A SURVEY OF FREE LOGICS[1]

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This survey of free quantification theories and free identity theories is intended to supplement two other studies, one by E. Bencivenga in *Handbook of Philosophical Logic*, Vol. 3, and another by K. Lambert in two of his papers published in *Inquiry* and *History and Philosophy of Logic* which survey and review some of the important contributions to the area. However these studies were intended not to give a comprehensive view of the results of research in the area but only to provide a context for presenting their author's own contributions. Thus they fall short of being exhaustive reviews of the literature. The present survey tries to present such a comprehensive account, paying attention to the diversity in the attempts to handle the problem as it arises in relation to quantification and identity. By not paying much attention to the differences in different reformulations of the standard quantification theory, Lambert and Bencivenga create the impression that this part of the free logicians' enquiry is unproblematic. The present survey shows that all known reformulations are defective on one count or another; they either invoke vacuous quantification, or require outer domains of interpretation, or multiply connectives, or resort to some such *ad hoc* device. All in all it appears that each of these technical triumphs covers some conceptual confusion. And when an unproblematic reformulation not succumbing to any of the known gimmicks is thought of, it so happens that it cannot be extended to include identity theory unless and until it is assumed that the domain of interpretation is non-empty. For a survey of all the non-standard theories of identity which the free logicians invoke, it comes to the fore that identity and existence go hand in hand, and that in the absence of one the other would not be available. This, in a sense, vindicates the classical relationship between the concepts of existence, identity and uniqueness, and calls into question the very efficiency of the research programme in free logic.

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1 Preliminaries

In specifying the prerequisites for what follows we shall be frugal and refrain from giving beaten track details which can be found in any well written text book such as [7] or [48]. While following the standard and familiar notation we shall use some convenient devices which would cut short needless elaborate explanations. We shall use $A(x_i)$ to stand for any well-formed formula in which x_i occurs bound, and Ax_i shall be used to stand for any well-formed formula in which x_i similarly occurs free at least once. A standing alone is neutral to the occurrence of x_i in it; thus A may contain x_i bound or free, or it may not contain it at all. $A, B, C, \ldots$ with or without subscripts shall be used as variable formulae. Ax_i and Ax_j, and similarly $A(x_i)$ and $A(x_j)$, shall be taken to be exactly alike except that the former contains x_i wherever the latter contains x_j. We shall use $\Delta, \Gamma, \Theta, \ldots$ for any finite (possibly empty) set of formulae; obviously when this set is a unit set, Δ stands for a formula. We assume the familiar presentation of the Standard Quantification Theory, (SQ) for short, in terms of the following axiom schemata and rule of inference.

Ax. 1 A, where A is a 2-valued truth-functional tautology.

Ax. 2 $(x_i)(A(x_i) \rightarrow B(x_i)) \rightarrow (A(x_i) \rightarrow (x_i)\, B(x_i))$

Ax. 3 $(x_i)\, Ax_i \rightarrow A_{x_j}$, where $i \leq j$

Ax. 4 $Ax_j \rightarrow (x_i)\, Ax_i$, where $i \leq j$

Ax. 5 $(x_i)(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)\, Ax_i \rightarrow (x_i)\, Bx_i)$

R1. If $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$.

We let (SQ+) be an extension of (SQ) such that it is like (SQ) except that it contains in its vocabulary individual constants a_i, and permits a replacement of x_i in Axiom 3 and Axiom 4 by a_j. (SQ=) and (SQ+=) are extensions of (SQ) and (SQ+) respectively and are obtained by augmenting the axiom sets by

Ax. 6 $(x_i)(x_i = x_i)$, and

Ax. 7 $(x_i = x_j) \rightarrow (Ax_i \rightarrow Ax_j)$.

We further assume that (SQ)s have been alternatively formulated with the help of Introduction and Elimination rules; we shall refer to such a formulation by (SQND), where ND is an abbreviation for 'natural deduction'. Finally, following Quine's advice, we shall exhibit only those structures which are relevant to either deduction or discussion.

It is obvious that the domains of interpretation of (SQ) and (SQ+) as well as (SQ=) and (SQ+=) are precisely the same. This domain, say D, comes fixed and its members are fully individuated. The terms of (SQ) are mapped onto D; and in virtue of this it has been thought that (SQ)s capture the rubrics of representation. In these systems existence and reference are married off to quantification. When a mapping $M(t_i)$ of a term t_i — be its an individual constant or an individual variable — obtains in D, this is to say that when it is the case that $M(t_i) \in D$, $M(t_i)$ is said to be the referent of the term t_i. It is assumed that $D \neq A$; and when $M(t_i) = d$ such that $d \in D$, the term t_i is said to be referring, and d is said to be existent. That

(*i*) the referent of a term be existent (in the sense specified here),

(*ii*) the reference of a term be constant (this is to say that it be the same throughout a given discourse), and

(*iii*) the referring of a term be unique,

are (the) three basic assumptions of (SQ)s. In virtue of this (SQ)s have come to be seen as the very criteria stipulating how we can tell whether a term is a referring one or not. Accommodating to terms which did not refer to the members of D in (SQ)-type systems thus naturally issued in several anomalies. Since Frege, Russell, and Hilbert there have been efforts to accommodate them too by allowing them to have a semblance of reference, and by letting the sentences in which they occur have some truth-value in terms of the deductive links these sentences are permitted to have with sentences in which terms referring to the members of D occur. Most of these attemps resulted in strategies requiring counter-intuitive assumptions. However there is no reason why logicians should invest so heavily in intuition, all the more so when it is commonsensical and not cultivated intuition that is at issue. In any case, Pegasus-hunting and Unicorn-haunted philosophers who took upon themselves the custodianship of common parlance felt uneasy with the way in which sentences containing non-referring terms are accommodated within the framework of (SQ)-type systems with interpretations of the kind mentioned above. So they demanded a radical revision in the syntax and the semantics of (SQ)s. They demanded a reformulation of (SQ)s setting them free from any or all of the requirements (*i*)–(*iii*) above. We shall call these resultant reformulations Free Quantification Theories, (FQ) for short.

Even those who thought that (SQ)s exhibit logical structure *per se* had a gripe concerning them; this was caused by the conception of logic to which they subscribed. This conception presupposed a sharp divide between the analytic and the synthetic components of theories, and an identification of the synthetic with the empirical. It was central to this conception that logic is not only an analytic theory but in fact a paradigm of analyticity itself. Russell [69] and Carnap [5] for instance felt uncomfortable with the following theorems of the (SQ)s.

Th. 1 If $\Delta \vdash (x_i)\, Ax_i$ then $\Delta \vdash (\exists x_i)\, Ax_i$, and

Th. 2 $\Delta \vdash (\exists x_i)(\sim Ax_i \vee Ax_i)$

They thought that these and similar theorems have some "empirical" content adversely affecting the purity of the (SQ)s which they thought to be exhausting the extension of the concept of logic. "If logic is to be independent of empirical knowledge", Carnap thought in (1), "then it must assume nothing concerning the existence of objects". Concerning the fact that in (SQ)s "not only sentences which are true in every domain but in every nonempty domain are demonstrable", Carnap thought that "in practice, this distinction is immaterial since we are usually concerned with non-empty domains. But if in order to separate logic as sharply as possible from empirical science, we intend to exclude from the logical system any assumption concerning the existence of objects, we must make certain alterations in" (SQ)s. He had two suggestions towards carrying out the intended alteration. As we shall see soon, both of these were repeatedly tried, but without much profit. These were first not to allow free variables to occur in theorems, and secondly to allow individual constants to be substituends only when they have referents in the domain of interpretation.

This Russellian uneasiness which Carnap shared and vented was induced by Wittgenstein [89]. In 1913 he wrote to Russell: "a Proposition like $(\exists x_i)(x_i = x_i)$ is, for example, really a proposition of physics. The proposition $(x_i)(x_i = x_j) \rightarrow (\exists x_j)(x_j = x_j)$ is not a proposition of logic. It is for physics to say whether anything exists." This uneasiness was reinforced by the conviction that logic is a theory which is a subtheory of all possible theories, and that it should hence be free from all possible specific assumptions impinging upon its purity and analyticity. Thus by the early thirties it had come to be believed even by those with cultivated intuition that "whether individuals exist or not", as Jaśkowski put it in (1), "it is better to solve this problem through other theories", and confine logical theory to what is common to all theories including those with empty domains of interpretation. In fact the first attempt to reformulate (SQ)s to set them free from the first requirement mentioned above came from Jaśkowski.

When existence is married off to quantification and $(\exists x_i)$ is read as 'there exists an x_i such that', and referential interpretation of quantifiers is accepted, Theorem 1 and Theorem 2 of (SQ)s listed above require that the domains of interpretation be non-empty. This has been interpreted to mean that (SQ)s have existential presuppositions. And as these theorems are supposed to be "logical truths", they are presumed to be forcing us to accept the existence of at least one entity. This has been taken to mean that (SQ)s have existential consequences. Free logicians, this is to say that those who identify logic with (FQ)s, tried to obtain for logic freedom from presuppositions and consequences of this sort by unlinking existence and quantification, or by forsaking the referential interpretation of the quantifiers, or by modifying the interpretations which (SQ)s have normally received, or by reformulating them in such a way that theorems like 1 and 2 above, and their consequences, as well as formulae of which they are consequences, are not set down as theorems. In this effort the central thrust of free logicians has been to capture the logical behaviour of the concept of existence; and the main concern of universally free logicians has been to delimit precisely the extension of the concept of logic in which it remains an exemplary analytic theory free from presuppositions and consequences that may be empirical. Of course there is an issue about the relationship between Universally Free Quantification theory and Free Quantification Theory, (UFQ) and (FQ) for short, which received some attention, but not as much as it should have. However, investigations into (FQ) as well as (UFQ) have thrown much light on the nature of singular terms, the concept of identity and its relationship with the concepts of existence and individuation. It is also the case that now we are in a better position to appreciate the issues involved in these conceptual links and above all the very mechanism of quantification. As such, free logic is not all about the relatively useless case of the null-individual which Quine, with or without tongue in cheek,tried to brush away.

2 Restrictions on the Rule of Inference

The initial attempt in the intended direction was due to Jaśkowski [30]. His formulation, which we shall refer to as (FQJ), short for 'free quantification theory of Jaśkowski, is couched in a nominalistic syntactical metalanguage which however is not important for our purpose; nor is it essential for his system. It can be reformulated using a neutral metalanguage, or a metalanguage which is indifferent to philosophical issues. (FQJ) is a subsystem of (SQND) and is arrived at by imposing the following conditions on the notion of a valid deduction in (SQND).

C1 If A is an assumption, A is not of the form Bx_i, C2 Where A, B and $A \rightarrow B$ occur as separate lines in a deduction, (i) either both A and B are, or (ii) neither A nor B is of the form Cx_i, and

C3 besides the rules of sentential logic, the only rules of inference permitted are (i) Universal Instantiation, and (ii) Universal Generalisation.

C1 and C2 can alternatively be stated as:

C1′ Only closed formulae be allowed to occur as assumptions, and

C2′ No rule of sentential logic be applied to formulae containing both free and bound occurrences of the same or different individual variables.

$(x_i)\,Ax_i \rightarrow (\exists x_i)\,Ax_i$ cannot be deduced in (FQJ). In this system generalisations of two-valued tautologies result in theorems, and standard laws of quantifier distribution come out valid. To note how (FQJ) blocks deductions of certain formulae which the free logicians consider as undesirable, consider the following deduction which is valid in (SQND) but not in (FQJ). In this deduction the line marked by a cross is the result of a move not permitted in (FQJ); it is arrived at by applying rules of sentential inference on a formula which contains both free and bound occurrences of x_i.

$$
\begin{array}{ll}
(x_i)\,Ax_i & \\
\quad Ax_i & \\
\quad (x_i) \sim Ax_i & \\
\quad\quad \sim Ax_i & \\
\quad (x_i) \sim Ax_i \rightarrow Ax_i & \\
\quad \sim\sim Ax_i \rightarrow \sim (x_i) \sim Ax_i & + \\
\quad Ax_i \rightarrow (\exists x_i)\,Ax_i & \\
\qquad\qquad (\exists x_i)\,Ax_i &
\end{array}
$$

The completeness of (FQJ) is contingent on the fact that in it a conditional in which the consequent is $(x_i)\,Bx_i$ and the antecedent is $(x_i) \sim A$, where A but not B is a truth-functional tautology, is a theorem. The following, for instance, is a valid deduction in (FQJ):

$$
\begin{array}{l}
(x_i) \sim A \\
\quad \sim A \\
\quad \vdots \\
\quad Bx_i \;\&\; \sim Bx_i \\
\quad Bx_i \\
\qquad (x_i)\,Bx_i
\end{array}
$$

The next attempt at a reformulation of (SQ) was made by Mostowski in [49]. (FQM) has the same set of axiom schemata as (SQ), but its rule of inference is of a restricted nature and is to be formulated as:

RIM: If $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$, provided B contains no variable, say x_i, free unless A contains at least one free occurrence of x_i.

Thus, though $(x_i)\,Ax_i \rightarrow Ax_i$ is a valid proof sequence in (FQM), it cannot be extended further to result in $(x_i)\,Ax_i \rightarrow (\exists x_i)\,Ax_i$. This means that the unrestricted nature, specifically the transitivity, of material implication is lost in it. Material implication may be said to be equivocal in (FQM), for it partially orders the set of formulae but does not order so the set of formulae that are theorems. Mostowski's original specification of (FQM) differs with (SQ) not only with respect to the rule of inference but also with respect to Axiom 4. In place of Axiom 4 it has Axiom 4′ $A \rightarrow (x_i)\,A$. As such, when A contains x_i bound, it permits vacuous quantifications. However, vacuous quantification is not required for its being sound and complete; so we pruned it by dropping that undesirable syntactical feature.

(FQJ) and (FQM) have an important feature in common, namely their soundness requires that in their interpretations open formulae have the same truth-value as their respective closures. This means that though $\models (x_i)A \equiv Ax_i$, it is not the case that $\vdash (x_i)A \equiv Ax_i$. Because of this we will have to think that equivocation of $\rightarrow$ is essential to their being complete. Where V and V^* are valuation functions in their interpretations over non-empty and empty domains respectively, both these systems require that

(*i*) $V(Ax_j) = V((x_i)\,Ax_i)$, and

(*ii*) $V^*(Ax_i) = V((x_i)\,Ax_i)$.

Because all open formulae come out valid when the domain of interpretation is empty, it happens that in these systems

$$V^*(Ax_i \rightarrow (\exists x_i)\,Ax_i) = V^*((x_i)Ax_i)$$

$$V^*(Ax_i) = V^*((x_i)\,Ax_i \rightarrow (\exists x_i)\,Ax_i).$$

3 Vacuous Quantification

If open formulae are to be barred from being theorems while accepting that $V^*(A) = V((x_i)A)$, vacuous quantification becomes essential for soundness. This is evident from Hailperin's (FQH) [20] and

(FQQ) of Quine [54]. These systems differ from (FQJ) and (FQM) in their semantics, but have syntactical similarities with them. They permit vacuous quantification like (FQM), and like (FQJ) they bar open formulae from being theorems. To take note of these systems we need two syntactical notions, namely the notion of a Universal Closure and the notion of an Alphabetical Closure. If A is Bx_i, the universal closure of A is $(x_i)\, Bx_i$, and if A is Bx_ix_j and if x_i and x_j occur in B in their alphabetical order, the alphabetical closure of A is $(x_i)(x_j)\, Bx_ix_j$. We shall use $(UC)A$ and $(ac)A$ as shorts for the universal and the alphabetical closures, respectively, of A. (FQH) was thought of as a simplification of (FQM), and (FQQ) as a simplification of (FQH). (FQH) is a reformulation of (SQ) as presented by church [7]; and (FQQ) is a reformulation of (SQ) as formulated by Quine [53]. (FQH) is determined by the following set of axiom schemata and rule of inference.

Ah. 1 $(ac)A$ where A is a two-valued truth-functional tautology.

