}_{n=1}^\infty$, then it is easy to see that $\widetilde{L}(E_n) \subset E_n$ and $M^-(\widetilde{L}|_{E_n}) = 2nN = d_n - N$. Hence Φ satisfies conditions (H1), (H2).

It follows from (A4) that Φ satisfies the condition (H3).

It follows from (A2) that

$$|\langle \nabla\varphi(z), w\rangle| = \left| \int_0^{2\pi} G_z(z,t) w\, dt \right| \le \int_0^{2\pi} |G_z(z,t)||w|\, dt \le C\|w\|,$$

for all $z, w \in E$ for some $C > 0$, so $\|\nabla\varphi(z)\| \le C, \forall z \in E$. Hence Φ satisfies the condition (H4).

In what following we show that Φ satisfies the condition $(H5)$, the idea is similar to that in [3].

Lemma 4.1. *Suppose* (A1), (A2) *and* (A5) *hold, then the functional* Φ *satisfies* (H5).

Proof. Assume that the sequence $\{x_j\}$ satisfies that $\{x_j^+ + x_j^-\}$ is bounded and $\|x_j^0\| \to \infty$ as $j \to \infty$, we will show that $\|\varphi''(x_j)\| \to 0$ as $j \to \infty$.

First, since $\{x_j^+ + x_j^-\}$ is bounded and $E \hookrightarrow L^2([0,2\pi], \mathbb{R}^{2N})$ is compact embedding, the sequence $\{x_j^+ + x_j^-\}$ contains in a compact subset of $L^2([0,2\pi], \mathbb{R}^{2N})$. Since $x_j^0 \in N(L_\infty)$, x_j^0 satisfies the equation

$$\dot{x}_j^0 = JA(t)x_j^0,$$

then we obtain

$$x_j^0(t) = x_j^0(t_0) + \int_{t_0}^{t} JA(s)x_j^0(s)\, ds,$$

thus

$$|x_j^0(t)| \le |x_j^0(t_0)| + \int_{t_0}^t |JA(s)||x_j^0(s)|\,ds \le |x_j^0(t_0)| + C\int_0^{2\pi} |x_j^0(s)|\,ds,$$

where $C > 0$ is a constant. It follows from Gronwall inequality that

$$\max\{|x_j^0(t)||t \in [0,2\pi]\} \le C' \min\{|x_j^0(t)||t \in [0,2\pi]\}. \tag{4.2}$$

Since $\|x_j^0\| \to \infty$ as $j \to \infty$ and $\dim N(L_\infty) < \infty$, according to the equivalence of the norm between finite dimension spaces, we have $\|x_j^0\|_{C^0} \to \infty$ as $j \to \infty$, that is $\max\{|x_j^0(t)| \mid t \in [0,2\pi]\} \to \infty$ as $j \to \infty$. By (4.2), one has

$$\min\{|x_j^0(t)| \mid t \in [0,2\pi]\} \to \infty \quad \text{as } j \to \infty.$$

Denote $\overline{x}_j = x_j^+ + x_j^-$. We claim that: for any $\varepsilon > 0$ there exists $M > 0$ such that

$$\operatorname{mes}\{t \mid \overline{x}_j(t) \ge M\} < \varepsilon.$$

If the claim is not true, then there exists $\varepsilon_0 > 0$ such that there is a subsequence of $\{x_j\}$ (still denoted by $\{x_j\}$) such that $\operatorname{mes}\{t \mid \overline{x}_j(t) \ge M_j\} > \varepsilon_0$, where $M_j \to \infty$ as $j \to \infty$. Since the sequence $\{\overline{x}_j\}$ contains in a compact subset of $L^2([0,2\pi],\mathbb{R}^{2N})$, there exists $x_0 \in E$ such that in $L^2([0,2\pi],\mathbb{R}^{2N})$ there is a subsequence of $\{\overline{x}_j\}$ (still denoted by $\{\overline{x}_j\}$), converges to x_0, thus, we have

$$\int_0^{2\pi} |\overline{x}_j(t)|^2\,dt \to \int_0^{2\pi} |x_0(t)|^2\,dt < \infty.$$

This contradicts to

$$\int_0^{2\pi} |\overline{x}_j(t)|^2\,dt \ge M_j^2\varepsilon_0 \to \infty, \quad j \to \infty.$$

The claim is proved.

