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ON THE ASYMPTOTIC BEHAVIOR OF STRONGLY DAMPED WAVE EQUATIONS

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Abstract. This paper is devoted to the asymptotic behavior of the semilinear strongly damped wave equation with forcing term only belongs to H^{-1}. Some refined decompositions of the solution have been presented, which allow to remove the quasi-monotone condition $f'(s) > -k$. The asymptotic regularity and existence of a finite-dimensional exponential attractor are established under the usual assumptions.

1. Introduction

We consider the following strongly damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + f(u) = g & \text{in } \Omega \times \mathbb{R}^+, \\ (u(0), \quad u_t(0)) = (u_0, v_0), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $g \in H^{-1}$ is time-independent, $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0) = 0$ and satisfies the following conditions:

$$|f'(s)| \leq C_0(1 + |s|^4) \quad \text{for all } s \in \mathbb{R} \tag{1.2}$$

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and

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.3}$$

λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

The global well-posedness of (1.1)–(1.3) in the natural energy phase space $H_0^1(\Omega) \times L^2(\Omega)$ was well known, for example, see [1], [2], [12]:

Lemma 1.1 ([1], [2], [12]). *Let $\Omega \subset \mathbb{R}^3$ be a smooth domain, $g \in H^{-1}$ be independent of time, $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0) = 0$ and satisfy* (1.2)–(1.3). *Then, for every $T > 0$ and every $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$,* (1.1) *has a unique weak solution*

$$u \in C([0,T], H_0^1(\Omega)), \quad u_t \in C([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)),$$

and the solution operator defines a continuous semigroup $\{S(t)\}_{t\geq 0}$ on $H_0^1(\Omega) \times L^2(\Omega)$. Moreover, $\{S(t)\}_{t\geq 0}$ satisfies the following Lipschitz continuity: for any $z_1, z_2 \in H_0^1(\Omega) \times L^2(\Omega)$ and any $t \geq 0$,

$$\|S(t)z_1 - S(t)z_2\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq e^{c_1 t}\|z_1 - z_2\|_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.4}$$

where the constant c_1 depends only on the size of $\|z_i\|_{H_0^1(\Omega)\times L^2(\Omega)}$.

The asymptotic behavior of solutions to equation (1.1) has been the object of extensive studies via attractors, e.g. see [1], [2], [4], [5], [9], [10], [12], [13], [17], [19] and the references therein, especially, the first result concerning existence of a global attractor in the critical case was proved by Carvalho and Cholewa [2] whereas fractal dimension of the global attractor in the critical case was estimated by Cholewa et al. [5].

Recently, asymptotic regularity for dissipative equations has been paid more attention, especially for the strongly damped wave equation, e.g. see [3]–[5], [9], [12], [13], [15] and so on for the relative results of (1.1).

For the case $g \in L^2(\Omega)$, the authors in [12] have proved that the global attractor is bounded in $H^2(\Omega) \times H^1(\Omega)$, and based on such regularity results, by using of the abstract framework developed in [6], they obtained further the existence of exponential attractor. In [13], the authors have proved that the global attractor is bounded in $H^2(\Omega)\times H^2(\Omega)$ as $g \in L^2(\Omega)$ when the nonlinearity $f(\,\cdot\,)$ satisfies $\liminf_{|s|\to\infty} f'(s) > -\lambda_1$, and the authors in [13] also pointed out further that one can prove the regularity of attractor when $f(\,\cdot\,)$ only satisfies (1.2) and (1.3), which have been realized recently in [16], [19]. In [9], the authors have studied a more general case, that is, a quasi-linear equation and the growth of nonlinear term can large than 5 under some additional structural conditions, also some asymptotic regularity and the existence of exponential attractor have been established.

For the case $g \in H^{-1}$, the corresponding results are not so abundant as that for $g \in L^2(\Omega)$.

We know that the solution of the elliptic equation

$$\begin{cases} -\Delta u + f(u) = g \in H^{-1}, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

in general only belongs to $H_0^1(\Omega)$ when $f(\,\cdot\,)$ satisfies (1.2) and (1.3). So, as $g \in H^{-1}$, we cannot expect any higher regularity for the first component u of (u, v) in attractor than $H_0^1(\Omega)$.

In [15], the author has proved some asymptotic regularity for the solution of (1.1) under the following additional conditions: $f \in \mathcal{C}^2$ with

$$|f''(s)| \le C(1 + |s|^3) \quad \text{for all } s \in \mathbb{R}, \tag{1.6}$$

$$f'(s) \ge -k \qquad \text{for all } s \in \mathbb{R}. \tag{1.7}$$

The quasi-monotone condition (1.7) can simplify the proof largely when we consider the long-time behavior, removing such condition has been seen as a "nontrivial progress" to some extent; For example, we see that the following "good" (smooth and linear growth) nonlinear function satisfies (1.2), (1.3) and (1.6), but not (1.7):

$$f(v) = \frac{\lambda_1}{2}\, v \cdot \sin v, \quad \text{for all } v \in \mathbb{R}.$$

Hence, our first main purpose of this paper is to remove the quasi-monotone condition (1.7) and establish a same asymptotic regularity as that in [15]. That is, in Section 3, by using of some refined properties of the stationary solution of (1.5) and combining with some skillful decomposition of (1.1) (see [18] for another application to nonclassical diffusion equation), we prove the following asymptotic regularity:

