Topological **M**ethods in **N**onlinear **A**nalysis
Volume 44, No. 1, 2014, 11–21

FUČÍK SPECTRUM IN GENERAL: PRINCIPAL EIGENVALUES AND INADMISSIBLE SETS

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Abstract. In this paper, we study the Fučík spectrum of a linear operator in a general setting. We illustrate the influence of various aspects on the structure of the Fučík spectrum. Mainly, we describe the inadmissible areas where the Fučík spectrum of a given operator cannot be located.

1. Introduction

There are many papers studying the structure of the so called Fučík spectrum for a particular self-adjoint differential operator (let us mention, e.g. [4], [8], [1], [2], [6]) or for finite-dimensional operators represented by real n-by-n matrices (e.g. [13], [9]). In our text, we formulate the Fučík spectrum problem in a general setting. We consider an arbitrary linear operator acting on a general ordered Hilbert space and we study how the choice of the space, its ordering and the operator properties influence the structure of the corresponding Fučík spectrum. We focus mainly on the general description of the inadmissible areas for the Fučík spectrum, i.e. the regions in $\mathbb{R}^2$-plane where the Fučík spectrum of a given operator cannot be located. We show that one of the essential properties is the normality of the considered operator. Our results are illustrated by a series of examples in both finite as well as infinite dimensional spaces.

2010 *Mathematics Subject Classification.* 47A75, 34L05, 35P30, 15A24.

Key words and phrases. Fučík spectrum, asymmetric nonlinearities, non-selfadjoint operators, normal operators, cone orderings.

The authors were supported by the Ministry of Education, Youth and Sports of the Czech Republic, Research Plan # MSM4977751301 and by grant # ME09109 (program KONTAKT).

2. Preliminaries

Let H be a Hilbert space over the field of real numbers $\mathbb{R}$, endowed with a scalar product $\langle\cdot\,,\cdot\rangle$ and a norm $\|\cdot\|$ induced by the scalar product. Let us consider a closed pointed convex cone K, i.e. K being a closed subset of H such that

(a) $\forall\, u, v \in K:\ u + v \in K$,
(b) $\forall\, u \in K\ \forall\, \lambda \in \mathbb{R}^+:\ \lambda u \in K$,
(c) $K \cap (-K) = \{0\}$,

where $\mathbb{R}^+ = [0, +\infty)$. Further, let us consider two order relations $\leq$ and $\ll$ on H induced by K by

$$u \leq v \Leftrightarrow (v - u) \in K,$$
$$u \ll v \Leftrightarrow (v - u) \in \mathrm{Int}(K).$$

Moreover, we take into account only such cones K for which $(H, \langle\cdot\,,\cdot\rangle, K)$ is a Hilbert lattice, i.e. for any $u, v \in H$ the following holds

(a) the supremum $u \vee v$ and the infimum $u \wedge v$ of u and v exist in H,
(b) $\|\,|u|\,\| = \|u\|$, where $|u| = u \vee (-u)$,
(c) $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$.

Thus, for any element $u \in H$, we can define its positive and negative parts by

$$u^+ = u \vee 0 \quad \text{and} \quad u^- = (-u) \vee 0.$$

Let us note that $u = u^+ - u^-$, $|u| = u^+ + u^-$ and u^+ is the orthogonal projection of u onto K. We will need the following auxiliary assertion (see [7]).

Lemma 2.1. *Let $(H, \langle\cdot\,,\cdot\rangle, K)$ be a Hilbert lattice. Then*

(a) $\forall\, u, v \in K:\ \langle u, v\rangle \geq 0$,
(b) $\forall\, u \in K \setminus \{0\},\ v \in \mathrm{Int}(K):\ \langle u, v\rangle > 0$,
(c) $\forall\, u \in H:\ \langle u^+, u^-\rangle = 0$.

Let us consider a linear operator $L\colon \mathrm{Dom}(L) \subset H \to H$ with $\mathrm{Dom}(L)$ dense in H and let us denote its spectrum by $\sigma(L)$. Further, we define the so called Fučík spectrum of L as follows.