For any $\varepsilon > 0$, by the above claim and (A5), there exist $M > 0$ and $X > 0$ such that $\operatorname{mes}\{t \mid \overline{x}_j(t) \ge M,\ t \in [0,2\pi]\} < \varepsilon$, and $\|G_{zz}(z,t)\| < \varepsilon$ for $|z| > X$. Denote $Y = \{t \mid \overline{x}_j(t) \le M,\ t \in [0,2\pi]\}$ and since $\min\{|x_j^0(t)| \mid t \in [0,2\pi]\} \to \infty$ as $j \to \infty$, there exists $j_0 > 0$ such that

$$|x_j(t)| = |\overline{x}_j(t) + x_j^0(t)| \ge |x_j^0(t)| - M > X$$

for $t \in Y$ and $j \ge j_0$. Moreover, by (A5), there exists $M_0 > 0$ such that $\sup\{\|G_{zz}(z,t)\| \mid (z,t) \in \mathbb{R}^{2N} \times \mathbb{R}\} \le M_0$. Hence for $j \ge j_0$, we have

$$\int_0^{2\pi} \|G_{zz}(x_j(t),t)\|^2\,dt = \int_Y \|G_{zz}(x_j(t),t)\|^2\,dt + \int_{[0,2\pi]\setminus Y} \|G_{zz}(x_j(t),t)\|^2\,dt \le 2\pi\varepsilon^2 + M_0^2\varepsilon.$$

By Hölder inequality and Sobolev inequality, if $j \geq j_0$ we have

$$\begin{aligned}|\langle\varphi''(x_j)v, v\rangle| &= \left|\int_0^{2\pi} G_{zz}(x_j(t), t)v \cdot v\,dt\right| \leq \int_0^{2\pi} \|G_{zz}(x_j(t), t)\||v|^2\,dt \\ &\leq \left(\int_0^{2\pi} \|G_{zz}(x_j(t), t)\|^2\,dt\right)^{1/2}\left(\int_0^{2\pi} |v|^4\,dt\right)^{1/2} \\ &\leq (2\pi\varepsilon^2 + M_0^2\varepsilon)^{1/2}\|v\|^2_{L^4([0,2\pi],\mathbb{R}^{2N})} \leq C'(2\pi\varepsilon^2 + M_0^2\varepsilon)^{1/2}\|v\|_E^2,\end{aligned}$$

where $C' > 0$ is a constant. Hence $\|\varphi''(x_j)\| \to 0$ as $j \to \infty$, Φ satisfies the condition (H5). $\square$

Note that L_∞ and L_0 are decided by $A(t)$ and $A_0(t)$ respectively, we can denote $j^-(A) := M_{\mathcal{E}}^-(L_\infty)$, $j^0(A) := M^0(L_\infty)$, $j^-(A_0) := M_{\mathcal{E}}^-(L_0)$, $j^0(A_0) := M^0(L_0)$. According to Theorem 3.1 we have the following result.

THEOREM 4.2. *Suppose that* (4.1) *satisfies conditions* (A1)–(A5)*, and* $j^-(A_0) < j^-(A)-1$ *or* $j^-(A_0) > j^-(A)+j^0(A)+1$*, then* (4.1) *has at least one nontrivial* 2π*-periodic solution.*

REMARK 4.3. There are many results about the existence of nontrivial solutions of Hamiltonian systems resonant at infinity, some used the Landersman–Lazer type conditions, which implies that the global compactness((PS)* condition) of the functional is guaranteed, see [12], [22], some used strong resonance condition, which implies that the functional satisfies the (PS)* condition apart from some exceptional levels, see [5], [19]. Here we use a different condition, which is similar as in [21] for second order elliptic equation. Under this condition, the (PS)* condition may fail at any level.