Theorem 1.2 (Asymptotic regularity). *Let $f \in \mathcal{C}^2(\mathbb{R})$ and satisfy (1.3) and (1.6), $g \in H^{-1}$ and $\{S(t)\}_{t\ge 0}$ be the semigroup generated by the weak solution of (1.1) in the natural energy space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$. Then, for each positive constant $\sigma < 1/2$, there exist a subset $\mathscr{B}_\sigma$, positive constant ν and a continuous increasing function $Q_\sigma(\,\cdot\,)\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that: for any bounded set $B \subset \mathcal{H}$,*

$$\operatorname{dist}_{\mathcal{H}}(S(t)B, \mathscr{B}_\sigma) \le Q_\sigma(\|B\|_{\mathcal{H}})e^{-\nu t} \quad \textit{for all } t \ge 0, \tag{1.8}$$

where $\mathscr{B}_\sigma$ satisfying

$$\mathscr{B}_\sigma = \{z \in \mathcal{H} : \|z - (\phi(x), 0)\|_{H^{1+\sigma}(\Omega)\times H^\sigma(\Omega)} \le \Lambda_\sigma < \infty\}$$

for some positive constant Λ_σ; and $\phi(x)$ is the unique solution of the following elliptic equation

$$\begin{cases} -\Delta\phi + f(\phi) + (3l + C_0)\phi = g(x) - g^\eta(x) & \text{in } \Omega, \\ \phi|_{\partial\Omega} = 0, \end{cases} \tag{1.9}$$

where the constants l, C_0 come from (3.1) *and* (1.2) *respectively, $g^\eta \in L^2(\Omega)$ such that $\|g - g^\eta\|_{H^{-1}} < \eta < (C_0 \cdot C'^6)^{-1/4}/4$. The constant Λ_σ and $Q_\sigma(\,\cdot\,)$ may depend on σ, but ν is independent of σ; C_0 comes from* (1.2) *and C' is the embedding constant of $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$.*

Our second purpose is to establish the existence of a finite-dimensional exponential attractor for equation (1.1) when $g \in H^{-1}$.

For the case $g \in L^2(\Omega)$, the existence of an exponential attractor has been obtained in several papers recently, e.g. Cholewa, Czaja and Mola [5], Pata and Squassina [12], Yang and Sun [19], Kalantarov and Zelik [9] and so on. In these papers, due to $g \in L^2$, the asymptotic regularity of solutions can arrive $H^2(\Omega) \times H^2(\Omega)$, consequently the nonlinear term $f(u)$ will belong to $L^\infty(\Omega)$ by the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, hence, the criterion for the existence of an exponential attractor devised in [6], [11] can be verified easily.

For the case $g \in H^{-1}$, as mentioned previously, the first component u of global attractor will only bounded in $H_0^1(\Omega)$. Hence the nonlinear term $f(u)$ only belongs to H^{-1}, this brings some essential difficulties in verifying the criterion for the existence of exponential attractor; for example, the method in [12], [19] can not directly apply to this case and further argument is needed. In order to overcome this difficulty, in [15], the author required $f(\,\cdot\,)$ satisfy some additional conditions which hold for the standard polynomial nonlinearities $f(\tau) = \tau|\tau|^4 +$ lower order terms.

In Section 4, based on the asymptotic regularity result Theorem 1.2 and some asymptotical regular decomposition, we prove the existence of exponential attractor under the same conditions as in Theorem 1.2, that is:

Theorem 1.3 (Exponential attractor). *Under the assumptions of Theorem* 1.2, *there exists a set $\mathcal{E}$, which is compact in $H_0^1(\Omega) \times L^2(\Omega)$ and satisfies the following conditions*:

(a) *$\mathcal{E}$ is positively invariant, i.e. $S(t)\mathcal{E} \subset \mathcal{E}$ for all $t \geq 0$;*
(b) *$\dim_F(\mathcal{E}, H_0^1(\Omega) \times L^2(\Omega)) < \infty$, i.e. $\mathcal{E}$ has finite fractal dimension in $H_0^1(\Omega) \times L^2(\Omega)$;*
(c) *there exist an increasing function $\widetilde{Q}\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha > 0$ such that for any subset $B \subset \mathcal{H}$ with $\|B\|_{\mathcal{H}} \leq R$ there holds*

$$\operatorname{dist}_{H_0^1(\Omega)\times L^2(\Omega)}(S(t)B, \mathcal{E}) \leq \widetilde{Q}(R)e^{-\alpha t} \quad \text{for all } t \geq 0;$$

(d) $\mathcal{E} = (\phi(x), 0) + \mathcal{E}_\sigma$ *with* $\mathcal{E}_\sigma$ *bounded in* $H^{1+\sigma}(\Omega) \times H^\sigma(\Omega)$ $(\sigma < 1/2)$, *where* $\phi(x)$ *is the unique solution of* (1.9).

Remark 1.4. Moreover, combining with the estimates about $\|\nabla u_t(t)\|$ and $\|u_{tt}(t)\|$ given in Pata, Zelik [13], as that in Yang, Sun [19], we indeed can prove a stronger attraction for the second component $u_t(t)$ of $(u(t), u_t(t))$. For example, we can improve the attraction in Theorem 1.3(c) to be $\operatorname{dist}_{H_0^1(\Omega)\times H_0^1(\Omega)}(S(t)B, \mathcal{E})$ provided that $t > 0$.

2. Preliminaries and notation

We first recall a Gronwall-type inequality, which will be used in the proof of Theorem 1.2, for the proof please see [8]:

Lemma 2.1. *Let* $\Lambda\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\varepsilon\Lambda(t) \le h(t)\Lambda(t) + k,$$

where $\varepsilon > 0$, $k \ge 0$ *and* $\int_s^t h(\tau)\,d\tau < \varepsilon(t-s) + m$ *for all* $t \ge s \ge 0$ *and some* $m \ge 0$. *Then,*

$$\Lambda(t) \le \Lambda(0)e^m e^{-\varepsilon t} + \frac{ke^m}{\varepsilon}, \quad \text{for all } t \ge 0.$$

Next we recall a criterion for the existence of exponential attractor that established in [6, Theorem 2.8]:

Lemma 2.2 ([6]). *Let* X *and* Y *be two Banach spaces such that* Y *is compactly embedded into* X *and let* B *be a bounded closed subset of* Y. *Operator* $S\colon B \to B$ *satisfies that following condition: there exist positive constants* ε *and* K *such that*

$$\|Sh_1 - Sh_2\|_Y \le (1-\varepsilon)\|h_1 - h_2\|_Y + K\|h_1 - h_2\|_X \quad \text{for all } h_1, h_2 \in B.$$