Ah. 2 $(ac)((x_i)(A \rightarrow B) \rightarrow ((x_i)A \rightarrow (x_i)B))$

Ah. 3 $(ac)((x_i)(x_j)A \rightarrow (x_j)(x_i)A)$

Ah. 4 $(ac)(A(x_i) \rightarrow (x_j)A)$

Ah. 5 $(ac)((A(x_i) \rightarrow (x_i)\, Bx_i) \rightarrow (x_i)(Ax_i \rightarrow Bx_i))$

R1h. **R1.** With the help of Ah. 5, Ah. 1 and **R1** it can be shown that $(x_i)((x_i)\, Ax \rightarrow (\exists x_i)\, Ax_i)$ comes out as a theorem; but $(x_i)\, Ax \rightarrow (\exists x_i)\, Ax$ cannot be proved in the system. This is also the case with (FQQ), whose axiomatic base constitutes

AQ. 1 $(uc)A$ where A is a two-valued tautology.

AQ. 2 $(uc)((x_i)(A \rightarrow B) \rightarrow ((x_i)\, Ax_i \rightarrow (x_i)\, Bx_i))$

AQ. 3 $(uc)(Ax \rightarrow (x_i)\, Ax)$

AQ. 4 $(uc)((x_i)\, Ax_i \rightarrow Ax_i)$

RIQ If $\vdash A$ and $(uc)(A \rightarrow B)$, then $\vdash B$.

The soundness of (FQH) and (FQQ) is preserved by accepting

(I) $V^*(Ax_i) = V^*((x_i)\, Ax_i)$,

(II) $V^*(A(x_i)) \neq V^*((x_i)\, A(x_i))$,

(III) $V((x_i)\, Ax_i) = V(Ax_i)$, and

(IV) $V((x_i)\,A(x_i)) = V(A(x_i))$,

where V and V^* are valuation functions in interpretations over non-empty and empty domains respectively. This implies that vacuous quantification, as Hochberg [28] remarked, "is not vacuous in the empty domain". For when A does not contain x_i free $(x_i)A$ may appear almost a trick, and one may wonder, like Wittgenstein, whom one might be tricking when he tricks in logic. To preclude vacuous quantification and yet be able to maintain that I and II above are correct, Hochberg suggested either abandoning the standard logic of sentences, specially

(1) $\vdash (A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$,

(2) $\vdash \sim\sim A \leftrightarrow A$, and

(3) the rule of modus ponens,

or else sacrificing definitional equivalences like

(1) $(\exists x_i)A \leftrightarrow \sim (x_i) \sim A$, and

(2) $(x_i)A \leftrightarrow \sim (\exists x_i) \sim A$.

Of course he noted that rejecting $(x_i)\,Ax_i$ materially implies $(\exists x_i)\,Ax_i$ as an alternative to either of these. But this alternative is the aim of all (FQ)s and each of them tries to achieve it in its own way. As such, it cannot be a solution to the problem at hand which is to get rid of vacuous quantification. As we noted (FQH) and (FQQ) incorporate vacuous quantification in order to bar

$$(x_i)\,Ax_i \rightarrow (\exists x_i)\,Ax_i$$

as a theorem. That the second alternative does not solve the problem should be evident from the fact that in intuitionist quantification theory the formula in question can be shown to be a theorem although the standard definitional equivalences that hold in (SQ) do not hold in it. The same can be said about the first suggestion. A restriction on the application of the rules of sentential logic, specially transpositions and double negation, is of no significance. In (FQJ), for instance, the applicability of the rules of sentential logic was restricted in one specific way, but the results were not commensurate. It required an identification of $V^*(Ax_i)$ and $V^*(\sim Ax_i)$ and an exclusion of open formulae from the set of theorems blocking the introduction of singular terms into the system. This might induce one to believe that the intended result may have to be gained by restricting the use of *modus ponens*. We have seen the results of one way of restricting this rule in (FQM).

There exists another system in which this rule is restricted differently; it is (FQS) of Schneider [72] and [73].

As our intention is not to stick to the chronological order but only to impose some order on the chaotic material we are handling, let us consider (FQG) of Goe [15] and [16] before we take up (FQS). The axiomatic base of (FQG) is

Ag. 1 Ah. 1

Ag. 2 $(ac)((x_i)(A \rightarrow Bx_i) \rightarrow (A \rightarrow (x_i)\, Bx_i))$

Ag. 3 $(ac)((x_i)\,(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)\, Ax_i \rightarrow Bx_j))$

RIg RIh

(FQG) is aimed at eliminating vacuous quantification from (FQH) and (FQQ). In (FQG) proof by assumption is permitted, but the assumption must be, as in (FQJ), only a closed formula. Thus like (FQH) and (FQQ) it does not permit open formulae to occur as theorems. Though its rule of inference is normal, the axioms are so well-chosen that even when $(ac)A$ and $(ac)(A \rightarrow B)$ are theorems, $(ac)B$ cannot be a theorem unless every free variable of A is also a free variable of B. In this respect it resembles (FQM). If (FQM) has the same set of axioms as (SQ) and differs from it in having a restricted rule of inference, (FQG) has the same rule of inference as (SQ) but a more stringent set of axioms. But $\rightarrow$ is equally equivocal in both of them; there is no isomorphism between its formational and transformational behaviours.

(FQS) however differs from each of these systems in one respect and agrees with them in another. Like (FQM), it requires $V^*(Ax_i)$ and $V^*(\sim Ax_i)$ to be the same; otherwise its soundness will be lost. In fact its semantics does not differ much from those of (FQJ) and (FQM). But its rule of inference is that of (SQ) and hence differs from those of (FQJ) and (FQM). By not permitting vacuous quantification it differs from (FQH) and (FQQ). In virtue of permitting open formulae to appear as theorems it is similar to (FQM).

Now let a formula A of (FQS) be such that no other variable occurs free in A. The soundness of (FQS) requires that

1. $V^*(Ax_i) = V^*((x_i)\, Ax_i) = V^*(Ax_i \rightarrow (\exists x_i)\, Ax_i)$
2. $V^*(Ax_i) = T$, and
3. $V^*((\exists x_i)\, Ax_i) = F$.

As a result, in (FQS) $\rightarrow$ is not only equivocal in the sense mentioned earlier, but also ceases to be a truth-functional connective when the domain of interpretation is empty.

Leblanc and Thomason [43] have constructed completeness proofs for ten systems of (FQ); in five of these, restrictions are placed on the rule of inference. However none of those five systems incorporate the kind of restriction placed on the rule of inference in (FQJ). Also proved complete by them is a system which in similar to (FQL) of Lambert [36] which can be axiomatised with the following axioms and rule of inference.

AL. 1 Ah. 1

AL. 2 $(x_j)(Ax.3)$

AL. 3 $Ax.2$

AL. 4 $(A(x_j) \rightarrow Bx_j) \rightarrow (Ax_j \rightarrow (x_j)\, Bx_j)$.

RIL R1.

Equivalently, we can accept Ax. 1 and make some alterations in the fourth axiom. Lambert is not clear on this point, though it affects the system in a significant way. In any case, this system has as its theorems only those formulae which either are closed tautologies or else are tautologies whose atomic subformulae are closed formulae. (FQH. 1) of [21] and (FQL) have the same set of theorems. (FQH. 1) is determined by Ah. 1, Ah. 2, AL. 4, **RIh** and $\vdash (uc)(Ax_i \rightarrow (x_i)\, Ax_i)$. Material implication is equivocal in this system too. On this count none of these systems can be considered as a proper reformulation of (SQ). These systems can at most be considerations as alternatives to (SQ).

4 'Exists' as a Predicate

Most of the systems which we shall group together in this section are either inspired by Leonard [45] or else have been developed as alternatives to his systems. Though they are intended to be taken as (FQ)s, strictly speaking they are not reformulations of (SQ) in the sense of being its subtheories. They are subtheories of a suitably augmented (SQ), say (SQ′) for short. Assuming that the vocabulary and the formulae of (SQ) are appropriately augmented to result in (SQ+), let (SQ+1) be the result of adding to (SQ+)

Ax. 0 $(\exists! x_j) \equiv (EP)(Px_i \;\&\; M{\sim}Px_i)$

as an additional axiom schema, or as a definitional equivalence. Here P is to be read as 'is possible', or 'possibly'. Leonard [45] suggested that (FQLE), which is determined by the set of axiom schemata and rule of inference of (SQ+1) with the only difference that Axiom 3 is replaced by

Ale. 3 $(x_i)(Ax_i \rightarrow (\exists! x_i \rightarrow Ax_j)$ for $i \leq j$,

be treated as a suitable reformulation of (SQ). But his opens its doors to quantification over predicates and to modal notions. There are reasons for considering it as a second order modal system; as such it is an uneconomical way of reformulating (SQ). But then, is there any better way of reducing the cost? Towards an answer to this Lambert [35] suggested (FQL.1), which is a reformulation of (SQ+2) arrived at by adding to the vocabulary of (SQ)s a monadic predicate $\exists!$ which is to be read as 'Exists'. (FQL.1) is to be axiomatised with the standard rule of inference and the following schemata.

ALI. 1 Ax. 1

ALI. 2 Ax. 2

ALI. 3 Ax. 5

ALI. 4 $(\exists\, x_i)Ax_i \rightarrow (\exists! x_i \;\&\; Ax_i)$

ALI. 5 $Ax_j \rightarrow (\exists! x_j \rightarrow (\exists x_i)\, Ax_i)$.

An equivalent to this is (FQML) of [38], which has the same rule of inference but a slightly different set of axiom schemata, namely,

AmL. 1 Ax. 1

AmL. 2 Ax. 2

AmL. 3 $(x_i)(A \rightarrow B) \rightarrow ((x_i)A \rightarrow (x_i)B)$

AmL. 4 $A \rightarrow (x_i)A$

AmL. 5 $(x_i)\, Ax_i \rightarrow (\exists! x_j \rightarrow Ax_j)$

AmL. 6 $(x_i)\, \exists! x_i$.

The completeness of (FQML) is worked out using supervaluations and partial interpretations due to van Fraassen. $(x_i)\exists! x_i$, which figures in AmL. 6, comes out as a theorem in (FQL.1). This sounds Eudoxian in the sense that it assumes what needs to be accounted for but has somehow become recalcitrant. The interpretation over the empty domain for which (FQL.1) comes out sound differs from the interpretation of (FQM) and (FQS). To spell out this difference, let V^* and V be as before. Then in the interpretations of (FQM) and (FQS) the following hold:

i) $V^*(A) = T$ when A is Bx_i

ii) $V^*(A) = \sim V^*(B)$ when A is $\sim B(x_i)$

iii) $V^*(A) = V(V^*(B(x_i)) \to V^*(C(x_i)))$ when A is $Bx_j \to C(x_i)$

iv) $V^*(A) = V(Bx_i \to Bx_i)$ when A is $(x_i)Bx_i$, and

v) $V^*(A) = V(Bx_i \ \& \sim Bx_i)$ when A is $(\exists x_i)\,Bx_i$.

In contrast to this, in interpretations of (FQL.1) over the empty domain, instead of (*i*)–(*v*) above, the following (1)–(6) hold:

1) $V^*(A) = V(A)$ if A is Bx_i

2) $V^*(A) = V(\sim V^*(B))$ if A is $\sim B$

3) $V^*(A) = V(V^*(B) \to V^*(C))$ if A is $B \to C$

4) $V^*(A) = V(Px_i)$ if A, where $\exists!x_i$ is a monadic predicate of (SQ+2) but not of (FQL.1)

5) $V^*(A) = V((x_i)(Px_i \to V^*(Bx_i)))$ if A is $(x_i)Bx_i$, and P is as in 4)

6) $V^*(A) = V((\exists x_i)(Px_i \ \& \ V^*(Bx_i)))$ if A is $(\exists x_i)Bx_i$ and P is as in 5).

A simplication of the assumption involved in (4)–(6) with reference to the monadic predicate results in a system which does not include in its vocabulary the predicate constant $\exists!$. But it will be one which contains in its vocabulary a dyadic predicate constant, namely $=$. This means that a modification of (FQL.1) is possible so that assumption involved in (4)–(6) can be lifted; but the result of such a modification would tantamount to be a reformulation of an extension of (SQ) and not (SQ) itself.

To note another kind of futile attempt to solve the problem at hand, let (SQ+3) be the result of augmenting the vocabulary of (SQ) by monadic predicates A^i, $i = 1, 2, 3, \ldots$ such that $A^i x_j$ is well-formed only if $i = j$. I [57] suggested a (FQ) with this device. The defect of this and similar systems — for example the one suggested by Scales [71] which we shall consider when we come to attributional logics — is that when they are extended by augmenting them with identity axioms, $\exists!$ and A^i become dispensable. These systems will in effect be the same as the (FQH) of Hintikka [26], with the minor difference that in these the monadic predicates can be eliminated in terms of self-identity, whereas in Hintikka's system they can be eliminated in terms of identity with others. That $\exists! x_i \equiv (\exists x_j)(x_j = x_i)$ comes out as a theorem in (FQL.1) with identity axioms added to it, was shown in [35]. Leblanc [40] hinted at the following modification in (FQML) so that the resultant (FQMLL) could, as he thought, easily be 'the most satisfactory "free logic" ... to have been proposed'.

(FQMLL) can be completely axiomatised by the following schemata and rule of inference:

AmLL. 1 Ax. 1

AmLL. 2 $A(x_i) \equiv (x_i)\, Ax_i$

AmLL. 3 $(x_i)(A \rightarrow Bx_i) \rightarrow ((x_i)A \rightarrow (x_i)Bx_i)$

AmLL. 4 $(x_i = x_j) \rightarrow (Ax_i \rightarrow Ax_j)$

AmLL. 5 $(x_i)(\exists x_j)(x_j = x_i)$

AmLL. 6 $x_i = x_i$

RImLL. 1 RI.

RImLL. 2 If $\vdash A$, then $\vdash (x_i)A$.

This modification is guided by two factors, the first of which is the eliminability of $\exists!$. The second factor is the belief that vacuous quantification is not, after all, a cardinal sin. In (FQML), $(x_i)A$ is provable as a theorem when A firstly does not contain a free variable, secondly is a theorem of (SQ), and thirdly is not a theorem of (FQML). For example, though, as in (FQQ), $(x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i$ is not a theorem, its universal closure $(x_i)((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i)$ turns out to be a theorem. Because of vacuous quantification, the soundness of (FQMLL) requires that

i) $V^*((x_i)\, A(x_i)) = V^*(A(x_i))$ and

ii) $V^*((x_i)\, Ax_i) \neq V^*(Ax_i)$.

Thus Leblanc's claim about the maximum satisfactoriness of his system seems to be untenable, and it is so even if we ignore the fact that it is a reformulation if (SQ=) and not (SQ). Leblanc together with Meyer gave two other systems of (FQ). One of them, in [42], shares the same features as (FQMLL); the second system, (FQLeM), in [41], is the outcome of an intelligent exploitation of a lemma in the completeness proof of (FQM) in [49]. Leblanc's claim does not hold even for this system since it contains an artificial condition on one of the axioms. Its axioms are the same as those of (SQ) except that instead of Axiom 3 it has

Alem. 3 $\vdash (\exists x_i)(Px_i \rightarrow Px_i) \rightarrow$ Ax. 3.

5 An Attempt to Overcome Some Defects

Kearns in his three alternative formulations of (FQ) tries to overcome some of the defects in the reformulations of (SQ) which we have noted so far. Of these, two are sketched in [31]. (FQK.1) does not permit open formulae as theorems unless they are truth-functional tautologies. The rules of inference and the axiom schemata of this system are:

Ak. 1 Ax. 1

Ak. 2 $(x_j)((x_i)\,Ax_i \rightarrow Ax_j)$

Ak. 3 $(x_j)(Ax_j \rightarrow (\exists x_i)\,Ax_i)$

Ak. 4 $(x_i)(x_j)(Ax_i \rightarrow Bx_j) \rightarrow (x_j)((\exists x_j)\,Ax_j \rightarrow Bx_j)$ provided x_i does not occur in B.