Definition 2.2. The Fučík spectrum of an operator $L\colon \mathrm{Dom}(L) \subset H \to H$ is the set $\Sigma(L)$ of all pairs $(\alpha, \beta) \in \mathbb{R}^2$, for which the problem

$$Lu = \alpha u^+ - \beta u^- \tag{2.1}$$

has a nontrivial solution u.

Throughout the text, we will illustrate our general results on two typical examples.

Example 2.3. Let $H = \mathbb{R}^n$, $n \in \mathbb{N}$, with a cone $K = (\mathbb{R}^+)^n$ (i.e. K is the positive orthant). That is, $u \in H$ is an n-dimensional vector $u = [u_1, \dots, u_n]^t$, $|u| = [|u_1|, \dots, |u_n|]^t$ and $u^\pm = [u_1^\pm, \dots, u_n^\pm]^t$ with

$$u_i^+ = \max\{u_i, 0\}, \quad u_i^- = \max\{-u_i, 0\}, \quad i = 1, \dots, n.$$

The linear operator $L\colon H \to H$ is represented by a real n-by-n matrix.

Example 2.4. Let $H = L^2(\Omega)$ with a cone $K = \{u \in H;\ u = u(x) \geq 0$ a.e. $x \in \Omega\}$. That is, $u \in H$ is a square-integrable function on Ω and

$$u^+(x) = \max\{u(x), 0\}, \quad u^-(x) = \max\{-u(x), 0\}, \quad \text{a.e. } x \in \Omega.$$

In this setting, we usually consider L to be a real linear differential operator.

3. Symmetry, trivial branches and principal eigenvalues

Lemma 3.1. *The Fučík spectrum $\Sigma(L)$ is symmetric with respect to the diagonal $\alpha = \beta$:*

$$(\alpha, \beta) \in \Sigma(L) \Leftrightarrow (\beta, \alpha) \in \Sigma(L).$$

Proof. The symmetry of the set $\Sigma(L)$ follows immediately from the fact that $(-u)^+ = u^-$, namely if $(\alpha, \beta) \in \Sigma(L)$ with u being a nontrivial solution of $Lu = \alpha u^+ - \beta u^-$ then $-u$ is the nontrivial solution of $L(-u) = \beta(-u)^+ - \alpha(-u)^-$.□

Lemma 3.2. *Let $\lambda \in \mathbb{R}$ be an eigenvalue of L such that there exists a nonnegative* ([1]) *eigenvector u corresponding to λ, i.e. $Lu = \lambda u$, $0 \leq u$. Then the Fučík spectrum $\Sigma(L)$ contains the "cross"*

$$\{(\alpha, \beta) \in \mathbb{R}^2 : (\alpha - \lambda)(\beta - \lambda) = 0\},$$

which is usually called the trivial Fučík branch.

Proof. Trivially, since the eigenvector u of L corresponding to λ satisfies $u = u^+$, then (2.1) has a nontrivial solution for $\alpha = \lambda$ and β arbitrary. If we replace u by $-u$, we obtain $\beta = \lambda$ and α arbitrary. □

Example 3.3. Let $H = \mathbb{R}^3$ be ordered by the cones K_{+++} and K_{++-}, respectively, where

$$\begin{aligned} K_{+++} &= \{u \in \mathbb{R}^3 : u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0\}, \\ K_{++-} &= \{u \in \mathbb{R}^3 : u_1 \geq 0,\ u_2 \geq 0,\ u_3 \leq 0\}. \end{aligned}$$

Let us consider two matrices

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 0 & -4 \\ 2 & 0 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 2 & -2 \\ 1 & -1 & 3 \end{bmatrix},$$

([1]) u nonnegative means $u \in K$.