Then the semigroup $\{S^n\}_{n=1}^\infty$ *defined on* B *has an exponential attractor* $\mathcal{M}$ *in* Y, *that is,* M *satisfying the following properties:*

(a) $\mathcal{M}$ *is compact in* Y *and its fractal dimension in* Y *is finite, i.e.*

$$\dim_F(\mathcal{M}, Y) < \infty;$$

(b) $\mathcal{M}$ *is semi-invariant with respect to* S, *i.e.* $SM \subset M$;

(c) $\mathcal{M}$ *attracts* B *exponentially, i.e. there exist* $C, \kappa > 0$ *such that*

$$\operatorname{dist}_Y(S^n B, \mathcal{M}) \le Ce^{-\kappa n} \quad \text{for all } n \in \mathbb{N}.$$

Here S^n *is the* n *times iteration of* S, *and* $\operatorname{dist}_Y(\,\cdot\,,\,\cdot\,)$ *means the Hausdorff semi-distance in* Y.

In the following we give the notation that we will use throughout this paper:

(1) $A=-\Delta$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, and consider the family of Hilbert spaces $D(A^{s/2})$, $s\in\mathbb{R}$ with the standard inner products and norms, respectively,

$$\langle\,\cdot\,,\,\cdot\,\rangle_{D(A^{s/2})}=\langle A^{s/2}\,\cdot\,,\,A^{s/2}\,\cdot\,\rangle\quad\text{and}\quad \|\cdot\|_{D(A^{s/2})}=\|A^{s/2}\cdot\|.$$

Especially, $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|$ denote the $L^2(\Omega)$ inner product and norm respectively;

(2) $\mathcal{H}^s=D(A^{(1+s)/2})\times D(A^{s/2})$, $s\in[0,1]$; Especially, $\mathcal{H}=\mathcal{H}^0$;

(3) $\xi_u(t)=(u(t),u_t(t))$ for any $t\geq 0$;

(4) $Q(\,\cdot\,)$, $Q_i(\,\cdot\,)\colon[0,\infty)\mapsto[0,\infty)$, $i=1,2,\ldots$ are continuous increasing functions;

(5) C, c_i $(i=1,2,\ldots)$ denote the general positive constant, which may different from line to line.

3. Asymptotic regularity

We first list some properties associated with the assumptions of (1.2)–(1.3), which will be used later in the decomposition of equation (1.1) for removing (1.7).

- From (1.3), we have that: there exist $M_1>0$ and λ with $0<\lambda<\lambda_1$, such that $f(s)s\geq-\lambda s^2$ for all $|s|\geq M_1$;
- From (1.2) and $f(0)=0$, we have $|f(s)s|\leq C_0(1+|s|^4)s^2$ for all $s\in\mathbb{R}$, where C_0 is the constant in (1.2);
- Take

$$l=1+C_0(1+M_1^4)+\lambda, \tag{3.1}$$

then

$$f(s)s+ls^2\geq s^2\quad\text{for all } s\in\mathbb{R}; \tag{3.2}$$

- From (1.2) and (3.1), we have that: for any $s\in\mathbb{R}$,

$$f'(s)+l\geq -C_0|s|^4. \tag{3.3}$$

3.1. Decomposition of the equations. Since the injection $i\colon L^2(\Omega)\hookrightarrow H^{-1}$ is dense, we know that for every $g\in H^{-1}$ and any $\eta\in(0,1)$, there is a $g^\eta\in L^2(\Omega)$ which depends on g and η such that

$$\|g-g^\eta\|_{H^{-1}}<\eta. \tag{3.4}$$

Hereafter *we always assume that $f(\,\cdot\,)$ satisfies* (1.3) *and* (1.6), $\eta\in(0,(C_0\cdot C'^6)^{-1/4})/4$.

Now, we decompose the solution $u(t)$ of (1.1) corresponding to initial data $(u_0,\,v_0)$ as follows:

$$(u(t),u_t(t))=S(t)\xi_u(0)=K_\eta(t)\xi_u(0)+D_\eta(t)\xi_u(0),$$

where $K_\eta(t)\xi_u(0) = (w^\eta(t), w_t^\eta(t))$ and $D_\eta(t)\xi_u(0) = (z^\eta(t), z_t^\eta(t))$ solve the following equations respectively,

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + f(u) - f(z) = g^\eta + (3l + C_0)z & \text{in } \Omega \times \mathbb{R}^+, \\ w(x,t)|_{\partial\Omega} = 0, \quad \xi_w(0) = (0,0), \end{cases} \tag{3.5}$$

and

$$\begin{cases} z_{tt} - \Delta z_t - \Delta z + f(z) + (3l + C_0)z = g - g^\eta & \text{in } \Omega \times \mathbb{R}^+, \\ z(x,t)|_{\partial\Omega} = 0, \quad \xi_z(0) = \xi_u(0), \end{cases} \tag{3.6}$$

where the constant l comes from (3.1).

Then, we decompose further the solution $z^\eta(x,t)$ of (3.6) as $z^\eta(x,t) = v^\eta(x,t) + \phi^\eta(x)$, where $\phi^\eta(x)$ is the unique solution of (1.9), and $v^\eta(x,t)$ solves the following equation

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v + f(z^\eta) - f(\phi^\eta) + (3l + C_0)v = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ v|_{\partial\Omega} = 0, \\ \xi_v(0) = \xi_u(0) - (\phi^\eta(x), 0). \end{cases} \tag{3.7}$$

3.2. A priori estimates. At first, from (3.2), for the solution $\phi(x) = \phi^\eta(x)$ of (1.9) we have that

$$\|\phi\|_{\mathcal{H}} \le \|g - g^\eta\|_{H^{-1}} \le \eta. \tag{3.8}$$

Secondly, for the solution of (3.6) we have the following estimates:

LEMMA 3.1. *There exists an increasing function $Q_1(\,\cdot\,)$ such that for any bounded set $B \subset \mathcal{H}$, the following estimate holds:*

$$\|D_\eta \xi_u(0)\|_{\mathcal{H}} \le e^{-c_1 t} Q_1(\|B\|_{\mathcal{H}}) + c_2 \|g - g^\eta\|_{H^{-1}}, \quad \text{for all } \xi_u(0) \in B, \tag{3.9}$$

where the positive constant c_1, c_2 depend on $\|B\|_{\mathcal{H}}$, but are independent of η.

