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REVIEW

RICHARD E. HODEL

This is an outstanding book in all respects: well written and well organized, carefully researched with numerous photos and a genuine pleasure to read. It gives an accurate and detailed description of all of Zermelo's contributions to mathematics, not just to set theory. At the same time the author captures his personal life with a fascinating discussion of Zermelo's personality traits and his serious health problems.

My review is organized as follows: Cantorian set theory; Hilbert's Program; Zermelo's work in set theory, 1904-1908; Zermelo's personal life and research in all areas of mathematics; Zermelo's personality. The discussion of all of these topics relies to a considerable extent on the book under review.

Georg Cantor(1845-1918) and Ernst Zermelo (1871-1953) will forever rank among the most important figures in set theory, the branch of mathematics that studies infinity and at the same time serves as the foundations of mathematics. Set theory is the creation of Cantor, and to Cantor we are indebted for such fundamental results and ideas as these: equipotent sets; $\mathbb{N}$ is equipotent to $\mathbb{Q}$ and also to the set of algebraic numbers; $\mathbb{R}$ is uncountable; cardinal and ordinal numbers and their arithmetic; well- ordered sets; Cantor diagonal arguments and $|A| < |P(A)|$; the continuum hypothesis (CH); the aleph function $\aleph$. Cantor, however, did not axiomatize his theory of sets; we shall refer to intuitive, non-axiomatic set theory as *Cantorian set theory.* Cantor's last research papers were published in 1895 and 1897, and in these two papers he left a number of assertions unresolved:

- CH ($2^\omega = \aleph_1$);
- given any two sets, one is equipotent to a subset of the other;
- every set can be well-ordered.

Hilbert's Program had considerable influence on the direction of Zermelo's research. Indeed, in a 1930 paper Zermelo wrote: "When I was a Privatdozent in Göttingen, I began, under the influence of D. Hilbert, to whom I owe more than to anybody else with regard to my scientific development, to occupy myself with questions concerning the foundations of mathematics, especially with the basic problems of Cantorian set theory." (See p. 28.)

During the period 1900-1930, Hilbert was very much interested in the foundations of mathematics. Among other goals, he outlined the following areas of research in logic and the foundations of mathematics. (1) Give axiom systems for various branches of mathematics; for example, the natural number system, the real number system, Euclidean geometry, and Cantorian set theory. (2) Give consistency proofs for these axiom systems.

Hilbert's interest in foundational questions began as early as 1899, the year in which he published *Grundlagen der Geometrie*, a modern version of Euclidean geometry. In this work Hilbert stressed the distinction between undefined terms and defined terms, between axioms and theorems. For Hilbert, the undefined terms were *point, line*, and *plane*, but he emphasized that one could uniformly replace these with *table, chair*, and *beer mug.* Hilbert also gave a consistency proof for his axiom system by constructing a model. However, this model was based on the real number system $\mathbb{R}$, and therefore he classified his result as a relative consistency proof.

In 1900, Hilbert gave his famous list of 23 problems at the International Congress of Mathematicians. The first two problems were in foundations.

Problem 1 Prove or disprove $2^\omega = \aleph_1$. In his discussion of this problem, Hilbert also asked for a proof that there is a well-ordering on the set $\mathbb{R}$ of real numbers.

Problem 2 Give an absolute consistency proof of the axioms for the real number system.

The paradoxes of set theory were another source of concern for Hilbert. Early on, Cantor was aware of certain paradoxes that arise in Cantorian set theory. For example, suppose that $\{x : x \text{ is a set}\}$ is itself a set, call it E. By Cantor's Theorem, $|E| < |P(E)|$. On the other hand, $P(E) \subseteq E$ and therefore $|P(E)| \leq |E|$, a contradiction. There were many other troublesome paradoxes that needed to be explained.

Here is a partial list; later we will see how Zermelo used his axioms for set theory to resolve these puzzles.

Richard's paradox This paradox arises by applying a Cantor diagonal argument to the countable set consisting of all real numbers that can be defined with a finite number of words in some fixed language. Berry gave a somewhat simpler version: the number described by the expression "the smallest natural number not described in less than fifty words."

