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Torkel Franzén
Gödel's Theorem: An Incomplete Guide to Its Use and Abuse
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REVIEW

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Gödel's Incompleteness Theorems, first revealed to the world 75 years ago, have by now become part of our general intellectual background. One can find references to them in discussions on architecture, photography, literary criticism, the "theory of everything," theology, Zen Buddhism, etc. This might seem astonishing at first, given that they are quite technical results in mathematical logic. However, by exhibiting some surprising and insurmountable limitations of formal systems, Gödel's theorems have often been the source of inspiration for talk about analogous limitations in other domains. While there is certainly nothing wrong with this, one frequently also encounters shameless abuses of the theorems, where they are misrepresented or where certain conclusions are attributed to them erroneously. To ameliorate this situation is Torkel Franzén's explicit aim, namely "to allow a reader with no knowledge of formal logic to form a sober and soundly based opinion of these uses and abuses" (p. iv). This aim is what makes Franzén's book unique. On the one hand, Franzén presents Gödel's results with a minimum of symbolism and technical jargon (e.g., Franzén uses the term "Goldbach-like statements" instead of "Π^1_0-statements"), but without compromising on clarity and precision, and, on the other hand, he discusses Gödel's theorems and their consequences in a wide variety of contexts.

It is one of the great merits of this book that it sensitizes the reader to the linguistic subtleties involved in describing Gödel's achievements. Franzén does an excellent job in explaining the relevant terminology, often mentioning terminological variants that occur in the literature, and he carefully points out where they differ from usage in ordinary language. In particular such terms as "theory," "formal system," "axiom," "consistency," "completeness," and "incompleteness" can easily lead to misapplications of Gödel's results if they are used with their informal meanings. Throughout the text Franzén quotes and discusses many examples of such abuses, which are culled from the literature and from Internet discussion groups. By taking into consideration also Chaitin's work on information-theoretic complexity,[1] Friedman's investigations of large cardinal axioms,[2] and the research area of *automated theorem proving*, Franzén touches upon contemporary issues in logic that otherwise only rarely find their way into books of an introductory character like this one.

Gödel's Theorem (used by Franzén as collective term for both the first and second incompleteness theorem) begins with a brief introductory chapter, which also includes an overview of Kurt Gödel's life and work. In Chapter 2, "The Incompleteness Theorem: An Overview," which takes up a third of the entire book, Franzén introduces the relevant definitions, presents the main themes and conclusions, and exposes common misunderstandings. It is followed by the remaining six chapters and one appendix, ranging between 10 and 20 pages each, in which particular issues are explored in more detail. Due to this organization of the material, there is certain amount of repetition in the later chapters. The index is useful for locating definitions of the technical terms used.

Chapter 3 is a clear exposition of central results on decidability and computability, and, based upon these, of Gödel's first incompleteness theorem. Franzén points out that the proof of Gödel's first theorem using computability theory shows, in contrast to Gödel's own proof, that no self-reference is necessary to obtain the result and that the theorem also holds for certain non-classical systems (intuitionistic and second order). A more formal presentation of this material is given in the Appendix. As Franzén points out, Gödel's theorem says nothing about the truth of the Gödel-sentence G, but only that if the theory in question is consistent, then G is true. In many cases, however, one simply does not know whether the theory at hand is consistent or not, and it is the neglect of this observation that lies at the bottom of many misinterpretations of Gödel's result.

In Chapter 4, "Incompleteness Everywhere," alleged applications of Gödel's first theorem to legal systems, theology, philosophy, human thought, and physics are analyzed. Franzén emphasizes that Gödel's

[1]E.g., Chaitin, G. J., *The Unknowable*, Springer-Verlag, Singapore, 1999.

[2]See the archives of the FOM (Foundations of Mathematics) mailing list, `http://www.cs.nyu.edu/pipermail/fom`.

incompleteness relates only to the "arithmetical component" of theories that include a certain amount of arithmetic—a fact that even eminent physicists have overlooked—and that non-mathematical Gödel-sentences are not applications of Gödel's results or its proof, but merely "considerations inspired by the incompleteness theorem." He also argues that the incompleteness theorem has not lead to an alleged "post-modern condition" in which mathematics branches-off into infinitely many incompatible systems that are equally well accepted. This would have been an appropriate place to add a few words on the historical development of mathematics, e.g., the debate about Zermelo's axioms of choice.