5. Application to elliptic system

Consider the following strongly indefinite elliptic system

$$\begin{cases} -\triangle u = F_v(x, u, v) & \text{in } \Omega, \\ -\triangle v = F_u(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Problem (5.1) has been studied in [6] for subquadratic F, in [7], [11] for superquadratic F. Here we are interested in the asymptotically quadratic case and introduce the following assumptions:

- (B1) $F(x, u, v) \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$;
- (B2) $F(x, u, v) = \frac{1}{2}a(x)u^2 + b(x)uv + \frac{1}{2}c(x)v^2 + G(x, u, v)$, where $a(x), b(x), c(x) \in C(\overline{\Omega}, \mathbb{R})$, and there exists $C > 0$ such that $|G_u(x, u, v)| + |G_v(x, u, v)| < C$, for all $(x, u, v) \in \Omega \times \mathbb{R}^2$;
- (B3) $F(x, u, v) = \frac{1}{2}a_0(x)u^2 + b_0(x)uv + \frac{1}{2}c_0(x)v^2 + G_0(x, u, v)$,

where $a_0(x), b_0(x), c_0(x) \in C(\overline{\Omega}, \mathbb{R})$, and $|(G_0)_u(x,u,v)| + |(G_0)_v(x,u,v)| = o(|u|+|v|)$ uniformly in x as $|u|+|v| \to 0$;

(B4) The equation

$$\begin{cases} -\triangle u = b_0(x)u + c_0(x)v & \text{in } \Omega, \\ -\triangle v = a_0(x)u + b_0(x)v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has no nonzero solution;

(B5) $\|D^2G(x,u,v)\| \to 0$ uniformly in x as $|u|+|v| \to \infty$.

Let $H_0^1(\Omega)$ be the usual Sobolev space and set $E := H_0^1(\Omega) \times H_0^1(\Omega)$. Then E is a Hilbert space with inner product given by

$$\langle (u,v),(u',v')\rangle = \int_\Omega (\nabla u \cdot \nabla u' + \nabla v \cdot \nabla v')\,dx, \quad \text{for all } (u,v),(u',v') \in E.$$

It follows from (B1), (B2) and (B5) that the functional $\Phi\colon E \to \mathbb{R}$ defined by

$$\Phi(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega F(x,u,v)\,dx$$

is of class $C^2(E,\mathbb{R})$ and critical points of Φ correspond to weak solutions of (5.1) [18].

Let $0 < \lambda_1 < \lambda_2 \le \ldots$ be the eigenvalues of the operator $-\triangle$ in $H_0^1(\Omega)$ and let $(e_n)_{n=1}^\infty$ be the corresponding orthonormal basis of eigenfunctions.

Set $E_n := \operatorname{span}\{(e_i,0),(0,e_j) : 1 \le i,j \le n\}$, $\mathcal{E} = \{E_n, n\}_{n=1}^\infty$.

For $(u,v),(u',v') \in E$, we define the linear operators $\widetilde{L}$, B_∞ and B_0 by

$$\begin{aligned} \langle \widetilde{L}(u,v),(u',v')\rangle &:= \int_\Omega (\nabla v \cdot \nabla u' + \nabla u \cdot \nabla v')\,dx, \\ \langle B_\infty(u,v),(u',v')\rangle &:= \int_\Omega (a(x)uu' + b(x)u'v + b(x)uv' + c(x)vv')\,dx, \\ \langle B_0(u,v),(u',v')\rangle &:= \int_\Omega (a_0(x)uu' + b_0(x)u'v + b_0(x)uv' + c_0(x)vv')\,dx. \end{aligned}$$