Burali-Forti paradox Suppose the ordinals form a set W; now W itself is an ordinal, as is its one-point extension $W + 1$; the ordinal $W + 1$ cannot be in W.

Russell-Zermelo paradox This famous paradox, which concerns the class $\{x\colon x \text{ is a set and } x \notin x\}$, is generally credited to Bertrand Russell, but it was discovered independently by Zermelo, and Hilbert referred to it as the *Zermelo paradox*.

Hilbert's paradox Consider the following two operations on sets, which are standard constructions in mathematics. (1) [addition] if A is a set, then $\bigcup A$ is a set; (2) [self-mapping] if A is a set, then $A^A = \{f : f \text{ is a function from } A \text{ to } A\}$ is a set. Now let $\mathcal{A}$ be the collection of sets defined inductively as follows: (1) $\mathbb{N} \in \mathcal{A}$; (2) if $\mathcal{B} \subseteq \mathcal{A}$, then $\bigcup \mathcal{B} \in \mathcal{A}$; (3) if $A \in \mathcal{A}$, then $A^A \in \mathcal{A}$. Now consider the set $U = \bigcup \mathcal{A}$. We have: $U \in \mathcal{A}$ and also $U^U \in \mathcal{A}$, therefore $U^U \subseteq \bigcup \mathcal{A} = U$; we now have $|U^U| \leq |U|$, a contradiction.

In 1904, the International Congress of Mathematicians was held at Heidelberg, and at that meeting Julius König announced a result that played an important turning point in the development of set theory. Namely, König claimed that $2^\omega \neq \aleph_\alpha$ for any α and that $\mathbb{R}$ is not well-orderable. His proof relied on two results, the first due to König, the second to Felix Bernstein: (1) if $\mathrm{cf}(\kappa) = \omega$, then $\kappa^\omega > \kappa$; (2) ${\aleph_\alpha}^\omega = \aleph_\alpha \times 2^\omega$. Suppose by way of contradiction that $2^\omega = \aleph_\alpha$. Apply (2) to $\aleph_{\alpha+\omega}$ to obtain ${\aleph_{\alpha+\omega}}^\omega = \aleph_{\alpha+\omega} \times 2^\omega = \aleph_{\alpha+\omega} \times \aleph_\alpha = \aleph_{\alpha+\omega}$. However, $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$, and therefore we have a contradiction of (1).

Cantor attended König's talk and was immediately skeptical of his proof. The proof is incorrect, and the error occurs in the use of (2): Bernstein's equality does not hold for cardinals of cofinality ω, precisely the case under discussion. The above incident is discussed in detail in the book under review, and the author gives a very interesting discussion of the roles played by Zermelo and Felix Hausdorff in uncovering the error.

The Heidelberg incident, together with the unsolved problems left by Cantor, were obvious motivations for Zermelo's subsequent research program in set theory. Within a month of the meeting, Zermelo wrote a letter to Hilbert in which he used the power set version of the axiom of choice to prove that every set can be well-ordered. His proof, which is so beautiful and natural, can be quickly summarized as follows. Let A be a set and let h be a function from $P(A)$ into A such that $h(B) \in B$ for every non-empty subset B of A. Call a subset E of A a γ-set if it satisfies these two properties: there is a well-order $<$ on E; for all $a \in E$, $h(A - \{x : x \in A \text{ and } x < a\}) = a$. Let $\mathcal{E}$ be the collection of all γ-sets; then $\bigcup \mathcal{E}$ is itself a γ-set and $\bigcup \mathcal{E} = A$; thus A is well-ordered as required. Zermelo also noted that his result shows that every cardinal is an aleph, that $\mathbb{R}$ can be well-ordered, and that $\kappa \times \kappa = \kappa$ for every infinite cardinal κ.

Zermelo's theorem, which used a new "principle of logic" that he called the *axiom of choice* (hereafter AC), was not immediately accepted. A significant number of well- respected mathematicians were critical of either the use of the axiom itself or of the proof. Among his critics were René Baire, Émile Borel, Guiseppe Peano, Henri Poincaré, Felix Bernstein, Henri Lebesgue, and Arthur Schoenflies. For example, Borel commented: "Any argument where one supposes an arbitrary choice to be made a non- denumerably infinite number of times [...] is outside the domain of mathematics." (See p. 60.)