Ak. 5 $(x_i)(A \rightarrow Bx_i) \rightarrow (A \rightarrow (x_i)\,Bx_i)$, provided A does not contain x_i, and contains no free variables.

Ak. 6 $(x_i)(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)\,Ax_i \rightarrow (x_i)\,Bx_i)$

RIk. 1 RI.

RIk. 2 Rule for change of bound variables.

Kearns has three valid claims concerning (FQK.1). Firstly, its theorems are valid in all domains; secondly A is a theorem only if it is valid in all domains and does not contain any free variable; and thirdly the addition of $(\exists x_i)(Ax_i \rightarrow Ax_i)$ as an axiom will produce a system in which each A which is valid in non-empty domains and which does not contain free variables will come out as a theorem. This is to say that (FQK.1) so augmented, will be coextensive to (SQ). There is, however, some ambiguity here, as there was in the case of (FQL). It is not clear whether only closures of tautologies containing free variables are theorems or only tautologous formulae whose atomic subformulae are theorems. If the latter is the case then $A \rightarrow A$ will be a theorem even if A is Bx_i. However, in the light of what we noted about this system, specially Kearns' claims, we might alter Kearns' axiom Ak. 1 and AL. 1. Then the claim would still be valid, and formulae like

1. $Bx_i \rightarrow Bx_i$ and
2. $(\exists x_i)\,Ax_i \rightarrow Ax_j$

which are valid in both the empty and the non-empty domains will cease to be theorems while their closures, namely

1.a $(x_i)(Bx_i \rightarrow Bx_i)$ and

2.a $(x_j)((\exists x_i)\, Ax_i \rightarrow Ax_j)$

would turn out to be theorems. In this respect this system is unsatisfactory. The ambiguity is the result of Kearns' own remarks. On the one hand he says that the axioms of his system contain "axioms common to propositional calculus", and on the other he holds that a "substitution in the tautology

$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

yields

$$((x_j)Ax_j \rightarrow Ax_i) \rightarrow ((Ax_i \rightarrow (\exists x_i)\, Ax_i) \rightarrow ((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i))."$$

Kearns' (FQK. 2) is a simplification of (FQQ) and is arrived at replacing Aq. 2 by Ak. 2.2 $\vdash (UC)((x_i)(A \rightarrow Bx_i) \rightarrow ((x_i)A \rightarrow (x_i)Bx_i))$ and RIQ by RI. As vacuous quantification is not permited, Ak. 2.2 indeed is Ak. 2.2a which is:

$$(UC)((x_i)(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)A \rightarrow (x_i)\, Bx_i)).$$

In this system

$$(x_i)((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i),$$

which is a theorem of (FQQ), cannot be proved. We can go on to

$$\vdash (x_i)((x_i)\, Ax_i \rightarrow Ax_i)$$

$$\vdash (x_i)((x_i)\, Ax_i \rightarrow Ax_i) \rightarrow ((Ax_i \rightarrow (\exists x_i)\, Ax_i) \rightarrow ((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i))$$

$$\vdash (x_i)((x_i)\, Ax_i \rightarrow Ax_i) \rightarrow ((x_i)((Ax_i \rightarrow (\exists x_i)\, Ax_i) \rightarrow ((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i)))$$

$$\vdash (x_i)((Ax_i \rightarrow (\exists x_i)\, Ax_i) \rightarrow ((x_i)\, Ax_i \rightarrow (\exists x_i)\, Ax_i))$$

and not beyond that, for we cannot make use of Ak. 2.2 because the consequent of the last sequent in the proof does not contain x_i free.

Kearns' (FQK.3) is more complicated in its formulation than his earlier systems, but we shall simplify it as much as possible. The definition of a formula in (FQK.3) has, besides the usual conditions, an additional condition to the effect that if A is well-formed and is

$(x_i)B$, then B must be a formula containing x_i free. If A is $B \rightarrow C$, then $B \rightarrow C$ should not contain both and free occurrences of the same variable. Further, when A is a propositional variable and $(x_i)B$ is a well-formed formula, then $A \mp (x_i)B$ is a well-formed formula where $\mp$ is any normal dyadic truth-functional connective. Its axioms and rules of inference are:

Ak.3.1 Ax. 1

Ak.3.2 Ak. 2

Ak.3.3 $\vdash (ac)$ Ak. 5

Ak.3.4 $\vdash (ac)((x_i)(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)Ax_i \rightarrow (x_i)\,Bx_i))$

RIk.3.1 RI

RIk.3.2 If $\vdash A$ and A contains a propositional variable B as a subformula, then $\vdash (ac)A^*$ where A is the result of replacing B in each and all occurrences of it in A by a formula C provided A and C do not have common variables occurring in them.

RIk.3.3 If $\vdash (ac)\,Ax_i$, then $\vdash (ac)\,Ax_j$

RIk.3.4 If $\vdash A$ then $\vdash (ac)A^*$, where A^* is the result of replacing each and all occurrences of an m-adic predicate occurring in A by an m-adic predicate $(m < n)$.

This system, though it contains a large number of complicated rules, does meet the claims of Leblanc. The only defect with it is in its placing a total ban on free variables; tautologous formulae which are naturally treated as "logical truths" will not be theorems of (FQK.3). That ban makes the introduction of singular terms into its framework a more complicated affair than is the case with some of the systems which we considered earlier. This is evident from Kearns' own theory of description which he adds to (FQK.3). Moreover, when free variables are treated in this way, a problem arises in connection with $\rightarrow$. What is the set which $\rightarrow$ is supposed to order? Is it the set of closed formulae, or is it the set of formulae itself? If it is the latter set, then $Ax_i \rightarrow Ax_i$ will be a truth which will not be a theorem, and if it is the former set, then the notion of an atomic formula needs to be so redefined that each atomic formula will be a closed formula and only closed formulae will be atomic. Similarly unclear is the notion of a propositional variable in (FQK.3).

6 Another Attempt to Overcome Another Set of Known Defects

A system which can be construed as a reformulation of (SQ) in the strict sense of the term, that is in the sense that it is a subtheory of (SQ), and which does not resort to vacuous quantification, does not restrict the rule of inference in way, and does not turn the truth-functional connectives into non-truth functional ones in interpretations over the empty domain, but does permit open formulae to occur as theorems without accepting that $V^*(Ax_i) = V^*(\sim Ax_i)$, is indeed forthcoming. My [61] gives one such system. This (FQR) has the same vocabulary as (SQ), and its formulae are the same as the formulae of (SQ). The set of axiom schemata and rules of inference determining (FQR) is:

Ar. 1 Ax. 1.

Ar. 2 $(x_i)(A(x_i) \rightarrow B(x_i) \leftrightarrow (A(x_i) \rightarrow (x_i)\, Bx_i)$

Ar. 3 $(x_i)(Ax_i \rightarrow Bx_i) \leftrightarrow ((x_i)\, Ax_i \rightarrow (x_i)\, Bx_i)$

Ar. 4 $(x_i)(Ax_i \,\&\, Bx_i) \leftrightarrow ((x_i)\, Ax_i \,\&\, (x_j)\, Bx_i$ where $i \leq j$

Ar. 5 $(\exists x_i)\, Ax_i \rightarrow Ax_j$, provided A does not contain as a subformula a formula of the form, for $i = j$, $(\exists x_j)\, Cx_j$

RIr. 1 RI.

RIr. 2 If $\vdash A$ then $\vdash A^*$ where A is just alike A^* except that A contains as a subformula B where or wherever A^* contains C as a subformula, and $B \leftrightarrow C$.

To handle the notion of soundness of (FQR), we let its interpretation and the interpretation of (SQ) be the same when the domain of interpretation is non-empty. The interpretation of Ar. 1 is the same whether the domain is empty or nonempty. This means that their sentential logics are the same. When a formula A is either a truth-functional tautology, or is valid under standard interpretation — that is under interpretation of (SQ) over non-empty domains — we shall say that A is *standard-valid, S-valid* for short. Now let M^* be a unary function with arguments as well as values in the formulae set of (FQR), thus of (SQ), such that

1. $M^*(A) = A$ if is atomic,

2. $M^*(A) = \sim M^*(B)$ if A is $\sim B$,

3. $M^*(A) = M^*(B) \to M^*(C)$ if A is $B \to C$,

4. $M^*(A) = (Bx_i \to Bx_i)$ if A is $(x_i)Bx_i$ where Bx_i is (Qx_i) $\ldots(Qx_{i-1})\ldots(Qx_n)C$ and C is atomic, provided for each j, (Qx_j) is $x_j)$ and $n \geq 0$.

5. $M^*(A) =\sim M^*((x_i) \sim Bx_i)$ if A is $(\exists x_i)Bx_i$ where Bx_i is $(Qx_1)\ldots(Qx_{i-1})\ldots(Qx_n)C$ and C is atomic, provided for each j, $)Qx_j)$ is either $(\exists x_j)$ or (x_j) and $n \geq 0$.

6. $M^*(A) = (Bx_i \to Bx_i)$ if A is $(x_i)Bx_i$, and Bx_i is molecular, and

7. $M^*(A) =\sim M^*(x_i) \sim Bx_i$ if A is $(\exists x_i)Bx_i$ and Bx_i is molecular. Where V and V^* are valuation functions over the non-empty and empty domains respectively, and A is a formula of (SQ) and hence of (FQR),

I. $V^*(A) = V(\mathbf{A})$ where $\mathbf{A}$ is the associated formula of A (i.e. the result of deleting the individual variables occurring in A and correlating the predicate letter occurring in it to a propositional variable $\mathbf{A}$) if A is atomic, and

II. $V^*(A) = V(M^*(A)$ if A is either a closure of an atomic formula, or is a molecular formula.

A formula of (FQR) is valid under interpretation over the empty domain if and only if $M^*(A)$ is S-valid; and if $M^*(A)$ is S-valid we shall say that A is *M-valid* (short for Mostowski-valid, as it is due to Mostowski [49]). A is said to be *universally valid, U-valid* for short, if and only if A is valid under interpretations over all domains, be they empty or non-empty. A is U-valid if and only if A is *i)* S-valid and *ii)* M-valid. This means that A is U-valid if and only if both A and $M^*(A)$ are S-valid.

Though the definition of M^* above is informed by [49], it is not the same as the definition of the corresponding function in (FQM). Mostowski leaves $M^*(A)$ indeterminate when A is atomic; he does not provide for any value-assignment specifically for atomic formulae. In his interpretation of (FQM), as in the interpretation of (FQS) of Schneider [72], an atomic formula as well as its negation both have the same truth-value when the domain is empty, i.e. they have no truth-value. This in a sense strips off the sign of negation its truth-functionality in interpretations over the empty domain and robs all legitimacy in taking the sign of negation as something that corresponds to denial. In fact in such interpretations the sign of negation functions equivocally as it behaves in one way when it occurs negating open formulae and in another way when it occurs negating closed formulae. The import of what is implicit in this equivocation is that there are two kinds of negation, namely, to borrow the terminology of [71], attributional negation and propositional negation, or internal negation and external negation. We shall consider these distinctions are totally extraneous to (SQ)s. On the other hand, the interpretation offered to (FQR) is such that neither $V^*(A) = V^*(\sim A)$, nor $V(A) = V(\sim A)$, and where $\mathbf{V}$ is either V or V^*, $\mathbf{V}(A) =\sim (\mathbf{V}(\sim A))$, and $\mathbf{V}(\sim A) =\sim \mathbf{V}(A)$; this shows that in (FQR), as in (SQ), there is just one unequivocal truth-functional negation. Further,

(i) $Ax_i \rightarrow (\exists x_i)\, Ax_i$

(ii) $(x_i)(Ax_i \rightarrow (\exists x_i)$

(iii) $(x_i)\, Ax_i \rightarrow Ax_i$ and

(iv) $(x_i)((x_i)\, Ax_i \rightarrow Ax_i)$

are all U-valid in (FQM) and (FQS), whereas only *ii*) and *iv*) are U-valid in (FQR).

The completeness of (FQR) can be established, as has been shown in [61], by using the above interpretation and working out an analogy to the completeness proof of (FQM) in [49]. This system has the virtue of (FQK. 3); the addition of a single axiom schema to the effect Ax. 0 $\vdash (\exists x_i)\, Ax_i$ gives us (SQ), showing where the shoe really pinches and how the set of theorems of (FQR) is a proper subset of the theorems of (SQ).

7 Singular Existence Assumptions

(FQR) is free from the requirement that its individual variables should have referents in the domain of interpretation. And if we add to its vocabulary individual constants a_i to result in (FQR+) and allow the resulting system to be one in which individual constants can replace individual variables, the soundness of the system is not affected. This will so because in it free variables in effect behave like individual constants; that is to say they are dummy names. This is not the case with a similar extension of (SQ), for it has existence assumptions with respect to individual variables as well as individual constants. These are to be called General Existence Assumptions and Singular Existence Assumptions respectively. Because of the general assumptions of (SQ), it its normal interpretation, 'x_i exists' comes out true for each

i; thus for a specific sense of exists, '$(x_i)(x_i$ exists)'is valid in (SQ)s. The identity of the extension of 'exists', the range of the individual variables, and the entire domain of interpretation, make 'exists' a redundant predicate in (SQ)s. But this apparently has an anomalous consequence, namely within the framework of (SQ)s it is not possible to deny the existence of an entity without either ending up with a contradictory sounding assertion or else without committing to the existence of some other entity. This will be the case even when the entity in question does not exist. Even so ardent a defender of (SQ)s as [54] admits that in saying: "to say that something does not exist or that something which is not, is clearly a contradiction in terms; hence '$(x_i)(x_i$ exists)' must be true." He does not mention, maybe due to its obviousness, that the consequent of the sentence quoted implies the antecedent. This indeed is the case with (SQ)s.

As a way out of this predicament, Lejewski [44] suggested that, instead of reformulating (SQ)s, the standard referential interpretation of (SQ)s be dropped, and substitutional intrepretation of the same be offered. To note the difference between these two kinds of interpretations, let the universe of discourse be $D = \{c_1 c_2\}$. Let the discourse be a simple one consisting of three individual constants a_1, a_2 and a_3 such that a_1 and a_2 refer to c_1 and c_2 respectively, and a_3 has no referent. Let this discourse presuppose the rest of the apparatus of (SQ*). Then on referential interpretation the following hold in this discourse:

$$V((\exists x_i)\,Ax_i) = V(Aa_1 \vee Aa_2) \quad \text{and} \quad V((x_i)\,Ax_i = V(Aa_1 \;\&\; Aa_2).$$

For "if we think", writes Quine [55],"of the universe as limited to a finite set of objects $a, b, \ldots, k$, we can expand the existential quantifications into alterations and universal quantification into conjunctions." On the other hand, if substitutional interpretation were to be given to quantifiers, then this discourse vindicates the following:

$$V((\exists x_i)\,Ax_i) \;=\; V(Aa_1 \vee Aa_2 \vee Aa_3) \quad \text{and}$$
$$V((x_i)\,Ax_i = V(Aa_1 \;\&\; Aa_2 \;\&\; Aa_3).$$

These expansions show that the range of x_i in the referential interpretation is $\{c_1, c_2\}$ and in the substitutional interpretation it is $\{a_1, a_2, a_3\}$. Hence whereas in the former $(\exists x_i)\,Ax_i$ will have to be taken to mean 'there exists an x_i such that Ax_i', in the latter it will have to be unlinked from the notion of existence snf be taken to mean 'for some x_i, Ax_i'. On this count, Lejewskian (SQ)s come out free from anomalies and existence assumptions. As Lejewski remarks in (I), this "interpretation seems to be nearer to ordinary usage, for somehow we do not believe that everything exists and do not see a contradiction in that something does not exists. If logicians say that things are the other way around, it is because they follow" referential interpretation.