Then $\widetilde{L}$ is a self-adjoint Fredholm operator of index 0, $\widetilde{L}(E_n) \subset E_n$, $M^-(\widetilde{L}|_{E_n}) = n$. B_∞ and B_0 are compact self-adjoint operator. Let

$$\varphi(u,v) = \int_\Omega G(x,u,v)\,dx, \qquad \varphi_0(u,v) = \int_\Omega G_0(x,u,v)\,dx,$$

then $\nabla\varphi$ and $\nabla\varphi_0$ are compact mappings. By (B2) and (B3), $\nabla\varphi(u,v) = o(\|(u,v)\|)$ as $\|(u,v)\| \to \infty$ and $\nabla\varphi_0(u,v) = o(\|(u,v)\|)$ as $\|(u,v)\| \to 0$. Denote $L_\infty = \widetilde{L} - B_\infty$, $L_0 = \widetilde{L} - B_0$, we can rewrite Φ by

$$\Phi = \frac{1}{2}\langle L_\infty(u,v),(u,v)\rangle - \varphi(u,v) = \frac{1}{2}\langle L_0(u,v),(u,v)\rangle - \varphi_0(u,v).$$

Then Φ satisfies conditions (H1) and (H2).

By (B4), Φ satisfies the condition (H3).

By (B2), for any $(u,v),(u',v')\in E$, one has

$$\begin{aligned}|\langle\nabla\varphi(u,v),(u',v')\rangle| &= \bigg|\int_\Omega(G_u(x,u,v)u+G_v(x,u,v)v)\,dx\bigg|\\ &\le\int_\Omega(|G_u(x,u,v)|+|G_v(x,u,v)|)(|u|+|v|)\,dx\\ &\le C\int_\Omega(|u|+|v|)\,dx\le C'\|(u,v)\|,\end{aligned}$$

where $C'>0$ is a constant. So $\|\nabla\varphi(u,v)\|\le C'$, for all $(u,v)\in E$. Hence Φ satisfies the condition (H4).

To prove Φ satisfies the condition (H5), we should reference the proposition from [14].

Proposition 5.1 ([14]). *Let Ω be an open domain in $\mathbb{R}^N$ and $M\in L^\infty_{\rm loc}(\Omega,\Lambda)$, where Λ is the linear space of $m\times m$ real symmetric matrices. If $u\in(H^1_{\rm loc}(\Omega))^m$ satisfies the inequality $|\triangle u|\le|Mu|$ and u vanishes on a subset W of Ω with positive measure, then u is identically zero in Ω.*

Now we prove that Φ satisfies the condition (H5).

Lemma 5.2. *Suppose* (B1), (B2) *and* (B5) *hold, then Φ satisfies the condition* (H5).

Proof. Let the sequence $\{(u_j,v_j)\}$ satisfies that $\{(u_j^+,v_j^+)+(u_j^-,v_j^-)\}$ is bounded and $\|(u_j^0,v_j^0)\|\to\infty$ as $j\to\infty$. We will show that $\|\varphi''(u_j,v_j)\|\to 0$ as $j\to\infty$.

First, since $\{(u_j^+,v_j^+)+(u_j^-,v_j^-)\}$ is bounded and $E\hookrightarrow L^2(\Omega)\times L^2(\Omega)$ is a compact imbedding mapping, the sequence $\{(u_j^+,v_j^+)+(u_j^-,v_j^-)\}$ should contain in a compact subset of $L^2(\Omega)\times L^2(\Omega)$. Since $(u_j^0,v_j^0)\in N(L_\infty)$, (u_j^0,v_j^0) should satisfies the following equation

$$\begin{cases}-\triangle u_j^0=b(x)u_j^0+c(x)v_j^0 & \text{in }\Omega,\\ -\triangle v_j^0=a(x)u_j^0+b(x)v_j^0 & \text{in }\Omega,\\ u_j^0=v_j^0=0 & \text{on }\partial\Omega.\end{cases}$$

Let $(\widetilde u_j,\widetilde v_j)=(u_j^0,v_j^0)/\|(u_j^0,v_j^0)\|$, then $(\widetilde u_j,\widetilde v_j)$ satisfies the equation

$$\begin{cases}-\triangle\widetilde u_j=b(x)\widetilde u_j+c(x)\widetilde v_j & \text{in }\Omega,\\ -\triangle\widetilde v_j=a(x)\widetilde u_j+b(x)\widetilde v_j & \text{in }\Omega,\\ \widetilde u_j=\widetilde v_j=0 & \text{on }\partial\Omega.\end{cases}$$