There were three sources of criticism: distrust of Cantorian set theory; use of Zermelo's new axiom; the close connection of certain steps in the proof to arguments that lead to paradoxes. In defense of his use of the new axiom AC, Zermelo pointed out that non-provability does not mean non-validity.

It is interesting to note that Zermelo was not the first to use this new principle. For example, Cantor used the axiom in his proof that every infinite set contains a subset of cardinality $\aleph_0$, and Richard Dedekind used it to show that every Dedekind finite set is finite. Even some of his critics such as Bernstein, Schoenflies, Borel, Baire, and Lebesgue used some version of the axiom. For example, the axiom is needed to show that the union of a countable number of measurable sets is measurable. But it was Zermelo's proof of well-ordering that showed for the first time the true deductive power of AC.

In 1908, Zermelo, in response to his critics, published another paper on well-ordering. First of all, he gave a second proof. Whereas his first proof can be described as a "bottom up" approach, his new proof was "top down." This new proof did not rely on previous results of Cantor, but was independent of any prior theorems of set theory and required fewer set-theoretic assumptions. In this second paper Zermelo also discussed the objections that were raised against the first proof and he gave an equivalent form of the axiom of choice in terms of pairwise disjoint collections of non-empty sets.

Eventually, there was a gradual acceptance of both the use of the axiom and its application that every set is well-ordered. By 1915, Wacław Sierpiński and the Polish school began using the axiom. Ernst Steinitz was a strong supporter of the axiom and its necessity in certain areas of mathematics; he said: "Many mathematicians still oppose the axiom of choice. With the increasing recognition that there are questions in mathematics which cannot be decided without this axiom, the resistance to it may increasingly disappear." (See p. 71.) By 1930, Zorn's Lemma, an equivalent form of the axiom, was deemed indispensable in topology and algebra. Fraenkel and Bar-Hillel characterized the axiom as "probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago." (See p. 60.)

Zermelo's next research project was to give an axiomatization of set theory. Several factors influenced his choice of axioms. First of all, during the period 1900- 1908, Zermelo taught courses in set theory in which he experimented with several lists of axioms. In addition, Zermelo wanted to restrict the axioms so as to avoid the well-known paradoxes given by Russell-Zermelo, Hilbert, Burali-Forti, and Richard. He also wanted to provide an axiomatic framework for his proof that every set can be well-ordered. Indeed, it seems clear that he was careful to choose enough axioms so that his 1908 proof of well-ordering could be executed.

There was yet another motivating factor, namely the work of Cantor and Dedekind. In 1872 and 1883, Dedekind wrote two influential essays, *Stetigkeit und irrationale Zahlen* and *Was sind und was sollen die Zahlen?*. In the first he introduced his famous Dedekind cuts as a method for constructing the real numbers from the rational numbers. In the second he showed how the natural numbers and their operations are modeled by sets. This was the beginning of the program in which the various number systems, indeed all of classical mathematics, are modeled in terms of sets. Thus, Zermelo was careful to include enough axioms to duplicate the pioneering work of Cantor and Dedekind.

Zermelo's approach was patterned after that used by Hilbert in his 1899 *Grundlagen der Geometrie*: he used two undefined terms, namely *sets* and *urelements*, and one binary relation, denoted by $\in$. Zermelo's axioms for set theory (slightly simplified from the original due to the fact that Zermelo had two undefined terms) are as follows.

Z1 (extensionality) If two sets have the same elements, then they are equal.

Z2 (separation) Let A be a set and let Q be a property of sets. Then the collection $\{x : x \in A \text{ and } x \text{ has property } Q\}$ is a set.

Z3 (null set) There is a set that has no elements.

Z4 (pair) If x and y are sets, then $\{x, y\}$ is a set.

Z5 (union) If x is a set, then $\cup x$ is a set.

Z6 (power) If x is a set, then $P(x)$ is a set.

Z7(infinity) There is an infinite set. More precisely: there is a set Z such that $\varnothing \in Z$ and $\{a\} \in Z$ for all $a \in Z$.