PROOF. Set $f_0(\,\cdot\,) = f(\,\cdot\,) + (3l + C_0)\,\cdot\,$. Then, from (3.2), we see that $f_0(\,\cdot\,)$ satisfying all conditions required in [19, Lemma 3.2], consequently the proof is same as the proof for [19, Lemma 3.2]. □

Denote $h(\,\cdot\,) = f(\,\cdot\,) + (3l + C_0)\,\cdot\,$, then we have the following a priori estimate:

LEMMA 3.2. *Let $\eta < \min\{1, (C_0 \cdot C'^6)^{-1/4}/(4(1 + c_2))\}$, then for any bounded set $B \subset \mathcal{H}$, there exists $T_1 = T_1(\|B\|_{\mathcal{H}}, \eta) > 0$ such that the corresponding solutions of* (1.9) *and* (3.7) *satisfy the following estimate*:

$$\frac{1}{2}\|\nabla v(t)\|^2 + 2\langle h(v(t) + \phi) - h(\phi), v(t)\rangle - \langle h'(\phi)v(t), v(t)\rangle \ge 0 \quad \text{as } t \ge T_1,$$

where ϕ and v are the solutions of (1.9), (3.7) *correspondingly, and C' is the embedding constant of $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$.*

Proof. From (3.3) we have that

$$h'(s) \geq 2l + C_0 - C_0|s|^4 \quad \text{for all } s \in \mathbb{R},$$

therefore,

$$\begin{aligned}
2\langle h(v(t)+\phi) &- h(\phi), v(t)\rangle \\
&\geq 2(2l+C_0)\|v(t)\|^2 - 2C_0\int_\Omega |rv(t)+\phi|^4|v(t)|^2\,dx \\
&\geq (4l+2C_0)\|v(t)\|^2 - 2C_0 2^4\int_\Omega (|v(t)|^4+|\phi|^4)|v(t)|^2\,dx \\
&\geq (4l+2C_0)\|v(t)\|^2 - 32C_0\cdot C'^6(\|\nabla v(t)\|^4+\|\nabla\phi\|^4)\|\nabla v(t)\|^2.
\end{aligned}$$

On the other hand, from (1.2) we have $|h'(s)| \leq C_0(1+|s|^4)+3l+C_0$, and then

$$\langle h'(\phi)v(t),\, v(t)\rangle \leq C_0\|v(t)\|^2 + C_0C'^6\|\nabla\phi\|^4\|\nabla v(t)\|^2 + (3l+C_0)\|v(t)\|^2.$$

Hence, from (3.8) and (3.9), by taking η small enough (e.g. $\eta \leq (C_0\cdot C'^6)^{-1/4}/(4(1+c_2))$) and T large enough (e.g. $e^{-c_1T}Q_1(\|B\|_{\mathcal{H}}) \leq \eta$), we have that

$$\begin{aligned}
\frac{1}{2}\|\nabla v(t)\|^2 &+ 2\langle h(v(t)+\phi)-h(\phi), v(t)\rangle - \langle h'(\phi)v(t),\, v(t)\rangle \\
&\geq \frac{1}{2}\|\nabla v(t)\|^2 + l\|v(t)\|^2 - 33C_0C'^6(\|\nabla v(t)\|^4+\|\nabla\phi\|^4)\|\nabla v(t)\|^2 \geq 0
\end{aligned}$$

for all $t \geq T$. □

Consequently, as $\eta < (C_0\cdot C'^6)^{-1/4}/(4(1+c_2))$, similar to Pata and Zelik [14] and Sun [15], for the solution of (3.7) we have:

Lemma 3.3. *Let $\eta < (C_0\cdot C'^6)^{-1/4}/(4(1+c_2))$, then there exist positive constant k_1 and increasing function $Q_2(\,\cdot\,)$ such that for any bounded set $B \subset \mathcal{H}$, the following estimate holds*:

$$\|(v(x,t),\, v_t(x,t))\|_{\mathcal{H}} \leq Q_2(\|B\|_{\mathcal{H}})e^{-k_1t}, \quad \textit{for all } t \geq 0,\ \xi_v(0) \in B. \tag{3.10}$$

Proof. At first, applying the usual multiplier method (e.g. see [2], [12]) we can obtain that there is a constant $M = M(\|B\|_{\mathcal{H}})$ such that

$$\|(v(x,t), v_t(x,t))\|_{\mathcal{H}} \leq M \quad \text{for all } t \geq 0,\ \xi_v(0) \in B. \tag{3.11}$$

Now, similar to [14], for $\varepsilon \in (0,1)$ to be determined later, define the functional

$$\Lambda(t) = \|\nabla v(t)\|^2 + \|v_t(t)\|^2 + \varepsilon\|\nabla v(t)\|^2 + 2\langle h(z)-h(\phi),\, v\rangle + 2\varepsilon\langle v_t, v\rangle - \langle h'(\phi)v, v\rangle.$$