Yet (SQ)s with substitutional interpretation have been faulted, for instance by Cohen [14], on two counts. One of his objections is that as quantifiers in (SQ)s under substitutional interpretation are decoupled from existence, neither (SQ) nor any of its extensiosn will have rubrics for defining 'x_i exists', and hence the notion of existence will not be representable in them unless 'exists' is added as a primitive predicate. This obviously is not acceptable to many. His second objection to the Lejewskian reading of the quantifiers is that it leads to the Paradox of Predication in the empty universe. What he means by this is that when the universe of discourse, or the domain of interpretation, is empty no term has a referent, and hence there is no way of assigning different truth-values to Ax_i and $\sim Ax_i$, or Aa_i and $\sim Aa_i$ where A is monadic. Why Cohen talks of difficulties in assigning different truth-values, and not of assigning truth-values as such, will be evident when his assumption that if a_i has no referent, Aa_j is *exhypothese* true is taken into cognisance. He passes off this assumption as an argument against Lejewski. But before considering it further let us note the possible alternative ways of assigning truth-value to 1) Ax_i and 2) $\sim Ax_i$ when A is monadic and the domain of interpretation is empty:

(*a*) $V^*(1) = V^*(2) = T$ (as in (FQJ), (FQM), and (FQS)),

(*b*) $V^*(1) = F$ and $V^*(2) = T$ (Cohen),

(*c*) $V^*(1) = T$ and $V^*(2) = F$ (Hughes and Londey),

(*d*) $V^*(1) = V^*(2) = F$ (as in (FQH), (FQQ), and (FQG))

(*e*) $V^*(1)$ and $V^*(2)$ both devoid of a truth-value (Frege and van Frassasen)

(*f*) $V^*(1) = V^*(2) = U$ where U is a truth-value which is different from T and F (Woodruff),

(*g*) If $V^*(1) = T$, $V^*(2) = F$, and if $V^*(1) = F$, $V^*(2) = T$ (Rao).

With the exception of (*g*), all these alternatives are loaded with metaphysical assumptions, and from a purely logical point of view are towards oriented. Thus Lejewskian systems can be faulted on the count of involving extralogical assumptions, and on the count that they make it imperative to accept 'exists' as a primitive predicate. Now to say of some predicate that it is primitive is the same as to say that it is descriptive, "marking off a descriptive property" to borrow a phrase from Routley [65]. Routley attempts to get rid off the anomalies of (SQ)s without treating existence as a descriptive property. The next section will be devoted to his and related attempts.

8 Definitions of 'Exists'

If a predicate is not definable either in (SQ) or in (SQ=), then it may be said to be a descriptive predicate. It may be the case that this criterion determining what a descriptive predicate is needs a weakening to the effect that a predicate is descriptive only if it is not definable in (SQ=) so that 'x_2 exists' will not be a descriptive predicate within the framework of (SQ=). The difficulty here is that 'descriptive predicate' has not received any fixed universally accepted meaning in logico-philosophical literature. Opting for the least contestable meaning of the term, we resorted to the above stipulative definition. If it is insisted that the term in question means something different from what we took it to mean, we can safely drop that controversial term altogether, and use say P^* to designate a set of predicates such that each and only a member of the set of P^* is a value satisfying the formula 'x is a predicate such that each member of the domain of interpretation marked off by the predicate x'. Thus without entering into terminological quibbles, let us note that each member of $\{P^*\}$ is eliminable in (SQ=) by proving that it is coextensive with the identity predicate. Now let a predicate which is not a member of $\{P^*\}$ be called an undefinable predicate. As $\exists!$ is not eliminable in Lejewskian interpretation of (SQ)s, it is an undefinable one for him.

Routley [65] aims at accepting that $\exists!$ is definable and hence is eliminable in terms of $=$, while having a system which is free from the anomalies mentioned in the previous section. Like Lejewski, he delinks quantification and existence. This necessitates an introduction of non-standard quantifiers. He diagnoses the trouble with (SQ)s as due to the fact that in them on the one hand existence is defined away in terms of quantifiers and identity, and on the other open formula $\exists!x_i$ is allowed to be satisfied by each, and only one member of the domain of interpretation. This is the same as to say that because of referential interpretation, the variables and constants of (SQ)s are used to refer to only existent entities and the quantifiers are used to quantify over only existent entities.

To lift these restrictions and yet to accept referential interpretation as well as the definability of $\exists!$ is tantamount to using variables and quantifiers neutrally, that is to say that without any commitment either to the effect that the values are existent, or to the effect that the values are non-existent. This is to say that a neutral use of variables and quantifiers does not assume either the emptiness or the non-emptiness of the domain of interpretation. In that case, quantifiers so used will not be functionally the same as the quantifiers as they figure in (SQ)s. Similarly, the role of the variables in (SQ)s will not be the same as their role when they are used neutrally. The reason for this is that their respective ranges will be disparate and not coextensive. To preclude confusions, we will use (x_i^*) and $(\exists x_i^*)$ for the two neutral quantifiers, taking them to be shorthand versions of 'for all x_i' and 'for some x_i' respectively and reading them without any existential commitment in contrast with the quantifiers occurring in (SQ)s, namely (x_i) and $(\exists x_i)$, which are assured to be loaed existentially and are taken to be abbraviation of 'for all existent x_i' and 'for some existent x_i'.

Routley's system of Neutral quantification (NQ) is determined by the following complete and consistent set of axiom schemata and rule of inference.

AxN. 1 A where A is a two-valued truth-functional tautology.

AxN. 2 $(x_i^*)(A(x_i) \rightarrow Bx_i) \rightarrow (A(x_i) \rightarrow (x_i^*)\, Bx_i)$

AxN. 3 $(x_i^*)\, Ax_i \rightarrow Ax_j, \quad \text{for} \quad i \leq j$

AxN. 4 $Ax_i \rightarrow (x_i^*)\, Ax_i$.

RIN RI.

The vocabulary of (NQ) includes among its predicates ∃! and the notion of a formula is defined in such a way that it does not involve vacuous quantification. The following definitional equivalence hold in (NQ):

D1 $(\exists^* x_i)\, Ax_i \equiv\sim (x_i) \sim Ax_i$

D2 $(\exists x_i)\, Ax_i \equiv (\exists^* x_i)(Ax_i \;\&\; \exists! x_i)$, and

D3 $(x_i)\, Ax_i \equiv\sim (\exists x_i) \sim Ax_i$.

(NQ+) is to be obtained from (NQ) by augmenting its vocabulary by individual constants and by replacing AxN. 3 by AxN. iii $(x_i^*)\, Ax_i \rightarrow At_j$ where t_j is x_j as in Ax. 3, or else is a_j where, for $i \leq j$, a_j is *consistent*. It is expected to be consistent not in the formal sense that it does not contain a contradictory pair of assertions, but in the ontic sense that there is a monadic predicate P, or a diadic predicate Q, such that $(x_i)B_i$ or $(x_i)\, Q_{x_i x_j}$ is valid and Pa_j or $Qx_i a_j$ is true.

(NQ+=) is the result of adding = to the vocabulary of (NQ+) and augmenting its axioms by Ax. 6 and

Ax. vii $(x_i = x_j) \to (Ax_i \to A'x_j)$ provided A and A' are alike except that A' contains x_j free in just one place corresponding to the free occurrence of x_i in A, *and* x_j is not bound in A.

In (NQ+=) $\exists!$ becomes eliminable; consequently a simplification of AxN. iii will be possible due to the following theorems:

i) $(x_i^*(\exists^* x_j)(x_j = x_i)$, and

ii) $\exists! x_i \equiv (\exists x_j)(x_j = x_i)$

Thus AxN. iii can be replaced by

AxN. 3′ $((x_i^*)\, Ax_i \;\&\; (\exists^* x_i)(x_i = x_j) \to At_j$ where for $i \leq j$, t_j is either x_i or x_j.

D. 2 can now be replaced by D. 2′ $\exists x_i)\, Ax_i \equiv (\exists^* x_i)(Ax_i \;\&\; (\exists x_i)(x_j = x_i))$. The replacement of AxN. 3′ is made possible by the fact that within the framework of (NQ+=) with the interpretation for which it is consistent and complete, the consistency of a term and self identity of the referent of the term go together. So if in (SQ)s self-identity and existence get telescoped, in (NQ)s self-identity and possibility get telescoped. Prior to a consideration of the semantics proposed by Routley for (NQ+=), let us note that

(1) if (x_i^*) is replaced by (x_i) in (NQ), and if the resultant system is provided standard interpretation, we will have (SQ), and

(2) if (x_i^*) is replaced by $x(x_i)$ in (NQ+) and the proviso in AxN. iii namely that a_j must be consistent be altered to that a_j must have a referent in the domains of interpretation, we will have (SQ+) with its standard interpretation; this will tantamount to incorporating the suggestion by Quine that we should make sure that a term be a referring one before we use ? as a substituend.

Routley rightly claims that there are several subsystems of (NQ+=) which are related with some of the (FQ)s that we have considered so far. Some of these are only subsystems of the relevant (FQ)s and hence are not of much interest except for the fact that in these systems, $x_i = x_i$ does not hold universally, but only when x_i is consistent. Thus though Socrates and Pegasus both would be self-identical roundsquares and squarecircles would not be that fortunate. There must be some property which enables Socrates and Pegasus to have this property, and in virtue of which roundsquares do not have this property. This makes one sceptical about Routley's effort. After all, we are naturally inclined to treat a predicate that sets apart one entity from another as a descriptive predicate. It is on this count that $\exists!$ was rejected to figure in "logic"; so how can self-identity be allowed to determine what is logical? Thus it is possible to legitimately demand a further reformulation of (NQ)s such that self-identity can be defined away. (Anyhow, for one of the subsystems of (NQ) which can be arrived at by replacing C^*x_i) by (x_i) and AxN. 3 by AxN. 3.1 $((x_i)\,Ax_i \ \&\ \exists!x_j) \rightarrow Ax_j$, for $i \leq j$, Routley claimed coextensiveness with Leonard's (FQLe); but [79] proved this system to be incomplete.)

In working out the interpretation of (NQ+=), Routley makes use of the distinction between what is possible and what is existent (or what is actual); this means that for him the set of existents is a possible empty proper subset of the set of possibles. The interpretation of (NQ+=) is carried over the model $\langle D, D^*, R, E^*\rangle$ where D is a non-empty set and constitutes the range of the variables and constants. D^* is a proper subset of D such that $(x_i)\exists!x_i$ is valid under interpretations over D^* *only*, and R is a set of relations defined over $D \cdot E^*$ is a monadic predicate defined over D and is correlated to $\exists!$. With this difference in the selection of models, the interpretation of (NQ+=) is carried over this model just as the interpretation of (SQ+=) is carried over $\langle D^*, R, \exists^*\rangle$. As $\exists^*$ and $\exists!$ are uniquely correlatable in the interpretation of (SQ+) over $\langle D^*, R, \exists^*\rangle$, there exists a monadic predicate $M!$ such that $(x_i^*)\,M!x_i$ would turn out to be valid in (NQ+=). $M!$ is a short for 'is a possible entity'. We know that the interpretation of (SQ+=) over $\langle D^*, R, I\rangle$ where I is a diadic predicate defined over D^* and is uniquely correlatable to =. Analogously the interpretation of (NQ+=) over $\langle D, D^*, R, E^*\rangle$, $\langle D, D^*, R, E^*, M^*\rangle$, $\langle D, D^*, R, E^*, I\rangle$, and $\langle D, D^*, R, J\rangle$ are identical. The analogy does not stop there for if $(\exists x_i)(x_i = x_i)$ valid in (SQ+=) under interpretations over $\langle D^*, R, I\rangle$. As such, (NQ+=) is committed to non-empty domains as much as (SQ+=) is; the only difference is that the latter is committed to non-empty domains of existent entities, and the former to non-empty domains of possible entities; it is not committed to non-empty domains of existent entities also for the subset D^* of D may be empty without adversely affecting the system. Then if (SQ+=) has existence-assumptions, (NQ+=) has possibility assumptions. If so, the choice between them is not a choice between ontology-free and ontologically committed logic; it is a choice between two ontologies. [66] holds that of the two ways of reformulating (SQ), namely either as (FQ)s or as (NQ)s, he is not sure that it is clearly decidable "which reformulation of classical logic is most preferable", but that (NQ)s are "richer than" (FQ)s.

Let us recall the motive behind the attempts at reformulating

(SQ)s. It is the desire to dispell the uneasiness felt in connection with 1) the requirement that individual variables should have referents, and 2) the requirement that individual constants should have referents. Besides that general uneasiness, there is some specific uneasiness, and it arises because these referents are required to be existent (in a uniquely defined sense of the term). It is obvious that if we get rid of our uneasiness with reference to 1) and 2), we will be rid of the uneasiness due to 3). A scrutiny of (NQ)s has shown that getting rid of the uneasiness due to 3) will not set us free from the uneasiness on account of 1) and 2). (NQ)s, as Routley formulates them, are strategies to set (SQ)s free from 3) only; and these require that the individual variables and constants to have referents. The minimum requirement that these should have referents is common to (SQ)s as well as to (NQ)s. Then, if the interpretation of terms which do not have existent entities as their referents leads to troubles in (SQ)s, the incorporation of terms which do not have even possible entities as their referents leads to problems in (NQ)s. If Pegasus is troublesome to (SQ)s, roudsquares are so for (NQ)s. The restriction placed on terms to be used as instances of Universal Instantiation in (NQ)s is as *ad hoc* or expediency oriented as is the suggestion of Quine, for instance. Surely, Routley will not be content with (*SQ*)s which are exactly like (SQ)s except that instead of Ax. 3 they include *AX.* 3 which reads:

> $(x_i)\,Ax_i \rightarrow Ax_j$ where, for $i \leq t_j$ is either x_j or a_j such that a_j has a referent uniquely correlatable to it, and is a member of the domain of interpretation.

The difference between Ax. 3 and *AX.* 3 is that the latter incorporates a proviso corresponding to the proviso in AxN. 3. If Routley were to be satisfied with (SQ)s, then there would be no need for (NQ)s. He will dissatisfied with them for the reason that, even after such a modification in Ax. 3, unintended consequences will not have been precluded. As Pegasus does not exist, one is naturally induced to believe that he can legitimately assert that Pegasus does not exist. But when once he makes an assertion to that effect — while subscribing to (SQ)s, that is — he will have asserted that Pegasus is not (say) Socrates, in fact that pegasus is not anything which is included in the domain of interpretation. This is not only unacceptable to Routley but also may not have been intended by him when he asserted that Pegasus does not exist. The same is the case with reference to an assertion of the impossibility of roundsquares within the framework of (NQ)s. If one asserts that roundsquares are impossible objects, (NQ)s will impute to him the assertion that a roundsquare is not Pegasus, say. Should we then reformulate (NQ)s using still more general quantifiers which are free from existential and possibility import? To these we can find objections which are similar to our objections to (NQ)s.

The upshot of this is that, as Lambert remarked, "nowhere in philosophy is there a more nuisance statement than Singular negative existential". Our discussion of (NQ)s has shown that by multiplying kinds of entities — possibles, existents, ... — we will not have set ourselves free from that nuisance. An alternative to (NQ)s which resort to such a multiplication was sketched by Cocchiarella [10] in his system (APQI+), which is an intermediary between (NQ)s and Attributional Systems which will be considered in the next section. (APQI+) has, like (NQ)s, two pairs of quantifiers, and is sound and complete with respect to interpretations over $\langle D, D^*, R, I\rangle$ where $D^* \subseteq D$, and both D^* and D are non-empty. The range of $(\bar{\bar{x}}_i)$ is D^*, and the range of (x_i) is D. In this system the following equivalences hold.

$$\sim (x_i) \sim Ax_i \equiv (\exists x_i)\, Ax_i, \;\sim (\bar{\bar{x}}_i) \sim Ax_i \equiv (\bar{\bar{\exists}} x_i)\, Ax_i$$

$$\exists! x_i \equiv (\exists x_i)(x_j = x_i), \quad \text{and} \quad M! x_i \equiv (\bar{\bar{\exists}} x_j)(x_j = x_i).$$

As $(\exists x_i)\, Ax_i \rightarrow (\bar{\bar{\exists}} x_i)\, Ax_i$ holds and $(\bar{\bar{\exists}} x_i)\, Ax_i \rightarrow (\exists x_i) Ax_i$ does not hold in this system, and as $D^* \subseteq D$ where both D^* and D are non-empty, D will have to contain at least two members only one of which is a member of D^*. Thus if (SQ)s require that at least one entity be there, (APQ)s require at least one existent entity and also at least one possible entity.