The Second Incompleteness Theorem and its consequences are discussed in the first section of Chapter 5. In this connection Franzén draws attention to the fact that the "certain amount of arithmetic" that is required for the application of the First Incompleteness Theorem is different than that required for the second one. The remainder of the chapter is devoted to the theorem's alleged skeptical consequences and a discussion of the role of consistency proofs. Franzén makes the interesting observation that most of the skeptical conclusions drawn from Gödel's theorems seem to be based in one way or another on the view that underlies Hilbert's program, and he discusses the misleading formulation that the consistency of a theory can only be proved by a "stronger" theory. He also uncovers the shortcomings of commonly found suggestive expressions, such as "going outside the system." A consequence of Gödel's theorems, indeed one of the most striking according to Franzén, is the inexhaustibility of mathematical knowledge by a single formal system.

The most philosophical chapter of the book is the sixth, where Franzén attempts to debunk Lucas' and Penrose's arguments for the non-mechanical character of the mind. Since an extensive analysis of these arguments is beyond the scope of Franzén's book, the level of argumentation does not reach too deep. Thus, this chapter serves largely the purpose of outlining the main positions in this debate.

That the term "completeness" is used with two different meanings (namely as "negation completeness," and as the sufficiency to derive every logical consequence of a set of axioms) is a recurrent source of confusion for beginning students of logic, since it leads to the seeming contradiction that Gödel proved both the completeness and the incompleteness of certain formal systems. The completeness of first-order logic is the topic of Chapter 7, which is standard material of a first course in meta-logic. Franzén adds to it few brief remarks about the connection to the incompleteness theorems.

Chapter 8 deals with the relation between incompleteness and the notions of complexity and infinity. In particular, the claim that the incompleteness theorems apply only to "sufficiently complex" theories is scrutinized. Franzén points out that there are complex theories to which Gödel's theorems do not apply, as well as relatively simple theories to which they do apply (e.g., Robinson arithmetic). In addition, the connections of a mathematically precise notion of complexity and incompleteness are critically discussed. Franzén also contrasts Chaitin's notion of "randomness" to randomness that exists in nature and concludes that they are not analogous, but that many of Chaitin's claims about randomness rely on such an analogy nonetheless. Finally, Franzén explores various dimensions of incompleteness of set theory (ZFC).

Since many of the quotations Franzén adduces are intended to convey commonly occurring ideas and arguments, he often refrains from giving specific references. While this is an understandable practice, I think that the book would have benefited from a greater number of references, in particular to further reading material. For example, Franzén asserts in his discussion of a proof for Fermat's last theorem that there "is some reason to believe, on general grounds, that an elementary proof of the theorem exists," without hinting at what these grounds are or where one could look for them (p. 15); similarly, without further comment Franzén speaks of "good grounds" for thinking that ZFC captures all of ordinary mathematics (p. 25). In a similar fashion, specific references to more extensive treatments of certain technical issues would be useful for those who are intrigued by remarks Franzén makes in passing, for example, where he says that the incompleteness theorem can be adapted to non-classical systems (p. 19), or where he leaves the verification of the undecidability of the Rosser sentence (a stronger version of the undecidable sentence constructed by Gödel) to the interested reader (p. 43). Given Franzén's general reluctance to offer references, the five entries in the bibliography to Chaitin's work seem disproportionate.

While it is difficult to judge whether the book's professed target audience, namely "a reader with no knowledge of formal logic," will indeed be able to follow the presentation or not, there is at least some anecdotal evidence. I do know of a student with little background in logic who found the book demanding, but very accessible, and who showed a clear grasp of its main ideas after having read it. Thus, it is fair to say that this book does achieve its aims. It presents Gödel's theorems in a clear way and discusses their applicability and non-applicability in many different contexts. It is a well-written and entertaining read, and it certainly provides a unique and outstanding starting point for learning about the richness and the import of arguably the most important results in mathematical logic in the twentieth century.

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