Then, from Lemma 3.2 and by taking ε small enough, we have

$$\Lambda(t) \geq \frac{1}{4}\|\xi_v(t)\|^2_{\mathcal{H}} \quad \text{for all } t \geq T_1,\ \xi_v(0) \in B,$$

where $T_1 = T_1(\|B\|_{\mathcal{H},\eta})$ is given in Lemma 3.2. Therefore, same as that in [15, Lemma 4.3], multiplying (3.7) by $v_t(t) + \varepsilon v(t)$ we have that (note that $z_t = v_t$ and $\phi_t \equiv 0$)

$$\frac{d}{dt}\Lambda(t) + \varepsilon\Lambda(t) + \Gamma + \frac{\varepsilon}{2}\|\nabla v(t)\|^2 = 2\langle \big(h'(z) - h'(\phi)\big) z_t, v\rangle,$$

where

$$\Gamma = 2\|\nabla v_t(t)\|^2 + \frac{\varepsilon}{2}\|\nabla v(t)\|^2 - 3\varepsilon\|v_t\|^2 - 2\varepsilon^2\langle v_t, v\rangle - \varepsilon^2\|\nabla v\|^2 + \varepsilon\langle h'(\phi),\, v^2\rangle.$$

It is easy to see that $\Gamma \geq 0$ as ε small enough, and from (1.6) we have

$$2\langle (h'(z) - h'(\phi)) z_t,\, v\rangle \leq \frac{\varepsilon}{2}\|\nabla v\|^2 + \frac{c}{\varepsilon}\|\nabla z_t\|^2\Lambda,$$

where the constant c depends only on $\|B\|_{\mathcal{H}} + \|\nabla\phi\|$. Hence, applying Lemma 2.1 and noticing $\Lambda(T_1) \leq Q_2'(\|D_\eta(T_1)B\|_{\mathcal{H}})$, we have that

$$\|(v(x,t),\, v_t(x,t))\|_{\mathcal{H}} \leq Q_2''(\|D_\eta(T_1)B\|_{\mathcal{H}})e^{-k_1' t}, \quad \text{for all } t \geq T_1,\ \xi_v(0) \in B,$$

which, combining with (3.11), implies the estimate (3.10) immediately. □

Remark 3.4. Note that the constant k_1 in Lemma 3.3 depends on both $\|B\|_{\mathcal{H}}$ and η (through T_1).

3.3. Proof of Theorem 1.2. After obtained Lemma 3.3, applying the methods introduced in [7], [12], [20], the remainder of the proof are completely same as that in [15, Lemmas 4.4–4.8] (Note that in [15], the quasi-monotone condition (1.7) only be used to establish the a priori estimate [15, Lemma 4,3]). More precisely, we can finish our proof by the following steps:

Step 1. At first, about the solution of equation (3.5), by the usual multiplier method, we can deduce the following estimates: For every bounded subset $B \subset \mathcal{H}$ and any $\sigma \in [0, 1/2)$, there exist positive constant ν_σ (which depends only on $\|B\|_{\mathcal{H}}$ and σ) and an increasing function $Q_{\sigma,\|g^\eta\|}(\,\cdot\,)$ such that

$$\|K_\eta(t)\xi_u(0)\|_{\mathcal{H}^\sigma} = \|(w^\eta(t), w_t^\eta(t))\|_{\mathcal{H}^\sigma} \leq Q_{\sigma,\|g^\eta\|}(\|B\|_{\mathcal{H}})e^{\nu_\sigma t}, \tag{3.12}$$

for all $t \geq 0$, $\xi_u(0) \in B$.

Step 2. Secondly, based on Lemma 3.3 and the estimate (3.12), we can decompose the solution $\xi_u(t) = (u(t), u_t(t))$ of (1.1) as following: for any $\varepsilon > 0$,

$$u(t) = v_1(t) + w_1(t), \quad \text{for all } t \geq 0,$$

where $v_1(t)$ and $w_1(t)$ satisfy the following estimates:

$$\int_s^t \|\nabla v_1(\tau)\|^2\, d\tau \leq \varepsilon(t-s) + C_\varepsilon \quad \text{for all } t \geq s \geq 0, \tag{3.13}$$

and

$$\|A^{(1+\sigma)/2} w_1(t)\|^2 \leq K_\varepsilon \quad \text{for all } t \geq 0, \tag{3.14}$$

with the constants C_ε and K_ε depending on ε, $\|\xi_u(0)\|_{\mathcal{H}}$ and $\|g\|_{H^{-1}}$.

Step 3. Finally, follows the idea of Zelik [20], the estimates (3.13)-(3.14) allow us to overcome the difficulty brings by the critical nonlinearity and obtain that

$$\|K_\eta(t)\xi_u(0)\|^2_{\mathcal{H}^\sigma} = \|(w^\eta(t), w^\eta_t(t))\|^2_{\mathcal{H}^\sigma} \leq J_{\|B\|_{\mathcal{H}},\|g^\eta\|,\sigma}$$

for all $t \geq 0$ and $\xi_u(0) \in B$, for some positive constant $J_{\|B\|_{\mathcal{H}},\|g^\eta\|,\sigma}$; and then we can obtain the exponential estimate (1.8) by applying the attraction transitivity lemma devised in [7]. The details are similar to that in [12], [15]. □

4. Exponential attractor

The main purpose of this section is to construct an exponential attractor of $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}$ by the abstract method devised in [6], [11].

We first give a decomposition results about $u(t)$, which will be used to construct the exponential attractor, its proof are similar as the proof of Theorem 1.2 (or see [15, Lemma 4.9] for a outline of its proof), here we omit it.

Lemma 4.1. *Under the assumption of Theorem* 1.2*, for each $\sigma \in [0, 1/2)$ and for any bounded* (*in $\mathcal{H}^\sigma$*) *subset $B_1 \subset \mathcal{H}^\sigma$, if the initial data $\xi_u(0) \in \phi(x) + B_1$, then the solution $u(t)$ of* (1.1) *also satisfies a similar estimate, more precisely, we have*

$$\|S(t)\xi_u(0) - (\phi(x), 0)\|^2_{\mathcal{H}^\sigma} = \|(u(t), u_t(t)) - (\phi(x), 0)\|^2_{\mathcal{H}^\sigma} \leq K_{B_1,\sigma},$$

for all $t \geq 0$, $\xi_u(0) \in \phi(x) + B_1$, where the constant $K_{B_1,\sigma}$ depends only on the $\mathcal{H}^\sigma$-bound of B_1 and σ.