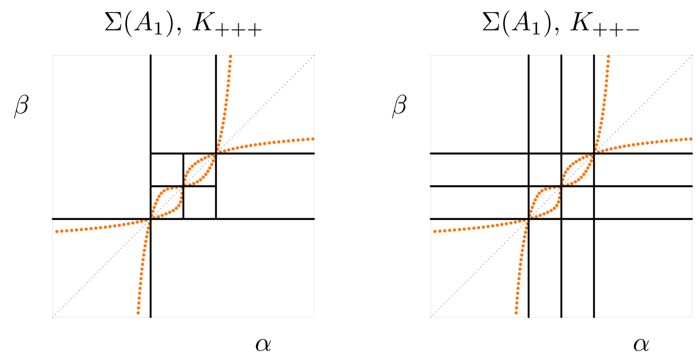

Figure 1. The Fučík spectrum of 3-by-3 matrix A_1 with respect to cones K_{+++} and K_{++-}.

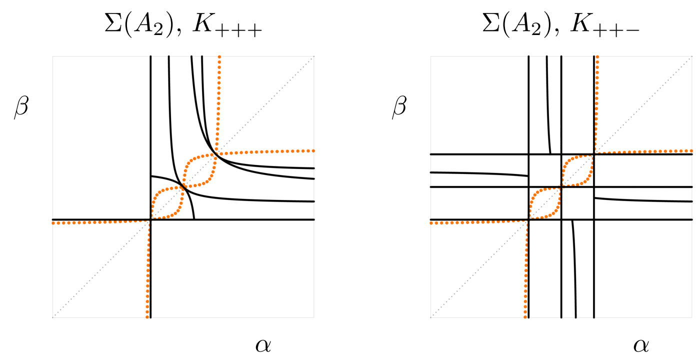

Figure 2. The Fučík spectrum of 3-by-3 matrix A_2 with respect to cones K_{+++} and K_{++-}.

both of which have three distinct real eigenvalues and their corresponding eigenvectors have first two components positive and the last component negative, i.e. all of them belong to K_{++-}. Figures 1 and 2 depict $\Sigma(A_1)$ and $\Sigma(A_2)$ with respect to cones K_{+++} and K_{++-}. Notice that in the case of K_{++-}, both sets $\Sigma(A_1)$ and $\Sigma(A_2)$ contain three trivial Fučík branches (cf. Lemma 3.2), moreover, the Fučík spectrum $\Sigma(A_2)$ contains also nontrivial branches which do not intersect the diagonal $\alpha = \beta$.

Definition 3.4. A real eigenvalue λ of the operator L is called the *principal eigenvalue*, if there exists a *strongly positive* ([2]) eigenvector u corresponding to λ, i.e. $Lu = \lambda u$, $0 \ll u$.

Lemma 3.5. *Let L^* be the adjoint operator to L and let $\lambda \in \mathbb{R}$ be the principal eigenvalue of L^*. Then the necessary condition for nontrivial solvability of the*

([2]) u strongly positive means $u \in \operatorname{Int}(K)$.

problem (2.1) *is* $(\alpha - \lambda)(\beta - \lambda) \geq 0$, *i.e.*

$$\{(\alpha, \beta) \in \mathbb{R}^2 : (\alpha - \lambda)(\beta - \lambda) < 0\} \cap \Sigma(L) = \emptyset.$$

Proof. Let u be a nontrivial solution of $Lu = \alpha u^+ - \beta u^-$ and let v be a strongly positive eigenvector of L^* corresponding to the principal eigenvalue λ, i.e. $0 \ll v$. Using the scalar product, we obtain

$$\langle Lu, v\rangle = \langle \alpha u^+ - \beta u^-, v\rangle. \tag{3.1}$$

The left-hand side of (3.1) can be written as

$$\langle Lu, v\rangle = \langle u, L^* v\rangle = \langle u, \lambda v\rangle = \langle u^+ - u^-, \lambda v\rangle.$$

Hence, (3.1) becomes

$$0 = (\alpha - \lambda)\langle u^+, v\rangle - (\beta - \lambda)\langle u^-, v\rangle. \tag{3.2}$$