9 Attributional Logics

The systems of reformulations of (SQ)s that will be considered in this section differentiate between what are called Propositional Negation and Attributional Negation, and hence are, in the terminology of Scales [71], Attributional Logics. In this sense of the term the system of Hughes and Londey [29] is an attributional logic, but we shall not take into account as it is sound and complete only for interpretations over the empty domain. It is the logic of the empty universe, which is of little interest.

We shall call systems which are free from existence assumptions with respect to individual variables as well as individual constants Universally Free System. Two universally free (AQ)s are due to Potter [50] and Scales [71]. Potter thinks that the defects of (SQ)s arise in those systems because the "use of the tilde ... has blurred its dual role, on the one as operating on propositions, and on the other as operating on predicates. In the former capacity, application of the tilde has the effect of forming the denial of the proposition so treated. In the latter capacity, treating predicates by applying the tilde forms another predicate. Confusion arises when the tilde is applied to singular proposition without distinguishing these two roles". If Potter credits Dummett for having discovered the possibility of distinguishing the sorts of negation corresponding to the two roles it performs, Scales rightly traces the source of this distinction to the medieval differentiation between *de dicto* and *de re* occurrences of (modal) operators, and also to the following passage in Aristotle's *Categoria*. "The statement 'Socrates is ill' is contrary of 'Socrates is well'. Yet we cannot always maintain ... that one statement must always be true and the other is false. But if Socrates does not exist, both one and the other are false. To say 'he is ill' will be false, and to say 'he is well' will be false if no Socrates so much as exists."

Of the two (AQ)s mentioned above, only (AQ+=S) of Scales [71] is fully developed and its principal metatheorems established. Potter's (AQ+P) on the other hand is sketchy; but he gives the minimum information required, and exploiting it we can specify it. In that direction we may note that Potter permits the elimination of names in favour of variables *à la* Quine; hence all that is needed is a specification of (AQIP). It should also be noted that (AQIP) existence is married off to quantifiers, and so, as in (SQ=), $\exists!$ can be dispensed with in terms of identity and quantifiers. In terms of syntax, (AQIP) differs with (SQ= only in incor[orating a notion of atomic formulae which is wider than the corresonding one in (SQ=). Where A is an atomic formula of (SQ), both A and $\sim A$ are atomic in (AQIP). In order to spell out the interpretation of (AQIP), let C be a unary function with arguments as well as values in the set of formulae of (AQIP) such that when A is a formula

$$C(A) = A \rightarrow A, \text{ if } A \text{ is } x_i = x_i, \text{ and otherwise } C(A) = A \text{ itself.} \quad (1)$$

When the domain of interpretation is empty, V^* and V are earlier, and A a formula,

i) if $C(A)$ is not A itself, then $V^*(A) = V(A)$, and

ii) if $C(A)$ is A itself, then

1. $V^*(A) = F$ if A is atomic
2. $V^*((x_i)\,Ax_i = F$,
3. $V^*(A) = V(V^*(B)\overline{\pi}\, V^*(C))$ if A is $B\overline{\pi}$ where $\overline{\pi}$ is diadic two-valued truth-functional connective, and
4. $V^*(A) = V^*(\sim V^*(B))$ if A is $\sim B$.

When the domain of interpretation is non-empty, (AQIP) is to be interpreted as (SQ=) is interpreted over non-empty domains.

Potter is not clear as to the way in which an atomic formula containing an occurrence of the tilde is to be interpreted. At one place he suggests that the relationship between the atomic formulae A and $\sim A$ is the same as the relationship between any two atomic formulae B and C, and at another place remarks that the values for which the atomic formulae A and $\sim A$ come out to be true may be "construed" to be constituting two classes α and β respectively such that β is "exclusive of any or all members of α". As to this ambiguity, that is as to whether the predicates of (AQIP) are related or are independent, [87] has argued that the former count the tilde is "unintelligible and certainly of no value claiming ordinary scientific denials", and if they are independent then the tilde is "useless".

There are two basic assumptions of attributional logics. The first of these is that to define a predicate over a set is not to define its complimentary predicate over that set, for there is no logical connection between the extensions of these two predicates. The second assumption is that unexemplified predicates and non-referring terms are functionally similar. So securing exemplication for each predicate is in effect equivalent to securing reference to each term. Potter is relying on the first assumption, and Scales is banking on the second. That Potter subscribes to the first assumption is evident from his explicit assertion that when A is atomic formula of (SQ), both A and $\sim A$ are atomic formulae of his system.

It can be noticed easily that in (AQIP) the law of excluded middle and the law of double negation do not hold in general. Though $\sim\sim A \vee \sim A$ and $\sim\sim\sim A \equiv \sim A$ hold without restrictions, $A \vee \sim A$ and $\sim\sim A \equiv A$ hold only when A is atomic. In fact the law of double negation cannot be proved as a theorem without violating the uniformity condition on the substitution of formulae for propositional variables in tautologies. Thus (AQIP) can be considered as an alternative reformulation of (SQ=) incorporating one of the alternatives suggested by [28]. And (AQIP) can be axiomatised by adding the following two axiom schemata to the axiomatic base of (SQ=); to specify these schemata, let C^* be a unary function with C^* argument as well as values in the set of formulae of (AQIP) such that when A is a formula of (AQIP)

1. $C^*(A) = A$ itself if A is atomic,

2. $C^*(A) = B$ if A is $\sim B$, and A is molecular, and

3. $C^*(A) = C^*(B) \rightarrow C^*(C)$ if A is $B \rightarrow C$

AX.1.i $(\exists x_j)(x_j = x_i)$ where A is a tilde free atomic formula, AX.1.ii $(A \rightarrow C^*(B)) \rightarrow (A \rightarrow (\exists x_i)(x_j = x_i)$ provided A does not contain x_j; $C^*(B)$ be tilde free, and A, B and $C^*(B)$ may be the same.

The identity theory of (AQIP) is standard in the sense that refexivity, symmetry, and transitivity of identity are preserved without restrictions. This is accomplished by restricting the set of terms to individual variables alone by incorporating the Russell-Quine strategy for eliminating names. But in (AQIP+) the universality of reflexivity of identity is lost, and the same is the case with some of the rules of qunatificational inference, for example the rule of universal instantiation. Thus its identity theory being non-standard for any implicit definition of the standard concept of identity should provide for reflexivity, symmetry, and transitivity, besides substitutivity of the identical without any restrictions. The following constitutes the axiom schemata and rules of inference of (AQIS+) of Scales. In AqS. 6 below the property abstractor lamda is to be read as 'the property A such that for only $x_k, \ldots, x_m$' where $k \leq m$.

AqS. 1 A where A is a two-valued truth-functional tautology,

AqS. 2 $(x_i)(Ax_i \rightarrow Bx_i) \rightarrow ((x_i)A \rightarrow (x_i)\, Bx_i)$,

AqS. 3 $(x_i)\exists!x_i$

AqS. 4 $(x_i)Ax_i \rightarrow (\exists!t_i \rightarrow A't_i)$ where A and A' are alike except that whereever x_i occurs free in A, a term t_i occurs free in A', and t_i may be the same as x_i,

AqS. 5 $A \rightarrow \exists!t_i$ provided A is an atomic formula and t_i occurs in A,

AqS. 6 $[\lambda(x_1 \ldots x_k)\, A't_1, \ldots, t_k] \equiv [(\exists!x_1 \,\&\, \ldots \,\&\, \exists!x_k) \,\&\, A(x_1, \ldots, x_k)]$ where A and A' are alike and are as in AqS. 4 and for each $i \leq k$, t_2 is free for x_i in A.

AqS. 7 $(x_i)(x_i = x_i)$

AqS. 8 $(t_i = t_j) \rightarrow (A \rightarrow A')$ for $i \leq j$, provided A and A' are alike except that A' contains t_j at only those places where t_i occurs in A.

RIS. 1 RI

RIS. 2 If Ax_i, then $(x_j)\, Ax_j$ where $I \leq J$.

$\exists!$ is to be retained only for convenience as it can dispensed with in terms of $=$. (I) $\exists!x_i \equiv (x_i = x_i)$ can be proved using AqS. 4 and 7.

Thus AqS. 3 can be dropped as from (I), $(x_i)(\exists! x_i \equiv (x_i = x_i))$ can be obtained; and this together with AqS. 2 and 7 yields AxS. 3. Thus in AqS. 4–6, $\exists!$ can be replaced by $t_i = t_i$.

The syntax of (AQIS+) is a bit complicated, though we simplified it a little. The notion of a formula in (AQIS+) as in (AQIP) is a complex one. To define it, let (SQ=+) be the result of augmenting the vocabulary of (SQ=) by names and descriptive phrases, and assume that the notion of a formula has suitably been specified for (SQ=+). Now

I. If A is an atomic formula of (SQ= +) then A is an atomic formula of (AQIS+),

II. if A is a formula of (SQ= +) and $x_1, \ldots, x_k$ are all the distinct individual variables that occur in A, then $\lambda(x_1, \ldots, x_k)A$ is an atomic formula of (AQIS+), and

III. A is a formula of (AQIS+) if and only if 1. A is an atomic formula of (AQIS+).

Freedom and bondage of the occurrences of x_i in A are assumed to have been defined as follows: an occurrence of x_i in A is a bound occurrence if and only if A is *i)* $(x_i^*)\, Bx_i$ or *ii)* $\lambda(x_i)\, Bx_i$, or *iii)* $\sim B$ and x_i is bound in B, or *iv)* $B \rightarrow C$ and x_i is bond in both B and C.

Evidently atomic formulae of this system may contain not only monadic connectives as in (AQIP) but also diadic connectives, or universal quantifiers, or the lambda operator.

To handle the semantics of (AQIS+), first note that (AQIP) and (AQIS+) agree on a point, namely that only proper subsets of their respective sets of formulae have ontological import. This is to say that in both these system no occurrences of an individual variable require the domain of interpretation to be non-empty. Only occurrences of a variable in formulae which do not have the tilde as their principal connective, and formulae which are equivalent to these formulae are assumed to be having ontological import. Where A is an atomic formula, for $\sim A$ not to be an atomic formula, the non-emptiness of the domain of interpretation is required. This much is common to these two systems, despite the fact that their respective notions of an atomic formula are different. As a result, when A is an atomic formula, $(x_i) \sim Ax_i \rightarrow \sim Ax_i$ and $at_i \rightarrow (\exists x_i)\, Ax_i$ come out valid, but $(x_i)\, Ax_i \rightarrow Ax_i$ and $\sim At_i \rightarrow (\exists x_i) \sim Ax_i$ turn out to be invalid.

As their interpretations are referential, this amounts to saying that not all occurrences of the individual variables and constants in these systems are referential. In (AQIS+) "an occurrence of an individual symbol ... in a formula A is referential if A is an atomic formula ... Otherwise the occurrence ... is non-referential", Scales remarks [71]. However, he does not state what constitutes an atomic formula of his system; in fact the term 'atomic formula' does not occur at all in his specification. But that does not pose any serious problem as he makes his intention clear. What he means by that term is captured by the definition offered above. Thus $\sim (\exists x_i)(x_i = x_i)$ does not have existential commitment, and hence that something does not exist can be captured in it as much as it can be captured in Routley's system. Though $(x_i) \sim (x_i = x_i)$ can be obtained from that formula, $(\exists x_i) \sim (x_i = x_i)$ does not follow from it. This implies that it is free from some anomalies. (AQIS+) does not follow from it. This implies that it is free from some anomalies. (AQIS+) does not require that the domain of interpretation be non-empty and in that sense is "liberated", to use Scale's own term. He claims for his system a middle position "between the standard predicate calculus and free logics", because it is "free in the sense that it has no existential presuppositions, but unliberated in that $Ax_i \rightarrow \exists! x_i$ is a theorem. Logics have been dubbed 'free' if they have no existential presuppositions, and if $Ax_i \rightarrow \exists! x_i$ is not a theorem ... (AQIS+) shows that these conditions are not coextensive. The semantic characterization seems to be preferable, in which case ... (AQIS+) ... would be a free logic."

One of the interpretations suggested by Scales is to be carried over to $\langle D, R \rangle$, where D is possibly empty and R is a set of relation defined over D. In order to simplify matters, let V+ be a valuation function in the interpretation over D, where D may or may not be empty; thus V+ is a neutral valuation function and is the same as V^* if D is empty and V if D is non-empty. Where A is a formula of (AQIS+). $V+(A)$ is to be defined as follows:

1. If A is a lambda-free atomic formula, then $V+(A) = T$ if and only if for each $i \leq j \leq K$ where $I(t_j)$ is the interpretation of t_j, i.e. what is assigned to it from the domain of interpretation; $I(t_j) \in D$ and $\langle I(t_i), \ldots, I(t_K) \rangle \in R$ where $t_i, \ldots, t_K$ are the individual variables or constants that occur in A.

2. If A is $\lambda(x_i, \ldots, x_K)\, Bt_2, \ldots, t_K$ then $V^+(A) = T$ if and only if, for each $i \leq j \leq K$, $I(t_j) \in D$, and $V^+(B'x_1, \ldots, x_K) = T$ where B' is just like B except that for each j, x_j occurs free in B' wherever t_j occurs B, and t_j is free for x_j in B.

3. If A is $(x_i)\, Bx_i$, then $V^+(A) = T$ if and only if for each t_i such that $I(t_i) \in D$, $V^+(B't_i) = T$ where B and B' differ only in the sense that the latter contains t_i wherever B contains x_i free, and t_2 is free for x_i in B.

4. If A is $\sim B$ then $V^+(A) = V(\sim V^+(B))$, and

5. If A is $B \rightarrow C$, then $V^+(A) = V(V^+(B) \rightarrow V^+(C))$.

The other interpretation worked out by Scales requies outer domains and hence is uneconomical besides being prone to a kind of Meinongianism. It also requires a differentiation between what can be called the domain of interpretation and the domain of valuation where the latter is wider than the former. In either of these interpretations, when the domain is empty, truth-valued assignment to atomic formulae and their denials seems to unnatural and counter-intuitive. In that case all atomic formulae would turn out to be false, and their denials true, Scale's justification for this is as follows: "if I say it is not the case that Jones is bald, I have attributed no property to Jones, but simply denied that he has a certain property; if Jones does not exist, then no property is truly attributable to him. Then when A is an atomic of containing a term t_2, when t_2 does not have a referent, $\sim A$ is vacuously ture, and as A and $\sim A$ are strong contradictories, A is vacuously false." Then if something does not exist — in any mode of existence — then any assertion about it is false and its denial true. This implies that truth-value is determined not only by existence and non-existence but also by the mode of existence and non-existence.

Moreover, within the framework of (AQIS+) there will be false self-identity statements whatever might be the mode of existence sharing which the referents of the referring terms do not exist. Even if we somehow swallow this sticky metaphysical pill, we still will be left with some dissatisfaction on account of the unintended consequences of (AQIS+). We wanted to delink logic and ontology by liberating "logical-truth" from the requirement that there be entities. Paradoxically, we arrived at a position where we are forced to accept that, while it is not necessary that there be entities, if there are any then they must share a unique mode of being and that they and they alone be self-identical. This is the same as to say that (AQIS+) assumes ontological fraternity. Unfortunately this is an extra-logical notion.