AC (choice) Let E be a set and let $\{A_t : t \in T\}$ be a pairwise disjoint collection of non-empty subsets of E. Then there is a subset B of E such that B interesects each set A_t in a singleton.

Zermelo was perhaps the first to realize the need for an explicit axiom that asserts the existence of an infinite set. In his paper he expressed the desirability of showing the consistency of his axiom system, but he admitted that he did not have a proof. For future reference, we list two additional axioms that were eventually added to Zermelo's list. Henceforth we will use the standard notation ZF for Z1-Z9 and ZFC for Z1-Z9 + AC.

Z8 (replacement) Let A be a set, and for each $x \in A$ let z_x be a set. Then $\{z_x : x \in A\}$ is a set.

Z9 (foundation) For every non-empty set A there exists $a \in A$ such that $a \cap A = \varnothing$.

The Axiom of Comprehension, repeatedly used in Cantorian set theory, states that the collection of all sets satisfying some property Q is a set. Zermelo realized that the Axiom of Comprehension was the source of the known paradoxes and that he needed to replace Comprehension with the more restricted Separation. There are two key restrictions here: (1) the process of selection is confined to a set that has already been constructed; (2) the defining criterion (property Q) must be stated in set-theoretic terms.

Zermelo explained how Separation blocks the Richard paradox (and other such semantic paradoxes) by observing that the second restriction of the axiom is violated. The other paradoxes, for example those due to Russell, Hilbert, Cantor, and Burali-Forti, all violate the first restriction. He even turned Russell's paradox to his advantage by showing that every set A has a subset B such that $B \notin A$; from this he concluded that there is no universal set.

Zermelo also used his axiom system to prove essentially all of the results from Cantorian set theory. In particular, he proved: $|A| < |P(A)|$; the Schröder-Bernstein Theorem; König's Theorem that if $\kappa_t < \lambda_t$ for all $t \in T$, then $\sum \kappa_t < \prod \lambda_t$. Finally, Zermelo showed that his axioms are sufficient to develop Dedekind's theory of cuts and his analysis of the natural number system and its arithmetic operations in terms of sets.

As with his introduction of AC, there were negative reactions to his axiomization. For example, Russell criticized the notion of definiteness, stating that Separation is "useless." He much preferred his theory of types as the best way to block paradoxes. In addition, other mathematicians did not immediately embrace his axiom system. For example, Felix Hausdorff, in his influential *Grundzüge der Mengenlehre* (1914), did not use Zermelo's axioms in his development of set theory. Finally, in 1915, it did play a role in Fredrich Hartogs' proof that cardinal comparability, which is solved using AC, is actually equivalent to AC.

Abraham Fraenkel began a correspondence with Zermelo in 1921 that led to the addition of the Axiom of Replacement to Zermelo's other axioms. Fraenkel pointed out that if A is the set whose elements are $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$,... , then one cannot prove the existence of the set whose elements are

$$A,\ P(A),\ P(P(A)),\ \ldots\ .$$

More generally, one cannot prove the existence of sets of cardinality $\aleph_\omega$. Independent of the work by Fraenkel, Thoralf Skolem also discovered this shortcoming of Zermelo's axioms, and likewise suggested the addition of Replacement. In 1925 John von Neumann added the final axiom of set theory, the Axiom of Foundation (more on this later).

The book under review is divided into 4 major chapters, each of which is devoted to a location where Zermelo spent a significant part of his life, and two short chapters.

1. Berlin 1891-1897

2. Göttingen 1897-1910
3. Zurich 1910-1921
4. Freiburg 1921-1953
5. A Final Word
6. Zermelo's Curriculum Vitae

Chapter 5 is an interesting discussion of Zermelo's personality, and Chapter 6 gives a ten-page overview of the highlights of his entire life. Here is a summary of the first four chapters.

BERLIN(1871-1897) Zermelo was born in Berlin on July 27, 1871. It is interesting to note that 1871 was the year in which Cantor wrote his paper on trigonometric series in which the solution required transfinite ordinal numbers; this paper is often regarded as the birth of Cantorian set theory. Zermelo became an orphan at an early age: his mother died on June 3 1878 when Zermelo was 7 and his father died on January 24 1889 when Zermelo was 18. Zermelo had five sisters, one older, four younger; all but one pre- deceased him.