For each fixed $\sigma \in (0, 1/2)$, denote

$$\mathcal{B}_\sigma = \overline{\bigcup_{t\geq 1} S(t)\mathscr{B}_\sigma}^{\mathcal{H}}, \tag{4.1}$$

where $\mathscr{B}_\sigma$ is the set obtained in Theorem 1.2. Then, from Lemma 4.1 we know that $\|\mathcal{B}_\sigma - (\phi(x), 0)\|_{\mathcal{H}^\sigma} < \infty$.

For any two initial data $\xi_{u_i}(0) \in \mathcal{B}_\sigma$ and the corresponding solution

$$(u^i(t), u^i_t(t)) = S(t)\xi_{u_i}(0), \quad i = 1, 2$$

set $(\widetilde{u}(t), \widetilde{u}_t(t)) = S(t)\xi_{u_1}(0) - S(t)\xi_{u_2}(0)$, then $(\widetilde{u}(t), \widetilde{u}_t(t))$ solves the following problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta\widetilde{u}_t - \Delta\widetilde{u} + f(u^1) - f(u^2) = 0, \\ \xi_{\widetilde{u}}(0) = \xi_{u_1}(0) - \xi_{u_2}(0), \\ \widetilde{u}|_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

Then we have the following result about the solution of (4.2), which is a key step in constructing the exponential attractor:

Lemma 4.2. *Let $\sigma \in (2/5, 1/2)$ be fixed and $\mathcal{B}_\sigma$ be defined by* (4.1)*. Then there exist a time $t^* > 0$ and positive constant K such that for any two initial data $\xi_{u_i}(0) \in \mathcal{B}_\sigma$, $i = 1, 2$, the following estimate holds*:

$$\|S(t^*)\xi_{u_1}(0) - S(t^*)\xi_{u_2}(0)\|_{\mathcal{H}^\sigma} \le \frac{1}{2}\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{H}^\sigma} + K\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{H}},$$

where both t^* and K depend only on the bounds $\|\mathcal{B}_\sigma - (\phi(x), 0)\|_{\mathcal{H}^\sigma}$ and σ.

Proof. Multiplying (4.2) by $A^\sigma(\widetilde{u}_t + \alpha\widetilde{u})$ and integrating oven Ω (where $\alpha \in (0, \lambda_1)$ is a small constant which will be determined later), we obtain that

$$\begin{aligned}
(4.3)\quad & \frac{d}{dt}(\|A^{\sigma/2}(\widetilde{u}_t + \alpha\widetilde{u})\|^2 + (1+\alpha)\|A^{(1+\sigma)/2}\widetilde{u}\|^2) \\
&\quad + 2\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + 2\alpha\|A^{(1+\sigma)/2}\widetilde{u}\|^2 \\
&\le 2\alpha\|A^{\sigma/2}\widetilde{u}_t\|^2 + 2\alpha^2\|A^{\sigma/2}\widetilde{u}\|\|A^{\sigma/2}\widetilde{u}_t\| \\
&\quad + 2\int_\Omega (f(u^2) - f(u^1)) \cdot A^\sigma \widetilde{u}_t\, dx + 2\alpha \int_\Omega (f(u^2) - f(u^1)) \cdot A^\sigma \widetilde{u}\, dx.
\end{aligned}$$

Note that for any $\varphi \in D(A^{(1+\sigma)/2})$ we have $\|A^{(1+\sigma)/2}\varphi\|^2 \ge \lambda_1\|A^{\sigma/2}\varphi\|^2$, so we can take α small enough such that, for all $\varphi_1, \varphi_2 \in D(A^{(1+\sigma)/2})$,

$$\begin{aligned}
(4.4)\quad & 2\alpha\|A^{\sigma/2}\varphi_1\|^2 + 2\alpha^2\|A^{\sigma/2}\varphi_1\|\|A^{\sigma/2}\varphi_2\| \\
&\qquad \le \|A^{(1+\sigma)/2}\varphi_1\|^2 + \alpha\|A^{(1+\sigma)/2}\varphi_2\|^2
\end{aligned}$$

and

$$(4.5)\qquad \alpha\|A^{\sigma/2}\varphi_1\| \le \frac{1}{2}\|A^{(1+\sigma)/2}\varphi_1\|.$$

In the following, we fixed α such that (4.4)–(4.5) hold, and begin to estimate the nonlinear term. At first, from (1.2) we have

$$\begin{aligned}
(4.6)\qquad |f(u^2) - f(u^1)| &\le C_0(1 + |u^1 + (1-\theta)u^2|^4)|u^1 - u^2| \\
&= C_0(1 + |u^1 + (1-\theta)u^2|^4)|\widetilde{u}|,
\end{aligned}$$

where $\theta \in (0, 1)$ depends on t, u^1 and u^2.

Now, applying the regular decomposition Lemma 4.1, for any $\varepsilon > 0$, we can decompose $u^i(t)$ ($t \ge 0$ and $i = 1, 2$) as $u^i = u^i_\varepsilon + \phi_\varepsilon$ with

$$(4.7)\qquad \|u^i_\varepsilon(t)\|_{H^{1+\sigma}} \le M_{\sigma,\varepsilon} < \infty, \quad \text{for all } t \ge 0,\ \xi_{u_i} \in \mathcal{B}_\sigma,$$

and

$$(4.8)\qquad \|\phi_\varepsilon\| \le \varepsilon,$$

where the constant $M_{\sigma,\varepsilon}$ depends only on the bounds $\|\mathcal{B}_\sigma - (\phi(x), 0)\|_{\mathcal{H}^\sigma}$ and ε. Therefore, by using of (4.6) and Hölder inequality, we have

$$
\begin{aligned}
(4.9)\qquad 2\int_\Omega &(f(u^2) - f(u^1)) \cdot A^\sigma \widetilde{u}_t\,dx \\
&\le 2C_0 \int_\Omega (1 + |u^1_\varepsilon + (1-\theta)u^2_\varepsilon + \phi_\varepsilon|^4)|\widetilde{u}| \cdot |A^\sigma \widetilde{u}_t|\,dx \\
&\le C(\|\widetilde{u}\|\|A^\sigma \widetilde{u}_t\| + \|\phi_\varepsilon\|^4_{L^6} \cdot \|\widetilde{u}\|_{L^{6/(1-2\sigma)}} \|A^{\sigma/2}\widetilde{u}_t\|_{L^{6/(1+2\sigma)}} \\
&\quad + \|u^i_\varepsilon\|^4_{L^{12/(1-\sigma)}} \|\widetilde{u}\| \cdot \|A^{\sigma/2}\widetilde{u}_t\|_{L^{6/(1+2\sigma)}}).
\end{aligned}
$$