Similarly (FQB) of Burge [4] is distinctly Aristotelian in the sense that it is based, like (AQS)s, on the assumption that predication and existence go hand in hand, which is to say that there are no unexemplified predicates. This assumption finds expression in one of the axioms of (FQB) which reads as:

AxB. 1 $At_1, \ldots, t_n \rightarrow [\exists y_i)(y_1 = t_1)]$ & $\ldots$ & $(\exists y_n)(y_n = t_n)$ where A is any predicate, including, for $n = 2$, the identity predicate, and where y_i, for $i = 1, \ldots, n$ is not free in A.

Thus $t_1 = t_2 \rightarrow [(\exists y_i)(y_1 = t_1)]$ & $(\exists y_2)(y_1 = y_2)$. This, in a sense, is what Wittgenstein [89] was suggesting in his letter to Russell in 1913 where he wrote that the real form of 'Socrates is human' is $(\exists y)(Hy \ \& \ y = a)$ where S is the subject and the whole of $(\exists y)(Hy \ \& \ y = a)$ is the predicate. Wittgenstein would suggest an assignment of falsity as the truth-value of $(\exists y)(Hy \ \& \ y = S)$ when S is non-referring. Similarly Burge suggests that an expression can be counted as an atomic predicate only if singular sentences containing it can be assigned falsity whenever it contains a term that has no referent in the domain of interpretaion. In (FQB) self-identity and existence come to be identified as such. A saving feature of this system is that its identity theory is standard. In virtue of this it can claim to be a better reformulation of (SQ+=) than, say, the system of van Fraassen and Lambert [85], which is too weak to yield substitutivity conditions. And it does not assign the same referent to all non-referring terms, as in Scott's [75]. In virtue of the fact that $(\exists y)(y = t_1) \ \& \ \ldots \ \& \ (\exists y)(y = t_n)$ as it occurs in Scott's system is an analogue of $At_1, \ldots, t_n$ in (FQB), virtual existential closures of atomic formulae of (SQI) are atomic analogues in (FQB). This is the import of $A \times B$. $t_1 = t_2$ in (FQB) receives normal reading as 't_1 is identical with t_2'', whereas in Scott's system it reads as 'anything identical with t_1 is identical with t_2'. Yet as $t_1 = t_2$ comes out false when t_1 or t_2 is devoid of reference, anomalies creep in. Taking t_1 as short for 'Pegasus' and t_2 as short for 'the winged horse caught by Bellorophon', we can note that on account of (FQB) we will have to treat 'Pegasus is the winged horse caught by Bellorophon' as false even though we trust that it is not. There exists in the literature another (AQ) which permits self-identity without restriction. It is developed at length by Grice in his [18]. Though it is interesting in its own way, it is not universally free but contains rubrics to incorporate into its frame individual constants which do not have referents. This is achieved by differentiating between what correspond to different roles of the tilde, and by an analogous differentiation between different roles of identity, one corresponding to existential identity as in (SQI), and another corresponding to non-existential identity as in (FQK) of Kearns [32]. Just as Hughes and Londey's system is an attributional logic of domains, Grice's system is an attributional logic of non-empty domains. All these (AQ)s incorporate either in their syntax or in their semantics the suggestion of Quine to avoid the troubles created by Universal instantiation by making sure that the term used in instantiating has a referent, or else that it is deductively dependent on a term which has a referent.

10 Truth-Value Gaps

Van Fraassen's strategy in his (1)–(5) to free (SQ)s from existence assumptions is aimed primarily at eliminating singular assumptions. As far as the general assumption is concerned, he accepts (FQQ) of Quine as a proper reformulation of (SQ), however working out an alternative interpretation of it which radically differs from the one which Quine provided for it. He thinks that 'free logic' should be read with Lambert as an abbreviation for logic free of existence assumptions with respect to its terms (singular and general) [37]. At the same time, he claims that, unlike the attempts to free (SQ)s which resulted in systems which are only subsystems or fragments of (SQ)s, adopting his strategy of supervaluation results in systems of which (SQ)s are subsystems. We shall show that accepting vacuous quantification becomes a necessary requirement for jointly sustaining all of these claims. In order to specify what, according to van Fraassen, is Free (SQ) — a term which better than (FQ) captures his intention — some clarifications are in order.

The conception of logical truth as coextensive with validity in standard interpretations of (SQ)s, according to [37] on the one hand "inclines one towards a conception of logical truth which presupposes nothing about the truth-values of statements. On the other hand, the classical codification of logical truths ... plus the Tarski truth-criterion demand the acceptance of an account of logical truth which definitely excludes truth-valueless statements". Free logic is aimed at a codification of all and only those "truths" which are so by virtue of the former account. (Though it does not have much bearing on the issue at hand, it should be noted that the Tarski-criterion is neutral to the problems here, and hence is delinked from them. Lambert is seeing a little more than is really there in the criterion. This criterion comes into the picture only when the intended meanings of the logical connectives have been spelled out, and the extra-logical symbols have been provided with referents. These referents need not be extraneous to the symbols, for they can be the symbols themselves — a brilliant insight which Hintikka subsequently exploited. In any case, the criterion does not limit the intended meanings. This is precisely why there can be Tarski-semantics for free logics too. Since Lambert's point can be restated without that reference to Tarski, we may as well ignore it in regard to his substantive points.) In the standard semantics of (SQ) as well as the semantics of all the (FQ)s and (AQ)s that we have so far considered, the notion of validity is characterised semantically and is defined in terms of interpretations over specified domains. Despite the fact that the construal of the notion of logical truth in terms of validity is common to all these kinds of notions, the notions of logical-truth in these systems are at variance because in the interpretation of (SQ)s validity is determined in terms of non-empty domains only, whereas in the other type of systems it is understood in terms of all domains, be they empty or non-empty. However all these systems have some common characteristics. At least there is one feature which is common to all the interpretations of these systems. There is also another characteristic which is naturally common to most of these systems. While characterising logical-truth in terms of validity, if it is presupposed in the semantics of all those systems that each formula has a truth-value, then it is also presupposed in the semantics of most of these systems — with an exception of (FQR) — that this truth-value can be fixed in the case of each formula. The various ways in which the assignment of a truth-value to atomic formulae is carried over in the semantics of those systems in the interpretations over (the) empty domain(s) "had on the whole been adopted *faute de mieux*, for lack of adequate theory", as [84] put it. In order to be adequate, any such semantic theory should satisfy, according to him, the following conditions:

(1) Logical-truth should be "a measure of validity".

(2) Logical-truth should not presuppose anything about truth-values of atomic formulae.

(3) Logical-truth should be valid in all domains, be they empty or non-empty.

(4) Validity should be characterised semantically.

(5) The interpretations should be completely extensional.

(6) Truth-value assignments should be carried out without going beyond the domain of interpretation.

(7) Classical propositional logic should hold without restrictions of any kind.

The import of (6) is that terms be mapped onto the domain set and not onto a set which can be constructed from the domain set or even an extension of the domain set. The semantics of the majority of the systems that we have either to considered are such that they violate one or another condition of adequacy as formulated by van Fraassen. To his list of conditions we may add another barring vacuous quantification. (FQR) however is the lone exception. But it cannot be hoped that van Fraassen would accept (FQR) as a satisfactory codification of logical-truth, for the interpretation for which it is sound and complete would still be unsatisfactory to him ever though it satisfies all of his conditions of adequacy and also the one that we intended to add. The reasons can be anticipated easily. In the interpretation of (FQR), although no fixed truth-value is assigned to atomic formulae or open formulae, it is assumed that they have a truth-value that the set of logical-truths is not affected by the truth-value they have have. Thus it excludes the possibility of an atomic formula remaining without truth-values. Further, although it does not contain individual constants in its vocabulary, if it were to be augmented with constants, it would be required to assume that each formula containing a singular term is either true or false, even when the domain of interpretation is empty. Its soundness and completeness are not affected whether such formulae are assigned the value truth or the value falsity. But van Fraassen, in the tradition of Aristotle, holds that a property can neither truly nor falsely be attributed to what does not exist, and hence a sentence about the non-existent is devoid of a truth-value. By parity of reasoning, even atomic formulae containing only variables in effect function as individual constants. Van Fraassen would hold that "though in any situation certain sentences are true and others false", it is not the case that in any given situation all sentences are either true or false. If this view of van Fraassen were to be incorporated into the semantics of (FQR), its soundness would be adversely affected because a truth-functional compound then would have to be treated as being without truth-value when one of its components is without truth-value. This is to say that the soundness of (FQR) is contingent up on the principle of bivalence, i.e., that for each formula A, A has at least one of the truth values T or F, and has at most one of those two values.

"But", van Fraassen [84] argues, "philosophers argued that the law of bivalence is itself not universally valid. Matrices have appeared to provide an easy way to construct perfectly intelligible non-bivalent languages. However, the use of matrices also introduces various complexities, and philosophers did not rush to embrace this method with which the logicians so agreably provided them. This may have been due in part, however, to the fact that it seems not to have been widely noticed (a) that most concepts in the semantic analysis of logic can be defined in terms of truth, without the use of the notion of falsity, or (b) that there are adequate matrices for classical propositional logic for which it is not the case that every element or its complement is designated". He shows that this is the case not only with propositional logic, but also with quantificational logic.

To show how most of the concepts in the semantic analysis of (SQ) can be defined in terms of truth alone, van Fraassen introduces the notion of *Supervaluation*. To explain this concept, let the valuation function Vc_i, $i = 1, 2, \ldots$ be called a *Classical Valuation function* if and only if

1. it is not the case that $Vc_i(A) = Vc_i(\sim A) = T(F)$, and
2. it is the case that either $Vc_i(A) = T$ or else $Vc_i(A) = F$.

This means that a valuation function is a classical valuation function if and only if it does not violate the principle of contradiction while adhering to the principle of bivalence. Further, let those and only those valuation functions which satisfy 2) above be called *Standard Valuation functions*.

The valuation function in the semantics of the different (FQ)s that we considered in the foregoing, and also the valuation function in the standard semantics of (SQ), are standard valuation functions. Only some of these functions are classical valuation functions. Other assign the same truth-value to both A and $\sim A$, for instance V^* as defined in the interpretations of (FQM) and (FQS). In both of these $V^*(A) = V^*(\sim A) = T$ when A is an open formula. Now let all those interpretation incorporating standard valuation function be called *standard interpretations*. Obviously only some of the standard interpretaions are classical interpretations. When the domains of interpretation is empty in the classical interpretations, there are two possible ways of assigning a truth-value given domains by invoking either the null set or an extension of the domain set. If this is to be eschewed, A will have to be left without a truth-value assigned to it. Van Fraassen's attempts using supervaluations go toward a determination of the set of logical truths, that is the set of formulae of (SQ) valid under standard interpretaion without tagging a truth-value to each of its formulae, or in other words, leaving truth-value gaps.

To note what a supervaluation is, let for $i < j$ Vc_i and Vc_j be classical valuation functions, and let Vs_i be a supervaluation function; then

1. $Vc_i(A) = Vc_j(A)$ if and only if $Vs_i(A)$ is devoid of a truth-value;
2. for each K, $Vc_K(A) = T$ if and only if $VsI(A) = T$:
3. $Vs_i(A) = T$ if and only if $Vs_i(\sim A) = F$, and
4. $Vs_i(A \rightarrow B) = F$ if and only if $(Vs_i(A) \rightarrow Vs_i(B)) = F$ which is so only if $Vs_1(A) = T$ and $Vs_i(B) = F$.

3 and 4 taken together are tantamount to saying that the truth-value of a compound under sepervaluations are uniquely determined by the

truth values of its compopents under supervaluations. Obviously, standard propositional logic and standard quatification theory are sound and complete for standard interpretations which are classical. Let the formulae of (SQ) valid under classical interpretations be classical logical truths; similarly let the formulae of (SQ) which are valid under supervaluations be called super logical truths. A is a super logical truth only if A is valid under supervaluations; and A is valid under supervaluations only if $Vs(A) = T$ under all interpretations of A. $Vs(A) = T$ if and only if $V(A) = T$ for all interpretation of (SQ) over non-empty domains, and Vs is the supervaluation function interpretations over non-empty domains.

Van Fraassen [83] shows that "the set of CLTs and SLTs is exactly the same. That this holds for linterpretations over non-empty domains is easy to see. If a formula is valid under all classical valuations, then it is valid under supervaluations, for a supervaluation reflects what is common to all classical valuations", and "what is common to classical valuations cannot transgress laws that hold for all classical valuations (in particular the laws of classical propositional logic". Let us see whether the same holds under interpretations over the empty domain. When the domain of interpretation is empty, there exist two classical valuation functions. Let them be $V'c_1$ and $v'c_2$ such that where A is an atomic formula one can choose either for $V'c_1(A) = F$ or $V'c_2(A) = T$. Let V's be the supervaluation function in interpretation over the empty domain. Obviously neither T nor F can be equated with $V's(A)$. As a valid formula of (FQ) is expected to be valid

I) under classical valuations over non-empty domains,

II) under supervaluations over non-empty domains,

III) under classical valuations over the empty domain, and

IV) under supervaluations over the empty domain wich is to say that $V's(A) = Vc_1(A) = V'c_2(A)$ should be the case in all interpretations of A over the empty domain — to identify classical logical-truths and supervaluational logical-truths, it becomes necessary for van Fraassen to accept a specific formulation of (SQ).

(SQ) as we specified it will turn out to be unsound under supervaluations. A will be devoid of a truth-value when one of its components is truth-valuenes s. In interpretations over the empty domain, all classical valuations assign F to $(\exists x_i)\, Bx_i$ and T to $(x_i)\, Bx_i$. But $(\exists x_i)\, Bx_i \equiv\sim (x_i) \sim Bx_i$ and van Fraassen accepts it. This implies that if $V's(Ax \cdot 4)$ is to be valid in interpretations over the empty domain, it will have to treated as standing for its own universal closure.

Thus, in order to make the set of CLTs and the set of SLTs identical, the axioms and the theorems of (SQ) will have to be taken as standing for their respective universal closures, and the required modification be made in the statement of the rule of inference. This means that the proper syntax of (SQ) is not the one in which we formulated it but the one given by Quine [53]. That formulation of Quine, (ML) as it is usually referred to, is not valid for interpretations over the empty domain. "However", van Fraassen [83] holds, "Hailperin and Quine have shown how a slight revision in the axioms of (ML) makes it valid for" the empty domain too. But, as we noted earlier, these revisions resulting in (FQH) and (FQQ) respectively involved vacuous quantification. Here is something which makes us wonder as to how much of sementics can be hidden behind syntax! And the systems of (FQ) which differ from these systems with respect to this feature turn out to be unsound on supervaluations. This provides a sufficient reason for adhering to the principle of bivalence and rejecting van Fraassen's semantics. (He gives in [83] another reason which motivated Leonard and Lambert to adhere to the principle of bivalence, and we shall come to it a little later.)

11 Some Other Not So Popular Systems

In this section we shall consider systems due to von Wright [86], Smiley [77], Woodruff [90], and Cocchiarella [10]–[13]. Von Wright's (FQW) though was not paid much attention — presumably because it was thought to be of limited interest as it was aimed at freeing only the monadic fragment of (SQ) — has an important idea incorporated in it. With respect to this idea it differs from all the systems that we noted till now. Von Wright wants a specification of what constitutes a logical-truth, not in terms of validity in all domains (that is model theoretically as van Fraassen requires any adequate semantic theory should be thought of), but by means of a "decision procedure, which in turn is to be specified in terms of truth-tables, turning the notion of logical-truth in propositional logic and the notion of logical truth in quantificational logic", "close parallels" in the terminology of Church [9]. According to von Wright the notions of logical-truth in propositional logic and in quantificational logic should be "equieffective". Von Wright goes much beyond that, and holds that quantificational logic "can be regarded as a special branch of the logic of propositions", though it is not isomorphic with the logic of propositions. The nonisomorphism between these two arises due to the fact that the notion of logical-truth in the logic of propositions is unilevelled whereas the corresponding notion in the other "is, so to speak, two levelled". (In his work on distributive normal forms for quantificational logic, Hintikka exhaustively elaborated this fecind idea and explored higher levels.) To explain this difference we need first to specify (FQW) and note its semantic interpretation. In this we shall follow Church's specification in a more familiar syntax of what "in effect is the same as" von Wright's own specification which has some dispensible assumptions leading to some eliminable misunderstandings of his essential point as in Cohen's ([14]) criticism of it. Church's specification in his [9] differs from von Wright's only in the sense that the former accepts what the latter calls the relational view of propositions, that is to say that a proposition establishes a relation between a number of *things*; and the latter accepts the Aristotelian view of propositions, that a proposition attributes a property to a *thing*. According to von Wright "every proposition can be analysed in both the way— these are universal but not unique view of propositions". Moreover, as he is concerned only with the monadic fragment of predicate logic, the distinction between these two views is of no consequence.