Zermelo obtained his Ph.D. from the University of Berlin in 1894 under the supervision of Hermann Schwarz, a student of Karl Weierstrass; the title of his thesis was *Investigations On the Calculus of Variations*. From 1894 to 1897 he was an assistant to Max Planck at the Berlin Institute for Theoretical Physics. During that time he began a series of papers that were critical of Ludwig Boltzmann's theory of statistical mechanics with applications to the Second Law of Thermodynamics.

GÖTTINGEN (1897-1910) In 1897 Zermelo began an appointment at Göttingen University. Zermelo had decided that he wanted to pursue research in theoretical physics and mechanics, and so in 1897 he wrote a letter to Felix Klein asking for advice; Klein was encouraging and Zermelo was offered a position at Göttingen. At that time Göttingen's main strength was in applied mathematics, but that was about to change: two years earlier, in 1895, Hilbert began his career at Göttingen. Initially, Hilbert's work was in algebraic number theory, but by 1900 his interests had shifted to the foundations of mathematics. As noted earlier in this review, Hilbert had the greatest influence on Zermelo, and the years 1900-1908 were his most productive years in set theory. Nevertheless, during the Göttingen years, Zermelo also continued his research in the calculus of variations (1903-1904). He enjoyed a close friendship with Constantin Carathéodory, with whom he wrote a book on the calculus of variations, and he also collaborated with Hans Hahn.

During the period 1905-1908, Zermelo suffered serious health problems; in particular, in 1906 he was diagnosed with tuberculosis of the lungs and on several occasions he was forced to cancel scheduled classes. However, by 1909 he appeared to be on the road to recovery. In that same year Hermann Minkowski, then at Göttingen, unexpectedly died, and Zermelo had hoped that he would get the vacant chair; instead, it went to Edmund Landau. The book under review gives a detailed and interesting discussion of the reasons why Zermelo had difficulty in securing a permanent position at Göttingen. However, his situation was resolved when in 1910 Zermelo was offered a full professorship at Zurich, preliminarily for a period of six years.

ZURICH (1910-1921) During the Zurich years, Zermelo did research in at least three different areas: algebra, set theory, and game theory. In a 1913 paper on algebra, he used AC to prove the existence of rings whose quotient fields are the real number field or the complex number field and whose algebraic numbers are algebraic integers. In a 1916 paper, Emma Noether extended these results.

During this period Zermelo wrote an unpublished paper in set theory in which he anticipated von Neumann's theory of the ordinal numbers. However, missing from his treatment was the theorem that every well-ordered set is order isomorphic to an ordinal. For a proof of this he needed the Axiom of Replacement. It was during his last year at Zurich when Zermelo began his correspondence with Fraenkel on Replacement.

Zermelo was an enthusiastic chess player, and in 1913 he published a paper in game theory in which he used set theory to analyze the game of chess. In 1912, Zermelo, at Russell's invitation, gave two talks at the 5th International Congress of Mathematicians, scheduled at Cambridge: one was on his paper on chess, the other on the foundations of mathematics. Game theory did not become an active area of research until 15 years later, with contributions by Dénes König, László Kalmár, and von Neumann. It is not clear to what extent the work of Zermelo was an influence, but the author of the book under review states that, at the very least, Zermelo deserves credit for "having written the first paper that mirrors the spirit of modern game theory." (See p. 132.)

During this period, Zermelo continued to have health problems, and in 1916 he was forced to retire with a pension. He remained in Zurich until 1921 and did considerable traveling during the period 1916-1921. In October 1921 Zermelo moved to Freiburg, Germany.

FREIBURG (1921-1953) During the early years in Freiburg, Zermelo had a rather tranquil life. He was retired with a reasonable pension and was active in mathematics with several projects: editing the collected works of Cantor (completed in 1932); research in the calculus of variations and game theory. He also had an honorary position at the University of Freiburg, where he would occasionally give lectures. In 1929, at the invitation of Sierpiński and Stefan Mazurkiewicz, Zermelo traveled to Poland where he gave a series of talks on logic and the foundations of mathematics. While there, he came in contact with a number of well-known mathematicians, including Kazimierz Kuratowski, Bronisław Knaster, Stefan Banach, Hugo Steinhaus, Jan Łukasiewicz, Alfred Tarski, and Stanisław Leśniewski.