Hence, noticing that $12/(1-\sigma) \le 6/(1-2\sigma)$ for each $\sigma \in (2/5, 1/2)$, then applying Cauchy–Schwarz inequality and Sobolev embedding $H^{1+\sigma}(\Omega) \hookrightarrow L^{6/(1-2\sigma)}$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we can deduce from (4.7)–(4.9) that

$$
\begin{aligned}
(4.10)\qquad 2\int_\Omega &(f(u^2) - f(u^1)) \cdot A^\sigma \widetilde{u}_t\,dx \\
&\le C(C_\varepsilon\|\widetilde{u}\|^2 + \varepsilon\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 \\
&\quad + \varepsilon\|A^{(1+\sigma)/2}\widetilde{u}\|\|A^{(1+\sigma)/2}\widetilde{u}_t\| + M^4_{\varepsilon,\sigma}\|\widetilde{u}\| \cdot \|A^{(1+\sigma)/2}\widetilde{u}_t\|) \\
&\le C_{\varepsilon, M_{\varepsilon,\sigma}}\|\widetilde{u}\|^2 + \varepsilon \cdot C(\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}\|^2).
\end{aligned}
$$

Similarly, we can obtain that

$$
\begin{aligned}
(4.11)\qquad 2\alpha\int_\Omega &(f(u^2) - f(u^1)) \cdot A^\sigma \widetilde{u}\,dx \\
&\le \alpha C_{\varepsilon, M_{\varepsilon,\sigma}}\|\widetilde{u}\|^2 + \varepsilon \cdot \alpha C(\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}\|^2).
\end{aligned}
$$

Consequently, insetting (4.10)–(4.11) into (4.3) and also using (4.4)–(4.5), we obtain

$$
\begin{aligned}
(4.12)\qquad \frac{d}{dt}&(\|A^{\sigma/2}(\widetilde{u}_t + \alpha\widetilde{u})\|^2 + (1+\alpha)\|A^{(1+\sigma)/2}\widetilde{u}\|^2) \\
&\quad + \|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \alpha\|A^{(1+\sigma)/2}\widetilde{u}\|^2 \\
&\le C_{\alpha,\varepsilon,\sigma,M_{\varepsilon,\sigma}}\|\widetilde{u}\|^2 + \varepsilon \cdot C_\alpha(\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}\|^2)
\end{aligned}
$$

for all $t > 0$. Note that α is fixed, and so we can take ε small enough such that

$$
\varepsilon \cdot C_\alpha(\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}\|^2) \le \frac{1}{2}(\|A^{(1+\sigma)/2}\widetilde{u}_t\|^2 + \alpha\|A^{(1+\sigma)/2}\widetilde{u}\|^2).
$$

Hence, we finally deduce that (using the embedding $\|\cdot\| \le C\|A^{1/2}\cdot\|$):

$$
(4.13)\qquad \frac{d}{dt}E_{\widetilde{u}}(t) + C_\alpha E_{\widetilde{u}}(t) \le C_{\alpha,\varepsilon,\sigma,M_{\varepsilon,\sigma}}\|\widetilde{u}\|^2 \quad \text{for all } t > 0,
$$

where $E_{\widetilde{u}}(t)$ is defined as

$$
(4.14)\qquad E_{\widetilde{u}}(t) = \|A^{\sigma/2}(\widetilde{u}_t(t) + \alpha\widetilde{u}(t))\|^2 + (1+\alpha)\|A^{(1+\sigma)/2}\widetilde{u}(t)\|^2,
$$

which satisfying (notice $\alpha \in (0, \lambda_1)$)

$$c_{\alpha,\lambda_1}(\|A^{\sigma/2}\widetilde{u}_t(t)\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}(t)\|^2) \le E_{\widetilde{u}}(t) \le C_{\alpha,\lambda_1}(\|A^{\sigma/2}\widetilde{u}_t(t)\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}(t)\|^2), \tag{4.15}$$

where the positive constants c_{α,λ_1} and C_{α,λ_1} are independent of t and $\widetilde{u}$. Then apply the Gronwall lemma to (4.13), we can obtain that

$$E_{\widetilde{u}}(t) \le e^{-C_\alpha t}E_{\widetilde{u}}(0) + C_{\alpha,\varepsilon,\sigma,M_{\varepsilon,\sigma}}e^{-C_\alpha t}\int_0^t e^{C_\alpha s}\|\widetilde{u}(s)\|^2\,ds, \tag{4.16}$$

which, combining with (4.15) and the Lipschitz continuity (1.4), implies that

$$\begin{aligned}\|A^{\sigma/2}\widetilde{u}_t(t)\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}(t)\|^2 &\le e^{-C_\alpha t}\cdot\frac{C_{\alpha,\lambda_1}}{c_{\alpha,\lambda_1}} \\ &\cdot(\|A^{\sigma/2}\widetilde{u}_t(0)\|^2 + \|A^{(1+\sigma)/2}\widetilde{u}(0)\|^2) + C_{\alpha,\varepsilon,\sigma,M_{\varepsilon,\sigma}}\cdot\frac{e^{2c_1t}-1}{2c_1\cdot c_{\alpha,\lambda_1}}\|\xi_{\widetilde{u}}(0)\|^2_{\mathcal{H}}.\end{aligned}$$