The syntax of (FQW) is the same as that of (SQ) except that it does not contain *i*) polyadic predicates, and *ii*) overlapping (or nested) quantifiers, that is formulae containing a quantifier within the scope of another quantifier. Now let A and A^* be formulae of (FQW) containing no free variables such that is in the prenex normal form, and A is arrived at from A^* by making use of $\sim (x_i) \sim — \equiv (\exists x_i) \sim —$, $(x_i) — \equiv \sim (\exists x_i) \sim —$, and other definitional equivalences holding in (SQ) "by driving the quantifiers inward as far as possible", such that A^* is a molecular formula whose ultimate constituent are $\cdot 2^K$ number of E-formulae $(\exists x_1)(P'_1 x_1 \ \& \ \ldots \ \& \ P'_K x_2), \ldots, (\exists x_n)\left(P'_{K_{x_n}} \ \& \ \ldots \ \& \ P'_{K_{x_n}}\right)$, where for $i \neq j$ P'_K are all the predicate letters occurring in A (and obviously in A^*). It is evident that $A \equiv A^*$. Thus A is "special kind of truth-function" where truth-value is uniquely determined by the truth-value of E-formulae occurring in A^*.

As each formula of (FQW) containing no free variables can be transformed into an equivalent formula in prenex form, it follows that it can be transformed into an equivalent formula in Herbrand normal form. Then for any formula A_i of (FQW), A_i can be transformed into an equivalent formula of (FQW), say A_j, such that A_j is either *i*) a truth-functional compound all of whose components are open formulae, or a truth-functional compound at least one component among whose components is a closed formula.

In case of (*i*), whether A_i is a logical truth or not is decided by testing for tautologyhood. In case of (*ii*), the truth-value of A_i is either (*a*) entirely determined by the way in which the truth-functional connectives connect the components of A_i, or else

(*b*) is determined by the way in which truth-functional connectives connect the components, of course in addition to the inner structure of the components which happen to be closed formulae. There is no need to consider the trivial case in which the first kind of dependence alone is sufficient to ensure logical truths because it is something which falls entirely within the realm of propositional logic. Von Wright attempts "to show that the second dependence also in terms of "truth-functions".

A_i may not be a tautology in the sense of (*ii*)–(*a*), and yet be logically true; otherwise quantificational and propositional logics will be "isomorphous". The set of tautologies in the sense of (*ii*)–(*a*) is only a proper subset of the set of logical-truths. For instance, to use von Wright's own example:

$$[(\exists x_i)(P_1' x_i \;\&\; P_2' x_i) \;\&\; (x_i)\,(P_2' x_i \rightarrow P_3' x_i)]$$
$$\rightarrow (\exists x_i)\,(P_1' x_i \;\&\; P_3' x_i)$$

though a logical truth, is not a tautology in the standard sense of the term. The essence of von Wright's approach consists of showing that, despite that, tautologyhood can be a measure of logical truth.

Von Wright shows this by letting the truth-value of a formula A_i for the case (*ii*)–(*b*) be determined by requiring that (1) A_i be a logical truth if and only if A_2^* is a truth-functional tautology, where A_i^* is the result of replacing each component A_i' in A_i, if A_i' does not contain a free variable, by its Herbrand normal form $A_i'^*$, and that (2) the truth-value of A_i^* be uniquely determined by $K + J$ components of A_i^*, where for $K \geq J \geq 0$. K is the number of the distinct E-formulae occurring in A_i^2, and J is the number of the distinct atomic formulae occurring in A_i^* outside the scope of the quantifiers. A_i^*, and hence A_i, is a truth-functional tautology, which is to say that is a logical truth if and only if A_i^* has the truth-value T in each of the 2^{K+J} possible value asignments to its components. From this decision procedure it emerges that the set of those and only those formulae of the monadic fragment of (SQ) which are logically true by being tautologies (be they unilevelled or twin levelled) and the set of those and only those formulae of the same fragment which are valid in all domains are precisely the same. This is to say that the notion of universal validity as it has been model-theoretically defined, and the notion of logical truth defined in terms of von Wright's effective procedure, are coextensive as far as the monadic fragment of (SQ) is concerned. Obviously, not all theorems of the monadic fragment of (SQ) are U-valid, nor are they logically true in the sense of von Wright.

The monadic fragment of (FQR) and (FQW) share a feature, namely they can be second-order duplicated such that if their first-order formulae do not require the terms to have referents, then their second-order formulae do not require the predicates occurring in them be ones that are exemplified or instantiated. Another system of (FQ) for which such a duplication is feasible is that of Cocchiarella [10], which frees (SQ)s by multiplying the domains of interpretation and by augmenting the syntax of (SQ)s with quantifiers over possible object domains. If Cocchiarella's system, like that of Routley's, is Meinongian, van Fraassen's system, as we noted, is Fregean insofar as truth-values are not assigned to formulae containing predicates not defined over the domain of interpretation. Russell [68] rejected both these ways of treating empty terms and unexemplified predicates. Despite all the differences that we noted, all these systems are two-valued, and on this score they can be grouped together.

The two system which we shall consider now are, on the other hand, three-valued, and are free only from singular existence assumptions. They are due to Smiley [77] and Woodruff [90] respectively. Smiley's system can be seen as an intermediary between van Fraassen's and Woodruff's system. It is, so to speak, a truncated three-valued system, and its major features are summed up by Smiley himself as follows:

1. The central feature of my theory is the division of sentences not into two categories (true and false) but three. Into the third category go, roughly speaking, all sentences directly involving non-denoting terms, or incompletely defined properties.

2. The theory is capable of two interpretations according as sentences in the third category are taken as being neither true nor false, or as being false.

3. The category of a compound sentence is a function of the categories of its components; the theory is formally 3-valued.

4. But this aspect of the theory should not be allowed to overshadow the fact that it can be developed purely in terms of the usual "*two* truth-values understood as usual and standing for truth and falsity ...; this development was carried out for the first interpretation of the theory."

Smiley carries out this interpretation by restricting the introduction of singular terms by Universal Instantiation by requiring that those terms satisfy an additional condition that they be referring. This means that non-referring terms are debarred from figuring in instantiation. Thus in (FQSm), though $(x_i)\,Ax_i$ is valid, $(x_i)\,Ax_i \rightarrow At_i$ is not valid when t_i is an individual constant with no referent. However At_i can be inferred from $(x_i)\,Ax_i$ on the assumption that $T(t_i = t_i)$, that is that t_i exists. Thus $[(x_i)\,Ax_i \ \& \ T(t_i = t_i)] \rightarrow At_i$ where T is a monadic truth-functional connective to be read as 'it is true that', and is to be defined as (1) $V(TA) = T$ if $V(A) = T$, and (2) $V(TA) = F$ if $V(A) = F$ where A is a formula of (SQ), and V the valuation function in the interpretations of (SQ). $V(A)$ is defined as usual based on interpretations over non-empty domains. Further, as in (SQ)s

(*i*) $V(\sim A) = T$ if $V(A) = F$,

(*ii*) $V(\sim A) = F$ if $V(A) = T$ and

(*iii*) $V(A \rightarrow B) = F(T)$ if $V(A) = T$ and $V(B) = F$ (either $V(A) = F$ or $V(B) = T$).

Let these connectives of (SQ)s be called *Primary Connectives*. Now in two-valued interpretations, $V(A) = T$ and $V(A) = F$ are the same; as a result, in the standard interpretations of (SQ)s, T is a redundant connective. In Smiley's system, $V(A) = T$ might mean either $V(A) = F$ or $V(A) =$ neither T nor F. It will mean the latter if the interpretation $I(A)$ of A is not defined over the non-empty domains D. If it means the latter, $\sim A$ too will be devoid of a truth-value adversely affecting the validity of standard propositional logic. The failure of standard propositional truths, as Smiley thinks, "enables us to mark out the boundary line, among the sentences containing terms that do not denote anything, between those that will fail to have truth-value on this account and those that will have ones nevertheless."

Smiley's strategy for marking off that boundary line consists first in treating "an occurrence of a term a in a sentence $A \ldots$ primary if it does not lie within the scope of any occurrence of the connective T; otherwise a's occurrence is *secondary*", and secondly using the connective "to define further one as follows:

$$\begin{aligned} \sim A &= df \approx TA \\ A \rightarrow B &= df\, TA \Longrightarrow TB.\text{"} \end{aligned}$$

Smiley calls these connectives *secondary*, and the original ones *primary*. "A sentence whose only connectives are secondary never lacks a truth-value ... The secondary connectives (even when sentences without truth-value are admitted) behave exactly as do connectives in the orthodox treatment (from which truthvalueless sentences are excluded); all tautologies in them are logical truths, the deduction theorem holds for $\Longrightarrow\!\!\!>$, contraposition holds for $\approx$, and so on. Very little of this holds good for the primary; for example $A \vee \approx A$ cannot be asserted to be a logical truth because of the possibility that A has no truth-values and the deduction theorem does not hold for $\rightarrow$ for the same reason. More important of all is the failure of contraposition: from the fact that $A \vdash B$ it does not follow $\sim B \vdash \sim A$, because A's having no truth-value while B has value F is compatible with $A \vdash B$ but not with $\sim B \vdash \sim B$." The relationship between primary negation $\sim$ (not) and secondary negation $\approx$ (it is not true that) is interesting, as A and $\approx A$ are only contraries, whereas A and $\sim A$ are contradictories.

Woodruff's specification of his system in [90], as he himself subsequently noted, is inconsistent. In his suggestion toward a modification of it to restore iconsistency, he sacrifies functional completeness. However, a restoration of Woodruff's system securing for it both consistency and functional completeness can be carried out precisely along the lines by which Smiley developed his system, with the difference that 'neither true nor false' will be replaced by 'has the truth-value U' where U is a third value. Hence his system, even after the needed modification, will be on par with that of Smiley. Both of them augment (SQ)s in more than one way, and as much cannot be taken as proper reformulations of (SQ)s.

12 Extension of (FQ)s

Hintikka [26] expressed doubted "as to how interesting an enterprise the study of ... presuppositionless logics will turn out to be in the long run." A study of the role of existential presuppositions in first order logic, he thought, would "soon exhaust all the general theoretical interest that is there in the area, if conducted as a purely syntactical and semantical enterprise. However, a philosophically satisfactory comparison between them will ... have to turn on a deeper conceptual analysis of the situation than the standard syntactical and semantical methods afford." That there is a large grain of truth in this will become evident when we attempt a comparison of the alternative identity theories. Reserving the "comparison" at issue for the next two sections, here we shall pay attention to one of those theories, namely the standard identity theory — (IS) for short — to note how identity *per se* is ontologically committed and hence will have to be treated as an extra-logical notion. Logicians may yet want to extend the boundaries of logic to include identity theory too, and may implore philosophers to ignore those ontological assumptions, as Hintikka for instance did try with reference to the inclusion of self-identity within the scope of logic. "Self-identity may seem to be", he argues in (2), "one thing we are justified in asserting of everything whatsoever. But if we forget our metaphysical ideas about self-identity, by which we so easily entrance ourselves in any way, the strangeness can perhaps be made to disappear. If you look instead at the semantical rules which govern the interpretation of our formulas ... you can see that the main purpose of the axiom $y = y$ is to make sure that our singular terms refer to one and the same individual at all occurrences (instead merely referring to individuals with some properties and relations)." Yet in his identity theory, which we shall call (IH), 'the horse caught by Bellerophon is the horse caught by Bellerophon' comes out true. But the same is not the case with 'Pegasus is Pegasus'. Obviously, here we are reading the copula as identity. But why should we be indifferent to the metaphysical assumptions? Hintikka's suggestions are tantamount to asserting that we better ignore them. Further, his suggestion that identity should be treated as a logical notion soley on account of "semantic rules which govern the interpretation of" identity formulae, collides with his view that we should "turn on to a deeper conceptual analysis of the situation than the standard synstactical and semantical methods afford." On "a hasty acceptance" of the "definitions of existence in terms of identity" which is to say on accepting $\exists!a_i \equiv (\exists x_i)(x_i = a_i)$ or $\exists!a_i \equiv a_i = a_i$ — Butchvarov [2] comments that "if we assume that the range of the quantifiers and the reference of names are restricted to existent things, then these two definitions are conceptually trivial. If we do not make these assumptions, then both definitions entail that everything, e.g. Pegasus, exists," as was shown by Lambert [34].

Extending (FQ)s to obtain (FQ=)s conceals several important issues. Further, such extensions are not of any philosophical interest nor they ever have any theoretical interest, since none of the possible extensions coincides with the logical structure of a significant theory unless and until it is further augmented by an assumption to the effect that the intended domain of interpretation is not empty. (This is to say that such an assumption constitutes the transcendental condition for the possibility of significant theories as such.) Further this implies that any (FQ) which is a proper subtheory of (SQ) is theoretically useless, and any (FQ) of which (SQ) is a proper subtheory has already enveloped the metaphysical assumptions in virtue of its right to be given logical status, in so far as it contains non-analytic truths, that is. It would be at most a prototheory. An extension of a (FQ) with which we are dissatisfied on some count or the other will be equally unsatisfactory and precisely on the same count. This raises the issue whether we can extend (FQR) in such a way that it emerges as a satisfactory reformulation of (SQ)! This does not seem possible given that due to the interpretation of (FQR) over the empty domain, the axioms of (IS) will not be M-valid. Nor does acceptance of the universal closures of the axioms of (IS), i.e., acceptance of Quine's (IQ), help; for within the framework of the resultant extension, substitutivity of the identicals will not be available. As such, the very purpose of extending (FQR) will be lost. Any plausible theory of identity should provide for 1) reflexivity, 2) symmetry, 3) transitivity, and 4) substitutivity. Substitutivity will be available, however, in a further extension of the suggested extension of (FQR) which could be arrived at by adding $(\exists x_i)\, Ax_i$ as an axiom, which is the same as to say in (SQ=) that (FQR) plus $(\exists x_i)\, Ax_i$ is (SQ).

In fact, in (FQR=) not even the reflexivity of identity will be available; it is only an addition of the axiom $(\exists x_i)\, Ax_i$ mentioned above that gives a "logical" support to (IS). That axiom securing the non-emptiness of the domain of interpretation, pushes to the fore what is concealed in (IS) and (IQ).

Quine has all these long years been hammering at one side of the relationship between existence and identity, namely that there is no existence without identity. The foregoing has shown the other side of the relationship; there is no identity without existence. It is precisely this mutual dependence betweeen identity and existence that enables us to define the latter in terms of the other in (SQ=), and it is this dependence which is implicit in the passage quoted above from Butchvarov that turns the definition into a patent triviality.

If we add to (FQR) the closures of the axioms of (IS), we will have — not a theoretically "inutile" extension, to misuse one of Quine's spicy phrases — but an extension which is extremely interesting for metaphysicians. It makes the extralogicality of identity perspicuous. Though the first-order (FQR) has built into it the necessary rubrics for its second-order duplication, (FQR=) will be devoid of them. The reason for this is simple; just as $x_i = x_i$ is not M-valid in the first-order (FQR=), so $P_i^J = P_i^T$ will not be M-valid in the second-order (FQR=). Obviously $(\exists P_i^J)(P_i^J x_i x_i$ will be neither a theorem nor an axiom in that system. Quine should not have any qualms about the second-order duplication of (FQR), since it would not individuate properties as the second-order duplication of (SQ) is presumed to do. This means that the relationship between 'being a value of a bound variable', 'existence', 'identity', and 'quantification' is a bit more complex than we can elicit from the appearances of (SQ)s, and needs to be given a second look to be properly grasped. Quine's criterion of ontological commitment, namely to be is to be a value of a bound variable, holds in the case of (SQ) because buit into it is the assumption that its variables do have values. This necessitates a modification in the statement of the criterion, for instance as: given that variables have values, to be is to be a value of a variable. Consequently we will have: given the existence of entities, to quantify is to entify. Variables and quantifiers are really "unproductive", they at most help us to find what there are, or are given to them; they do not produce their domains.