After a period of approximately 20 years, Zermelo renewed his interest in set theory and the foundations of mathematics, no doubt stimulated by work of Fraenkel, Skolem, von Neumann, and Gödel and also by his contact with the Polish mathematicians. As already noted, Fraenkel and Skolem independently noted the need to add the Axiom of Replacement to the axioms for set theory. In addition, they were critical of the Axiom of Separation, specifically the ambiguity in describing a "definite property." Both offered solutions, but Skolem's is the more precise and is the one in current use; he was the first to suggest that a first-order language is the best way to give a precise formulation of Replacement and Separation.

During the 1920's von Neumann made a number of important contributions to set theory. To begin with, the modern definitions of cardinal and ordinal numbers are largely due to him, as is the important idea of transfinite recursion and definitions by transfinite recursion. Von Neumann proved the result that every well-ordered set is order isomorphic to an ordinal. He was also interested in finding a categorical set of axioms for set theory. As a first step he wanted to eliminate sets that do not naturally occur in mathematics, for example, sets x such that $x \in x$, and more generally sets x for which there is a sequence $\langle x_n : n \in \omega \rangle$ with $x_0 = x$ and $x_{n+1} \in x_n$ for all $n \in \omega$. His solution was to introduce the Axiom of Foundation. During this period von Neumann also introduced the cumulative hierarchy $\{V_\alpha : \alpha \in ON\}$, constructed as follows:

$$V_0 = \varnothing;\ V_{\alpha+1} = P(V_\alpha);\ V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}.$$

Von Neumann used Foundation to prove that every set is in the hierarchy, thus giving a clean description of the universe of all sets and showing that the hierarchy gives a bottom-up construction of the universe of all sets. He also used the hierarchy to prove the result (stated in modern terms) that if ZFC without Foundation is consistent, then ZFC is consistent. This was the first consistency proof using the method of inner models. As an interesting aside, von Neumann used the notation ZF (approximately) to refer to the axioms Z1-Z9, and Zermelo, in his research, continued this practice. Thus Skolem did not receive the recognition that he deserves for his contributions to axiomatic set theory; it is not unfair to say that ZFS would be more appropriate.

Zermelo also obtained results on the hierarchy, and perhaps discovered it independently. In any case, he proved the result that if κ is a strongly inaccessible cardinal, then V_κ is a model of ZFC. Zermelo appreciated the back-and-forth relationship between the hierarchy and ZFC. On the one hand, the construction motivates the choice of the axioms of ZFC and emphasizes the importance of the Power Set, Union, and Replacement Axioms. Alternatively, given an intuitive understanding of the hierarchy, it then serves as the canonical model of the axiom system ZFC.

Zermelo rejected Skolem's solution of expressing the axioms of ZFC in a first-order language since it leads to what is known as *Skolem's paradox*. The Löwenheim- Skolem Theorem states that if a theory with a countable number of first-order sentences has a model, then it has a countable model. Thus, if we assume that ZFC (first-order version) has a model, then it has a countable model. Let M be a countable transitive model of ZFC. Now ZFC proves the existence of uncountable sets, and therefore there exists $x \in M$ with x uncountable. We also have $x \subseteq M$, and this appears to give a contradiction. The paradox is resolved by the observation that the cardinality of a set in M depends on the functions that are in M. For Skolem, the Löwenheim-Skolem Theorem showed that ZFC does not serve for the foundations of mathematics. But for Zermelo, this conclusion was unacceptable; the axiom system ZFC should describe Cantorian set theory with its endless collection of sets having larger and larger cardinality.

Gödel's Incompleteness Theorems were yet another challenge to Zermelo's view of mathematics. According to Gödel, any consistent and recursive list of (first-order) axioms with sufficient strength to develop recursive arithmetic is incomplete and cannot prove its own consistency. Therefore, the traditional approach in mathematics [formal systems with a first-order language, proofs with finitely many steps] has inherent limitations with regard to establishing mathematical truth.