Since $\|(\widetilde u_j,\widetilde v_j)\|=1$, there exists $(\widetilde u,\widetilde v)\in E$ such that $(\widetilde u_j,\widetilde v_j)$ weakly convergence in E and strongly convergence in $L^2(\Omega)\times L^2(\Omega)$ to $(\widetilde u,\widetilde v)\in E$, and $(\widetilde u,\widetilde v)$ satisfies the following equation

$$\begin{cases} -\triangle\widetilde{u} = b(x)\widetilde{u} + c(x)\widetilde{v} & \text{in } \Omega, \\ -\triangle\widetilde{v} = a(x)\widetilde{u} + b(x)\widetilde{v} & \text{in } \Omega, \\ \widetilde{u} = \widetilde{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $(\widetilde{u},\widetilde{v}) \neq 0$, according to Proposition 5.1 $(\widetilde{u},\widetilde{v})$ is not equal to zero almost everywhere in Ω. Thus by $\|(u_j^0,v_j^0)\| \to \infty$ as $j \to \infty$, $|u_j^0| + |v_j^0| \to \infty$ almost everywhere in Ω. Using this fact and (B5) we can follows similarly to the proof of Lemma 4.1 and obtain

$$\int_\Omega \|D^2G(x,u,v)\|^{N/2}\,dx \to 0 \quad \text{as } j \to \infty.$$

Now by Hölder inequality and Sobolev inequality, one has

$$\begin{aligned} |\langle \varphi''(u_j,v_j)v,v\rangle| &= \bigg| \int_\Omega D^2G(x,u,v)v\cdot v\,dx \bigg| \\ &\leq \bigg(\int_\Omega \|D^2G(x,u,v)\|^{N/2} \big)^{2/N} \bigg(\int_\Omega |v|^{2N/(N-2)} \bigg)^{(N-2)/N} \\ &\leq \bigg(\int_\Omega \|D^2G(x,u,v)\|^{N/2} \big)^{2/N} C'\|v\|^2, \quad \text{for all } v \in E. \end{aligned}$$

Hence $\lim_{j\to\infty} \|\varphi''(u_j,v_j)\| = 0$, then we proved Φ satisfies the condition (H5). $\square$

Note that L_∞ is determined by $a(x)$, $b(x)$ and $c(x)$, L_0 is determined by $a_0(x)$, $b_0(x)$ and $c_0(x)$, we can denote

$$i_\infty^- := M_{\mathcal{E}}^-(L_\infty), \qquad i_\infty^0 := M^0(L_\infty), \qquad i_0^- := M_{\mathcal{E}}^-(L_0), \qquad i_0^0 := M^0(L_0).$$

According to Theorem 3.1 we have the following result.

Theorem 5.3. *Suppose that* (5.1) *satisfies the conditions* (B1)–(B5), *and* $i_0^- < i_\infty^- - 1$ *or* $i_0^- > i_\infty^- + i_\infty^0 + 1$, *then* (5.1) *has at least one nontrivial weak solution.*

Remark 5.4. In [12], the authors studied the existence of nontrivial solutions for elliptic system (5.1) under the Landesman–Lazer type condition, which implies that the functional satisfies the (PS)* condition. However, under the condition of Theorem 5.3, the (PS)* condition may fail at any level.

Acknowledgements. The authors are grateful to the referees for their careful reading of the paper and helpful suggestions.

References

[1] A. Abbondandolo, *A new cohomology for the Morse theory of strongly indefinite functionals on Hilbert spaces*, Topol. Methods Nonlinear Anal. **9** (1997), 325–382.

[2] ———, *Morse theory for asymptotcally linear Hamiltonian systems*, Nonlinear Anal. **39** (2000), 997–1049.

[3] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorem and applications to nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012.

[4] K.-C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problem*, Birkhäuser, Boston, 1993.

[5] K.C. Chang, J.Q. Liu and M.J. Liu, *Nontrivial periodic solutions for strong resonance Hamiltonian systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 103–117.