Since $0 \ll v$ and $0 \leq u^\pm$, then $\langle u^\pm, v\rangle \geq 0$ and $\langle u^+, v\rangle + \langle u^-, v\rangle = \langle |u|, v\rangle > 0$ (see Lemma 2.1). It means that terms $\langle u^+, v\rangle$ and $\langle u^-, v\rangle$ are not both zero. Thus (3.2) implies $(\alpha - \lambda)(\beta - \lambda) \geq 0$. $\square$

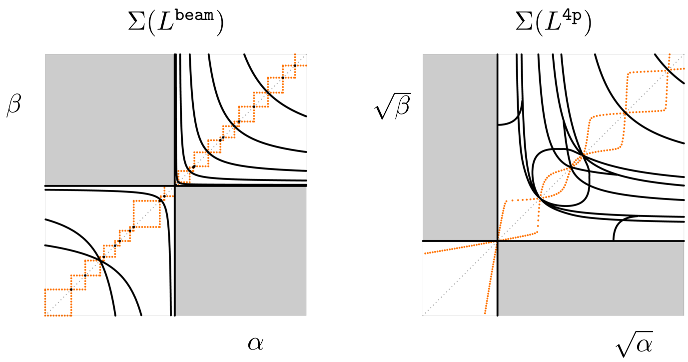

Figure 3. Inadmissible shifted quadrants (grey areas) and known parts of the Fučík spectra of the beam and the four point operators.

Example 3.6. Let us consider the setting from Example 2.4. Let $L^{\mathtt{beam}}$ be the beam operator defined in [11, p. 304] and let $L^{\mathtt{4p}}$ be the four-point operator defined in [5, p. 41]. $L^{\mathtt{beam}}$ is the self-adjoint partial differential operator of the fourth order with the principal eigenvalue $\lambda = -1$ and $L^{\mathtt{4p}}$ is the non-selfadjoint ordinary differential operator of the second order with $\lambda = 0$ being the principal eigenvalue of the corresponding adjoint operator. Figure 3 depicts the inadmissible shifted quadrants and known parts of the Fučík spectra $\Sigma(L^{\mathtt{beam}})$ and $\Sigma(L^{\mathtt{4p}})$. Let us note that the complete description of $\Sigma(L^{\mathtt{beam}})$ is not explicitly known, but its structure plays an important role in many applications, e.g. in models of suspension bridges (see [10] and [3]).

4. Inadmissible sets

Theorem 4.1. *Let ε be a fixed real number which belongs to the resolvent set of L, i.e. $\varepsilon \notin \sigma(L)$. Further, let $d := \|(L-\varepsilon I)^{-1}\|^{-1}$, where I is the identity on H. Then for $(\alpha,\beta) \in (\varepsilon - d, \varepsilon + d) \times (\varepsilon - d, \varepsilon + d)$, the problem* (2.1) *has only a trivial solution, i.e.*

$$(\varepsilon - d, \varepsilon + d) \times (\varepsilon - d, \varepsilon + d) \cap \Sigma(L) = \emptyset.$$

Proof. Let us start with the problem (2.1) and write it in the equivalent form

$$(L-\varepsilon I)u = (\alpha - \varepsilon)u^+ - (\beta - \varepsilon)u^-.$$

Since $\varepsilon \notin \sigma(L)$, we can apply the inverse operator $(L-\varepsilon I)^{-1}$ to obtain

$$u = (L-\varepsilon I)^{-1}((\alpha-\varepsilon)u^+ - (\beta-\varepsilon)u^-). \tag{4.1}$$