Zermelo, who did not fully appreciate the subtle difference between semantics and syntax in formal logic, claimed that there was a flaw in Gödel's proof. He informed Gödel of his discovery, and Gödel responded by writing a ten-page letter in which he gave a careful description of his methods and pointed out the source of Zermelo's confusion. Unfortunately, Zermelo was unwilling to concede and the correspondence ended.

To summarize: to a very great extent, Zermelo's research in set theory during the 1930's was a reaction to intuitionism, Hilbert's proof theory, and the research of Skolem and Gödel. Zermelo rejected the limitations imposed by first-order logic due to the results of Skolem and Gödel. His solution was to replace finitary, first-order logic with infinitary languages (allow infinitely long conjunctions and disjunctions) and infinitary logic (allow infinitely long proofs). Unfortunately, this approach put him out of the mainstream of research in mathematical logic.

In January 1933, the Nazis seized power in Germany, and this had unpleasant consequences for Zermelo. The philosopher Martin Heidegger, a strong supporter of the new regime, was appointed as the new Rector at the University of Freiburg, and one of his early decisions was the requirement that lectures be opened with the Hitler salute. Zermelo's refusal to comply with this directive eventually led to his dismissal from the University. At this time Zermelo began a gradual retreat from academic life, and he moved from the city to a residential area in the foothills of the Black Forest. Here he came in contact with a much younger woman, Gertrud Seekamp (1902-2003), and on October 14 1944 they were married. Zermelo died on May 21, 1953 at the age of 81.

ZERMELO'S PERSONALITY To say the least, Zermelo had an interesting and unusual personality. He has been described in a wide variety of ways: irascible, strange, solitary, nervous, unwilling to compromise; but also helpful, a stimulating conversationalist with strong personal convictions, and generous. He could be quite sarcastic at times; for example, he once wrote to a friend about his critics of his well-ordering proof: "With regard to the question of well- ordering the Göttingen "Count" [Bernstein] has suffered a spectacular defeat. König and Peano have written me obliging letters, and Jourdain is also retreating. Only Schoenflies gabbles on. Borel keeps silent." (See p. 76.) A research assistant in Freiberg named Helmuth Gericke said "sometimes [he] even insults his friends." (See p. 73.)

His research career was characterized by an unusual amount of controversy; for example, his disagreements with Boltzmann on statistical mechanics, Fraenkel and Skolem on definiteness in Separation, Gödel on incompleteness, Brower on intuitionism, and even Hilbert on the primacy in logic of formal systems based on first-order languages. Throughout his later years he felt that he did not receive adequate recognition for his pioneering work in set theory. He had a number of close friends, including Carathéodory, Erhard Schmidt, Gerhard Hessenberg, and also many admirers, for example Paul Bernays and Knaster.

Zermelo was not considered a good teacher for average students, but he was an inspiration for mathematically mature students. Zermelo had a wide variety of interests in the humanities, including music theory, philosophy, and classical poetry. Here is a quote from Wilhelm Süss in support of re-appointment of Zermelo to honorary professor in 1946: "Yet Zermelo is by no means only the withdrawn scholar who knows nothing else but science. Only those who have got into closer contact with him, certainly know about his enthusiasm and the good understanding by which he has occupied himself, for instance, with the classical world. Only very few people know for example his translations of Homer. Where he has confidence, he shows a childlike and pure nature and - despite an often hard destiny - a generous heart." (See p. 260.) As confirmed by his wife Gertrud, Zermelo had difficulty in dealing with the problems of everyday life; this was in contrast to the self-confidence that he had in dealing with issues in mathematics.

In summary, this book is not just a "must have" for specialists in set theory and its history. It is also recommended for logicians and historians of mathematics in general. Professors may confidently recommend its acquisition by their college or univeristy library. Ebbinghaus' book and G. Moore's *Zermelo's Axiom of Choice* together give us a wonderfully clear picture of the illustrious mathematical career of Ernst Fredrick Ferdinand Zermelo.

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Duke University, Durham, N.C. 27708
E-mail address: hodel@math.duke.edu