[6] D.G. Costa and C.A. Magalhães, *A variational method to subquadratic perturbations of elliptic systems*, J. Differential Equations **111** (1994), 103–122.

[7] D.G. de Figueiredo and P. Felmer, *On superqadratic elliptic systems*, Trans. Amer. Math. Soc. **111** (1994), 99–116.

[8] Y.X. Guo, *Morse theury for strongly indefinite functional and its applications*, Doctoral thesis, Institute of Mathematics, Peking University, 1999.

[9] ———, *Nontrivial periodic solutions for asymptotically linear Hamiltonian systems with resonance*, J. Differential Equations **175** (2001), 71–87.

[10] ———, *Nontrivial solutions for resonant noncooperative elliptic systems*, Comm. Pure Appl. Math. **53** (2000), 1335–1349.

[11] J. Hulshof and R. van der Vorst, *Differential systems with strongly indefinite variational structure*, J. Func. Anal. **114** (1993), 32–58.

[12] W. Kryszewski and A. Szulkin, *An infinite dimensional Morse theory with applications*, Trans. Amer. Math. Soc. **349** (1997), 3181–3234.

[13] E. Landesman and A.C. Lazer, *Nonlinear perturbations of linear eigenvalues problem at resonance*, J. Math. Mech. **19** (1970), 609–623.

[14] Z.L. Liu, J.B. Su and Z.-Q. Wang, *Solutions of elliptic problems with nonlinearities of linear growth*, Calc. Var. Partial Differential Equations **35** (2009), 463–480.

[15] A. Masiello and L. Pisani, *Asymptotically linear elliptic problems at resonance*, Ann. Math. Pura Appl. **4** (1996), 1–13.

[16] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer–Verlag, New York, 1989.

[17] A. Pomponio, *An asymptotically linear non-cooperative elliptic system with lack of compactness*, Proc. Roy. Soc. Londondon A **459** (2003), 2265–2279.

[18] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, AMS Conf. Ser. Math., vol. 65, 1986.

[19] A. Salvatore, *Periodic solutions of asymptotically linear systems without symmetry*, Rend. Sem. Math. Univ. **74** (1985), 147–161.

[20] M. Solimini, *Morse index estimates in minimax theores*, Manus. Math. **63** (1989), 421–453.

[21] J.B. Su and Z.L. Liu, *Bounded resonance problem for elliptic equations*, Discrete Contin. Dyn. Syst **19** (2007), 431–445.

[22] A. Szulkin, *Cohomology and Morse theory for strong indefinite functionals*, Math. Z. **209** (1992), 375–418.

[23] A. Szulkin and W.M. Zou, *Infinite dimensional cohomology groups and periodic solutions of asymptotically linear Hamiltonian systems*, J. Differentail Equations **174** (2001), 369–391.

[24] P.A. Zeng, J.Q. Liu and Y.X. Guo, *Computations of critical groups and applications to asymptotically linear wave equation and beam equation*, J. Math. Anal. Appl. **300** (2004), 102–128.

[25] W. M. Zou, *Computations of the cohomology groups with applications to asymptotically linear beam equtions and noncooperative elliptic systems*, Comm. Partial Differential Equations **27** (2002), 115–147.

Manuscript received March 12, 2013

Guanggang Liu
Department of Mathematics Science
Liaocheng University
Liaocheng 252000, P.R. CHINA
and
College of Mathematics
Jilin University
Changchun 130012, P.R. CHINA

E-mail address: lgg112@163.com

Shaoyun Shi
College of Mathematics
Jilin University
Changchun 130012, P.R. CHINA
and
Key Laboratory of Symbolic Computation
and Knowledge Engineering
of Ministry of Education
Jilin University
ChangChun University 130012, P.R. China

E-mail address: shisy@mail.jlu.edu.cn

Yucheng Wei
College of Mathematics
Jilin University
Changchun 130012, P.R. CHINA
and
Department of Mathematics
Hechi University
Yizhou 546300, P.R.CHINA

E-mail address: ychengwei@126.com