If we denote the right-hand side of (4.1) by Tu, then the operator $T\colon H \to H$ satisfies

$$\begin{aligned}
\|Tu - Tv\|^2 &= \|(L-\varepsilon I)^{-1}((\alpha-\varepsilon)(u^+ - v^+) - (\beta-\varepsilon)(u^- - v^-))\|^2 \\
&\le \|(L-\varepsilon I)^{-1}\|^2 \|(\alpha-\varepsilon)(u^+ - v^+) - (\beta-\varepsilon)(u^- - v^-)\|^2 \\
&\le \frac{1}{d^2}\,((\alpha-\varepsilon)^2\|u^+ - v^+\|^2 + (\beta-\varepsilon)^2\|u^- - v^-\|^2 \\
&\quad + 2|\alpha-\varepsilon||\beta-\varepsilon|(\langle u^+, v^-\rangle + \langle u^-, v^+\rangle)) \\
&\le \frac{1}{d^2}\,\max\{(\alpha-\varepsilon)^2, (\beta-\varepsilon)^2\}(\|u^+ - v^+\|^2 + \|u^- - v^-\|^2 \\
&\quad + 2(\langle u^+, v^-\rangle + \langle u^-, v^+\rangle)) \\
&= \frac{1}{d^2}\,\max\{(\alpha-\varepsilon)^2, (\beta-\varepsilon)^2\}\|u - v\|^2.
\end{aligned}$$

Here we use the facts that $\langle u^+, u^-\rangle = \langle v^+, v^-\rangle = 0$ and that $\langle u^+, v^-\rangle \ge 0$, $\langle u^-, v^+\rangle \ge 0$ (see Lemma 2.1). Hence, for $(\alpha,\beta) \in \mathbb{R}^2$ satisfying

$$\max\{|\alpha-\varepsilon|, |\beta-\varepsilon|\} < d,$$

the operator T is contractive, and using the Banach Contraction Principle, we obtain that the problem (4.1) – and thus also problem (2.1) – has a unique (trivial) solution. $\square$

Corollary 4.2. *The set*

$$\Pi(L) := \left\{ (\varepsilon - t d(\varepsilon), \varepsilon + t d(\varepsilon)) \in \mathbb{R}^2 : \right.$$
$$\left. d(\varepsilon) = \frac{1}{\|(L-\varepsilon I)^{-1}\|},\ t \in (-1,1),\ \varepsilon \in \mathbb{R} \setminus \sigma(L) \right\}$$

is an inadmissible set for the Fučík spectrum $\Sigma(L)$, i.e. $\Pi(L) \cap \Sigma(L) = \emptyset$. Let us note that $\Pi(L)$ is the open set.

Example 4.3. Let us consider the setting from Example 2.4 and the η-parameter differential operator with nonlocal boundary condition, $\eta \in [0,1)$,

$$\begin{aligned} L^\eta u(x) &:= -u''(x), \\ \operatorname{Dom}(L^\eta) &:= \left\{ u \in H^2(0,\pi) : u(0) = 0,\ (1-\eta)u(\pi) + \eta \int_0^\pi u(x)\,dx = 0 \right\}. \end{aligned}$$

For $\eta = 0$, the operator L^η is the self-adjoint differential operator with Dirichlet boundary conditions. For $\eta \in (0,1)$, the operator L^η is non-selfadjoint with spectrum $\sigma(L^\eta)$ made only of real distinct eigenvalues. Moreover, for $\eta = 1$, the domain $\operatorname{Dom}(L^\eta)$ is not even dense in $H = L^2(0,\pi)$. The detailed analytic description of the Fučík spectrum $\Sigma(L^\eta)$ is provided in [12]. Figure 4 illustrates the structure of $\Sigma(L^\eta)$ and the inadmissible set $\Pi(L^\eta)$ for various settings of η.