Hence, we can finish our proof by taking t^* as a time which satisfies

$$2\sqrt{e^{-C_\alpha t^*}\cdot\frac{C_{\alpha,\lambda_1}}{c_{\alpha,\lambda_1}}} < \frac{1}{2}$$

and taking K as

$$K^2 = C_{\alpha,\varepsilon,\sigma,M_{\varepsilon,\sigma}}\cdot\frac{e^{2c_1t^*}-1}{2c_1\cdot c_{\alpha,\lambda_1}}. \qquad \square$$

Set $\widehat{\mathcal{B}_\sigma} = \mathcal{B}_\sigma - (\phi(x),0) \subset \mathcal{H}^\sigma$, and define the shift operator T on $\widehat{\mathcal{B}_\sigma}$ as follows:

$$\begin{cases} T\colon \widehat{\mathcal{B}_\sigma} \to \widehat{\mathcal{B}_\sigma}, \\ T(\xi_u(0) - (\phi(x),0)) := S(t^*)(\xi_u(0)) - (\phi(x),0), \quad \text{for all } \xi_u(0) \in \mathcal{B}_\sigma. \end{cases} \tag{4.17}$$

Then, we can see that T satisfies all of conditions in Lemma 2.2 with $X = \mathcal{H}$, $Y = \mathcal{H}^\sigma$ ($\sigma \in (2/5, 1/2)$) and $B = \widehat{\mathcal{B}_\sigma} \subset Y$. Hence, we know that the semigroup $\{T^n\}_{n=1}^\infty$ has an exponential attractor $\widehat{\mathcal{M}} \subset \widehat{\mathcal{B}_\sigma}$ which satisfying

$$\begin{cases} \widehat{\mathcal{M}} \text{ is compact in } \mathcal{H}^\sigma \text{ and } \dim_F(\widehat{\mathcal{M}}, \mathcal{H}^\sigma) < \infty, \\ T\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}, \\ \exists\, c,\kappa > 0, \quad \mathrm{dist}_{\mathcal{H}^\sigma}(T^n(\widehat{\mathcal{B}_\sigma}), \widehat{\mathcal{M}}) \le ce^{-\kappa n} \quad \text{for any } n \in \mathbb{N}. \end{cases} \tag{4.18}$$

Now, set $\mathcal{M} = \widehat{\mathcal{M}} + (\phi(x),0)$, then from (4.17)–(4.18) we can see that $\mathcal{M} \subset \mathcal{B}_\sigma$ satisfying

$$\begin{cases} \mathcal{M} \text{ is compact in } \mathcal{H}^\sigma \text{ and } \dim_F(\mathcal{M}, \mathcal{H}^\sigma) < \infty, \\ S(t^*)\mathcal{M} \subset \mathcal{M}, \\ \exists\, c,\kappa > 0, \quad \mathrm{dist}_{\mathcal{H}^\sigma}(S(nt^*)(\mathcal{B}_\sigma), \mathcal{M}) \le ce^{-\kappa n} \quad \text{for any } n \in \mathbb{N}, \end{cases} \tag{4.19}$$

which certainly implies that

$$\begin{cases} \mathcal{M} \text{ is compact in } \mathcal{H} \text{ and } \dim_F(\mathcal{M},\mathcal{H}) < \infty, \\ S(t^*)\mathcal{M} \subset \mathcal{M}, \\ \exists\, c_0, \kappa > 0, \quad \operatorname{dist}_{\mathcal{H}}\big(S(nt^*)(\mathcal{B}_\sigma), \mathcal{M}\big) \le c_0 e^{-\kappa n} \quad \text{for any } n \in \mathbb{N}. \end{cases} \tag{4.20}$$

In order to passing from the discrete semigroup $\{S(nt^*)\}_{n=1}^\infty$ to the continuous case $\{S(t)\}_{t\ge0}$, we need the following Lipschitz continuity:

Lemma 4.3. *The mapping $(t, \xi_u(0)) \mapsto \xi_u(t)$ is Lipschitz continuous on $[0,t^*] \times \mathcal{B}_\sigma$.*

Proof. For any $\xi_{u_i}(0) \in \mathcal{B}_\sigma$, $t_i \in [0,t^*]$, $i = 1,2$, we have

$$\begin{aligned} &\|S(t_1)\xi_{u_1}(0) - S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}} \\ &\qquad \le \|S(t_1)\xi_{u_1}(0) - S(t_1)\xi_{u_2}(0)\|_{\mathcal{H}} + \|S(t_1)\xi_{u_2}(0) - S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}}. \end{aligned}$$

This first term has been estimated, e.g., see [12, Theorem 2]; for the second term, we have

$$\begin{aligned} \|S(t_1)\xi_{u_2}(0) - S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}} &\le \bigg| \int_{t_1}^{t_2} \bigg\| \frac{d}{dt}(S(t)\xi_{u_2}(0)) \bigg\|_{\mathcal{H}} \bigg| \\ &\le \bigg\| \frac{d}{dt}(S(t)\xi_{u_2}(0)) \bigg\|_{L^\infty(0,t^*;\mathcal{H})} |t_1 - t_2|, \end{aligned}$$

and note that $\big\| \frac{d}{dt}(S(t)\xi_{u_2}(0)) \big\|_{L^\infty(0,t^*;\mathcal{H})}$ has been estimated in [13]. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By using of the attraction transitivity lemma devised in Fabrie et al. [7] and taking $\mathcal{E} = \bigcup_{t\in[0,t^*]} S(t)\mathcal{M}$ (where $\mathcal{M}$ is given in (4.20)), then we can verify from (1.8), (4.20) and the Lipschitz continuity given in Lemma 4.3 directly that $\mathcal{E}$ is an exponential attractor of $\{S(t)\}_{t\ge0}$ in $\mathcal{H}$ and satisfies all conditions in Theorem 1.3. □

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