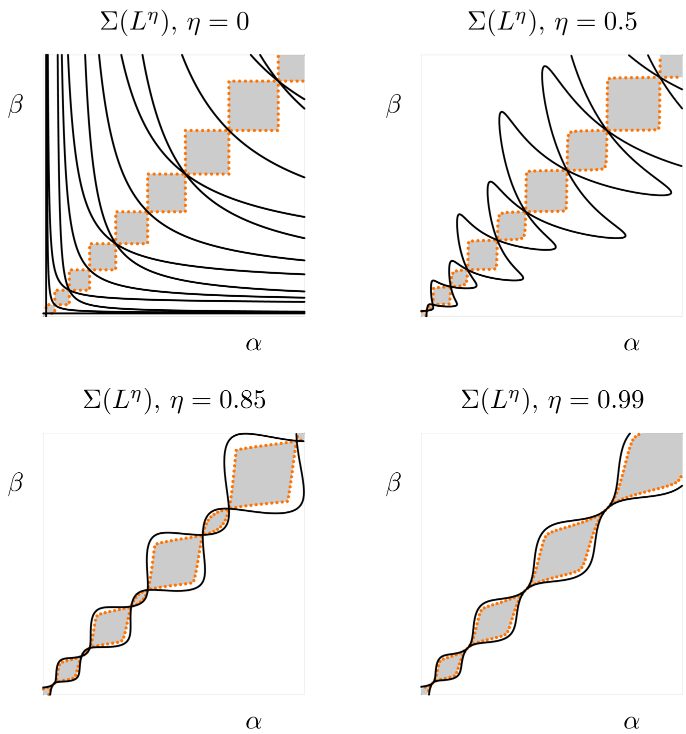

Figure 4. Inadmissible set $\Pi(L^\eta)$ (grey areas) of the operator L^η, $\eta = 0, 0.5, 0.85, 0.99$.

Example 4.4. Let us consider the setting from Example 2.3 and the following matrices

$$A_3 = \begin{bmatrix} 0 & -24 & -80 \\ 0 & 10 & 12 \\ 0 & 1 & 14 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & -6 & -5 \\ 0 & 10 & 3 \\ 0 & 4 & 14 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 8 & -2 & -4 \\ 4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} 8 & 2 & 4 \\ -4 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}.$$

The Fučík spectra $\Sigma(A_i)$ as well as the inadmissible sets $\Pi(A_i)$ for $i = 3, 4, 5, 6$ are shown in Figures 5 and 6.

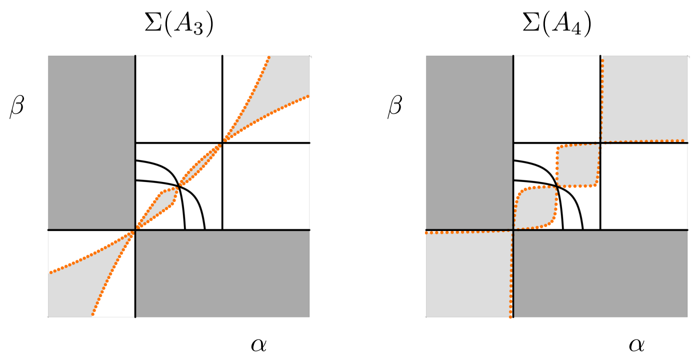

Figure 5. Inadmissible sets $\Pi(A_3)$ and $\Pi(A_4)$ (light grey areas) and inadmissible quadrants (dark grey areas) of 3-by-3 matrices A_3 and A_4.

Notice that $\Sigma(A_3) = \Sigma(A_4) \wedge \Pi(A_3) \neq \Pi(A_4)$, and, on the other hand, we have $\Sigma(A_5) \neq \Sigma(A_6) \wedge \Pi(A_5) = \Pi(A_6)$.

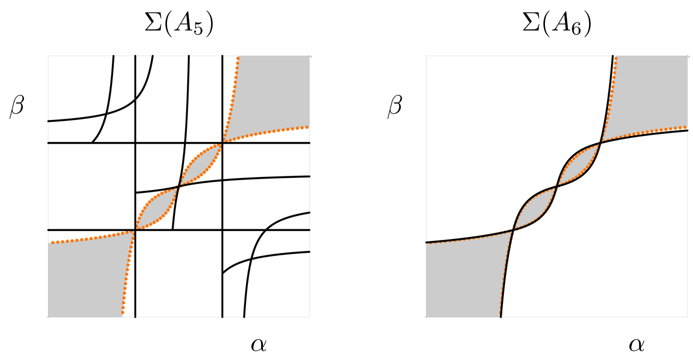

Figure 6. Inadmissible sets $\Pi(A_5)$ and $\Pi(A_6)$ (grey areas) of 3-by-3 matrices A_5 and A_6.

Moreover, zero eigenvalues of both matrices A_3 and A_4 are the principal eigenvalues of their adjoints ([3]). Hence, the corresponding Fučík spectra cannot be located in the second and fourth quadrants colored in dark grey in Figure 5.

([3]) We have $A^* = A^t$.

Remark 4.5. Let L be a normal operator (i.e. $L^*L = LL^*$) with spectrum $\sigma(L)$. Let ε be a fixed real number such that $\varepsilon \notin \sigma(L)$. Then the operators $(L - \varepsilon I)$ and $(L - \varepsilon I)^{-1}$ are normal as well and

$$\|(L - \varepsilon I)^{-1}\| = \frac{1}{\operatorname{dist}(\varepsilon, \sigma(L))}.$$

Thus, for a normal operator L, the inadmissible set $\Pi(L)$ has the following form

$$\Pi(L) = \{(\varepsilon - td(\varepsilon), \varepsilon + td(\varepsilon)) \in \mathbb{R}^2 : \\ d(\varepsilon) = \operatorname{dist}(\varepsilon, \sigma(L)),\ t \in (-1, 1),\ \varepsilon \in \mathbb{R} \setminus \sigma(L)\}.$$

Corollary 4.6. *Let L be a normal operator and let $\lambda, \widetilde{\lambda} \in \sigma(L) \cap \mathbb{R}$ be such that $\lambda < \widetilde{\lambda}$. Moreover, let $\sigma(L) \cap B(\varepsilon_0, r) = \emptyset$, where $B(\varepsilon_0, r) := \{z \in \mathbb{C} : |z - \varepsilon_0| < r\}$ with $\varepsilon_0 = (\lambda + \widetilde{\lambda})/2$ and $r = (\widetilde{\lambda} - \lambda)/2 > 0$. Then for $(\alpha, \beta) \in (\lambda, \widetilde{\lambda}) \times (\lambda, \widetilde{\lambda})$, the problem* (2.1) *has only a trivial solution, i.e.*

$$(\lambda, \widetilde{\lambda}) \times (\lambda, \widetilde{\lambda}) \cap \Sigma(L) = \emptyset.$$

Proof. Theorem 4.1 together with Remark 4.5 imply that the problem (2.1) has only a trivial solution for any $(\alpha, \beta) \in (\varepsilon - d, \varepsilon + d) \times (\varepsilon - d, \varepsilon + d)$, with $d = \operatorname{dist}(\varepsilon, \sigma(L))$ and $\varepsilon \notin \sigma(L)$ being arbitrary real number. If we take

$$\varepsilon = \varepsilon_0 = \frac{\lambda + \widetilde{\lambda}}{2},$$

then according to the assumption $\sigma(L) \cap B(\varepsilon_0, r) = \emptyset$, we have $d = \operatorname{dist}(\varepsilon, \sigma(L)) = r = (\widetilde{\lambda} - \lambda)/2$, and hence, $(\varepsilon - d, \varepsilon + d) = (\lambda, \widetilde{\lambda})$. $\square$

Corollary 4.7. *Let L be a normal operator with $\sigma(L) \subset \mathbb{R}$. Then the problem* (2.1) *has only a trivial solution for any*

$$(\alpha, \beta) \in \Pi(L) = S_{-\infty} \cup \Bigg(\bigcup_{\lambda \in \sigma(L)} S_\lambda \Bigg) \cup S_{+\infty},$$

where S_λ are squares given by

$$S_\lambda := \begin{cases} (\lambda, \widetilde{\lambda}) \times (\lambda, \widetilde{\lambda}) & \text{if there exists } \widetilde{\lambda} > \lambda : [\lambda, \widetilde{\lambda}] \cap \sigma(L) = \{\lambda, \widetilde{\lambda}\}, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $S_{-\infty} := (-\infty, m) \times (-\infty, m)$, $S_{+\infty} := (M, +\infty) \times (M, +\infty)$ with $m = \inf \sigma(L)$, $M = \sup \sigma(L)$.

Proof. Since the spectrum $\sigma(L)$ contains only real numbers and no complex ones, we can apply Corollary 4.6 for any two $\lambda, \widetilde{\lambda} \in \sigma(L)$ satisfying $\lambda < \widetilde{\lambda}$, $(\lambda, \widetilde{\lambda}) \cap \sigma(L) = \emptyset$. In the case of $\lambda = \sup \sigma(L)$ or $\lambda = \inf \sigma(L)$, we use the arguments from the proof of Corollary 4.6 with ε tending to $+\infty$ or $-\infty$, respectively. $\square$

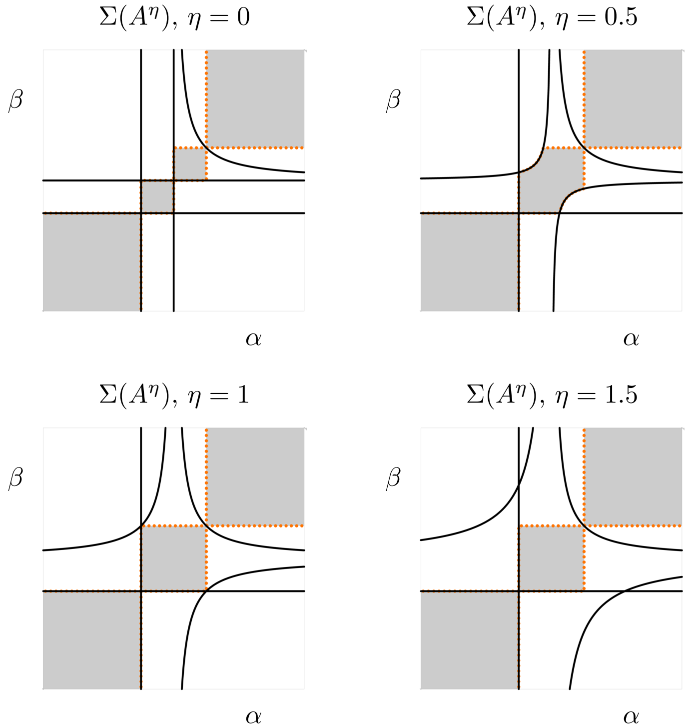

FIGURE 7. Inadmissible set $\Pi(A^\eta)$ (grey areas) of the 4-by-4 matrix A^η, $\eta = 0, 0.5, 1, 1.5$.

EXAMPLE 4.8. Let us consider the setting from Example 2.3 and the normal η-parameter 4-by-4 matrix, $\eta \in \mathbb{R}$,

$$A^\eta = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \eta & 0 \\ 0 & -\eta & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The spectrum $\sigma(A^\eta)$ consists of four eigenvalues $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm \eta \mathrm{i}$. The Fučík spectrum $\Sigma(A^\eta)$ and the inadmissible set $\Pi(A^\eta)$ for various settings of the parameter η are depicted in Figure 7. Notice that (cf. Corollary 4.6)

$$\sigma(A^\eta) \cap B(0,1) = \emptyset \;\Leftrightarrow\; |\eta| \geq 1.$$

REMARK 4.9. Let us note that dashed orange curves in all presented Figures 1–7 correspond to the boundary of the inadmissible set $\Pi(L)$ of the considered operator L. Let us sum up all particular operators used in the paper:

(a) self-adjoint operators $L^{\mathtt{beam}}$ and L^η, A^η with $\eta = 0$,
(b) normal but non-selfadjoint operator A^η with $\eta \in \mathbb{R} \setminus \{0\}$,
(c) non-normal operators $L^{\mathtt{4p}}$, L^η with $\eta \in (0,1)$ and A_i, $i = 1, \ldots, 6$.

Finally, let us point out the optimality of the set $\Pi(L)$ illustrated in Figures 6 and 7: some parts of the Fučík spectrum $\Sigma(L)$ trace the boundary of the inadmissible set $\Pi(L)$.

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Manuscript received February 1, 2012

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