Since neither the first-order (FQR) nor its second-order duplication requires non-empty domains or "existence" of entities — be they individuals or properties — quantification in those systems does not result in entification. But neither (IS) nor (IQ) can be appended to those systems; and this should be sufficient to question the legitimacy of giving logical status to identity. On this ground identity will have to be kept outside the province of logic, at least in so far as it is believed that the axioms of (IS) are taken to exhaustively characterise that concept. Quine, however, might not be convinced of this. He might say (as he did in [56] contesting Munitz's view that identity of universals is one thing and identity of individuals another): "This I deplore. Identity is as aloofly logical as quantification. It has its complete axioms, it has its general set theoretic definition by exhaustion of primitive predicates. When we do propound identity conditions for bodies, or persons, or classes, we are using the prior concept of identity in the special task of classifying the term 'body', or 'person', or 'class'; for an essential part of the clarification of a term is clarification of the standard by which we individuate its denotata."

According to Quine, identity is not only "aloofly logical" but also universal for it has its "complete axioms". The linkage between quantification and identity which we noted above implies that identity could be aloofly logical like quantification only if existence of entities is assumed In the light of this, the reasons adduced by Quine in support of his belief about the logicality of identity get vitiated. When logic itself is delinked from existence as in free logic, the "logicality" of quantification is not affected, but identity comes out with extralogical affiliation. As Quine himself is conciliatory on the set-theoretic definition identity, we may leave it at that and move on to consider the schematic definition of identity as a measure of its logicality. It is shared and defended by several others, for instance by Martin [46]. "The notion of identity", he writes, "may be introduced into some L(anguage) by definition. Thus suppose there are only two primitive predicates of L, 'P', and 'Q' of one and two places respectively. Suppose further that not both of P and Q are universal or null, or that one is universal and the other null. ('Universal' and 'null' here are of course ambiguous, but what is intended is clear enough.) We may then define '$x = y$' as $(Px \equiv Py$ & $(2)(Qx_2) \equiv$ & $(2)(Qzx \equiv Qzy)$ without further do, achieving here with the full effect of identity within this L.... And of course this definition generates L's with any finite number of primitive predicates, under suitable assumptions concerning their nullity and universality. Because as a matter of fact the primitive predicates of most of L's satisfy the requirement concerning nullity and universality — and if they do not, can be suitably transformed into such as do — we see that identity is already combined in quantification theory as developed for these L's."

First a word about the tacit assumptions of this (pseudo-)Leibnizian vision of identity. It assumes (*i*) that ontology and ideology are conceptually independent, or that ontology is autonomous in the sense that individuals come out on their own and need not be individuated by properties, (*ii*) that indiscernibility (theoretical or cognitive, or variantly semantic or epistemic) is an epistemic measure of identity, and (*iii*) that indiscernibility, like identity, is transitive. As it is not necessary for the purpose at hand, we shall not comment on any of these assumptions. Yet it should be pointed out that Martin seems to be suggesting that not only identity in effect is indiscernibility but also that indiscernibility in essence is equality, and hence identity is equality. This should be so in as such as what he is capturing in his definition is equality; worse is that he is claiming for it much more than even mathematicians would claims. To note what practicing mathematicians hold on the same theme we might cite Robinson [67]. "We shall say", he remarks there, "that a relation of equality is defined in a given system of axioms if it includes a relation $E(x, y)$ which is symmetrical, reflexive, and transitive, and such that for every relation $F(x_1, \ldots, x_n)$ includes in the system, it can be proved that equal objects can be substituted for one another as arguments;

$$(x_1)\ldots(x_n)(y_1)\ldots(y_n)\ [(E(x_1y_1)\ \&\ \ldots\ \&\ E(x_ny_n))$$
$$\rightarrow (F(x_1, \ldots, x_n) \rightarrow F(y_1, \ldots, y_n))$$

... The definition of equality, while it does not exhaust the full meaning of identity, will be adequate for our purposes."

Martin's reduction of identity to equality, and equality to indiscernibility in effect reduces the rule of substitution of terms to the rule of simplification in propositional logic. Even if this is beside the point, "identity is contained in" (SQ), as Martin and Quine think, echoing Wittgenstein of the *Tractatus*, because all that is required for this encapsulation is built into (SQ). On of these requirements is that predicates are universal, another is that some of the predicates are null, and the third is that it not be the case that some of the predicates are null and others are universal. This implies that

1) no predicate is defined over the entire domain of interpretation

2) each predicate is defined over a proper subdomain of the domain of interpretation, and

3) no predicate is defined over the empty domain.

When the domain of interpretation is permitted to be empty, these conditions do not hold. Systems like (FQR) for instance do not have the requisite apparatus for defining identity along the lines suggested by Martin. Nor is it possible, within their framework, to construct a schema which will have the "full effect of identity". This *pari passu* holds in the case of the second-order duplication of (FQR) too. Let S be the schema in question. Obviously S is a formula both of (FQR) as well as its second-order duplication. To claim that quantification theory encapsulates identity theory because $x_i = x_j$ and S are in effect the same, and that "identity is aloofly logical as quantification", is tantamount to claiming that $(x_i = x_j) \equiv S$ is a valid formula of the system in which it is a formula. Even if it is valid, it is not valid in (SQ), as Martin claims, but in (SQ=). (SQ) on Martin's reading does not contain universal predicates. Moreover if (SQ) does contain what in effect is identity, why extend it to result in (SQ=)? What is accomplished in extending (SQ) is to smuggle into it a set-theoretic truth. $(x_i = x_j) \equiv S$ is the quantificational counterpart of the set-theoretic truth which can be stated in many ways, one of which is: a set S has as many and no more and no fewer members of S', where all and only those members of S' are the distinct members of all and only the proper subsets of S. Identity has as much to do with cardinality as it has with existence. This implies that a decision on the logicality of identity requires greater analysis, and at one point or other in that analysis we come face to face with set theory and its ontological assumptions. The set-theoretic concept of identity and the concept of identity in the standard extended quantification theory are the same. It might also be the case that the meaning of the two concepts is the same insofar as S as a formula of Martin's version of (SQ) and $x_i = x_j$ are in effect the same. But S as a formula of (FQR) does not meet the conditions laid down by Martin. As a formula in (FQR), S will not be the same as $x_i = x_j$. Thus, if (FQR) is the ontology-free logical part of (SQ), neither can identity be tagged to it, not is it the case that identity is contained in it in any relevant sense of the term.

The logicality of identity is a consequence of the logicality of (SQ) into which is built all the requirements for treating identity as a logical concept. This is why identity is treated as a logical notion on the strength of (SQ); the defences of Quine and Martin are cases in point. But this should not induce us to believe, with Goe [17], that (SQ) "is superfluously string". Far from being so, what Goe thought to be superfluous is actually required for the intended use of (SQ). Even a cursory look at the history of (SQ) will show why it was developed to have the strength it has.

In spelling out why he thinks that identity is included in logic, Quine remarks in [56] that it is due to "its persistent recurrence in all sorts of theories and its relevance to all sorts of universe of discourse ... (it is) customarily considered under the head logic." So the final legitimiser is custom, and the final justification pragmatic! If persistent recurrence is to be a measure of logicality, (IS) ought to have been treated as relevant only to extensional discourse; it should have been made clear that the standard concept of identity does not figure in intensional discourse. On the contrary, Quine debunked intensional discourse because it flouts the rules of (IS). Quine might not be allergic to *post hoc propter hoc* arguments, and hence might deplore this marginal intervention. In any case, even if he is right about his characterisation of identity, it is certain that the justification he gives falls short of being satisfactory.

13 Non-Standard Identity Theories.

The upshot of the foregoing, in brief, is that (IS) provides for substitutivity, reflexivity, symmetry, and transitivity of the identicals provided the entities which satisfy the identity conditions it embeds exist and those that do not satisfy them do not exist. This means that the range of those conditions and the realm of the existents is the same. The import really may be stronger than this, and may amount to the assertion that whatever satisfies those conditions is an entity, whatever does not satisfy them is a non-entity! In this section we shall bring forth the import of some non-standard theories of identity — (InS) for short — which is that identity is not, contra Quine, a univocal concept, and that identity conditions for entities require not only that they exist, but also that they exist in a specific mode. Thus (InS) show that there are as many notions of identity as there are modes of existence.

The import of a specific (InS) depends upon (*i*) the nature of the values of bound variables in $(\exists x_i)Ax_i$ which it requires, and (*ii*) how the members of the set of the substituends for x_i not referring to any of those values are interpreted. This is evidently the same as to say that the import of a (InS) is relative to its semantic interpretation. To highlight the import of various (InS)s we shall group them according to how they fare with reference to the axioms of (IS). As (IS) is supposed to capture all the formal properties of the standard concept of identity, and as the axioms of (IS) are sufficient to define the properties of identity, it follows that an identity, and as the axioms of (IS) are sufficient to define theory in which any of those axioms does not hold without restrictions contains (or incorporates) a concept of identity which is different from the concept captured by (IS).

= is treated either as a predicate constant, or else as a connective. Let us first take the case of those systems which treat is as a diadic predicate. Unrestricted reflexivity of identity holds in all and only those systems in which (*i*) = is treated as a predicate, and (*ii*) Universal Instantiation holds without restrictions (namely in all those systems in which $(\exists x_i)Ax_i$ is assumed to be true, and all terms are permitted to be substituends). The definability of existence in terms of identity and quantifiers is rooted in this. (SQ=) has as its theorems both $(\exists x_i)(x_i = x_i) \rightarrow (x_i = x_i)$, and $(x_i = x_9) \rightarrow (\exists x_i)(x_i = x_i)$. This implies that

1. to be a sustituend of a bound variable,

2. to have a referent, and

3. to be an argument of a reflexive predicate, all are the same with reference to (SQ). Alternatively,

   to be a value,

   to exist, and

   to have a reflexive property,

   all are the same in the universe which is a model of (SQ).

Hence to say that Pegasus does not exist in such a universe is the same as to say that in that universe Pegasus is not Pegasus, meaning that it is not unique, because it could be something else. What is implicit in this is that in (SQ=) not only existence, but also uniqueness, is a universal property — something which Frege missed, but was noted and exploited by Russell. Whatever existsis unique, and whatever is unique exists, reminding us of Aristotle's equation of being one and being existent. That (SQ=) assumes existence as well as uniqueness follows from the fact that $(x_i)(\exists x_j)(x_K)$ $[((x_i = x_j \,\&\, (x_j = x_K)) \rightarrow (x_i = x_K)]$ is a theorem in it. Thus within its framework, non-referring terms cannot be differentiated; we cannot tell Pegasus from Centaur. (Most of the "theories of descriptions" — see [6] — are devices with which the classical logicians attempted to provide uniqueness to non-referring terms while adhering to (SQ=).)

Some purists, however, contend that there is a genuine Pegasus-problem, but providing a solution to it is not the business of logicians for it is not essentially a problem with which they need to be concerned in so far as they are logicians. They also maintain that if it is somehow believed that the problem is a result of the policies which they as logicians followed, then they would revise those policies so as to make their position independent of the Pegasus-problem, at least to the extent that their policies do not foreclose any of the possible solutions to it. Van Fraassen is one of them. He holds "that disputes about truth-value of statements containing non-naming nouns are philosophical in nature and that logic should be neutral in regard to them. Van Fraassen, as Goe [17] commented, is trying to absolve logicians from being responsible for the trouble. To maintain logical neutrality, van Fraassen modifies (following Strawson) his logical position by dosowning the principle of bivalence, and thereby refraining from assigning a truth-value to 'Pegasus = Centaur', or 'Pegasus is Centaur'. He allows 'Pegasus = Pegasus', but does not permit arriving at 'Pegagus is a winged horse' from 'everything is a winged horse', for the truth-value of the former is logically independent of the truth-value of the latter; their "truth-makers are logically independent of each other", as he would say. This is the same as to say that he would not allow 'Pegasus' to be a substitued. But why? Because to allow that would be equivalent to assuming that Pegaus exists; and that is what he wants to be uncommitted about. This could be the ontological fear of a logician; but he is not unkind to Pegaus. He accepts the permisibility of mutual substitution of 'Pegasus' and 'Centaur', provided they are identical. But whether they are identical or not, he would insist, is an extra-logical issue. (Individuation is not, but identity is, a logical notion, and this is an important idea which is usually forgotten in thinking about identity or individuation except by those who belong to the Geach-Quine tradition.) They would be identical if they were numerically one; enumerative equality and not indiscernibility is the measure of identity. Thus the (InS) of van Fraassen [81] with its neutralism goes a long way to accommodate 'Pegasus'. But paradoxically, the wery problem to be solved enters through the back door; let us see how this happens.

The (InS)s of van Fraassen, and of Lambert and Meyer [38], are aimed at designing an identity theory to accommodate terms like 'Pegasus' by bringing into existence a new universe in which everything winged and is also equine. Van Fraassen's system makes the aim explicit in stating that it contains individual constants, and Lambert and Meyer's system make provision for an inclusion of individualconstants. The reason for this insignificant difference, in the words of Lambert and Meyer, is that at the level of pure (rather than applied) logic, it is unnecessary to discriminate between a free variable functions as an "ambiguous name" ... But issues have been clouded from *Principia* on (at least) by insistence that the only appropriate actual names are names of the actuals. If these two systems of (InS) succeed, it is because they are appended to quantification theories which are in themselves unsatisfactory in virtue of containing those rubrics in their respective frameworks. Let us illustrate this point first with reference to van Fraassen's system which is appended to (FQIn) and which permits the inference of Ax_i from $(x_i)Ax_i$ only on the assumption that $\exists!x_i$. It is true that in this system $\exists!x_i$ is not definable in terms of $=$ and the quantifiers, whence 'Pegasus = Pegasus' could be true despite the fact that '$\exists$! Pegasus' is false. But given that in any universe in which everything is a winged hourse, if we accept (FQLM), we will be able to move from 'everything is a winged horse' to 'Pegasus is a winged horse' only if '$\exists$! Pegasus' is true. And $\exists$! Pegasus' is true if and only if 'P (Pegasus)' is true, where P is a monadic predicate uniquely correlatable to a property which each and all things in the universe exemplify. This property obviously cannot be being a winged horse; and whatever might be that property, which is hidden in the semantics of (FQLM), should now be evident. It is that it is sound and complete on the assumption that the domain of interpretation is either empty or else (if it is not empty, that is) all the members of the domain have a common property. But what about domains whose members constitute only imperfect communities? Obviously the universe could be so constituted that each thing in it has a property in common with some other thing but not with all other things. To put it differently the domain of interpretation of (FQLM) is required to be a class and not a set.

Lambert and Meyer deny existence to Pegasus by denying it a place in the extension of the universal property just as Quine would achieve the same by denying it a place in the extension of a relation (the relation of self-identity, to be specific). If the latter is unacceptable, so should the former be; there is no reason why we should prefer on to the other. And Lambert and Meyer think that Quine's policies are untenable. "(I)t is ridiculous", they write in [38], "that from $x = x$ the logician may assert 'Ceaser=Ceaser', withold comment on 'Pegasus = Pegasus' ... and ring up his archeological colleague with respect to 'Romulus = Romulus'. To be sure, Quine's convincing refutation of the myth of analyticity disposes of the corresponding myth that the logician may wholly be unmoved by the facts of life, but we do not expect him to mutter while reading the paper over his morning coffee: "By God, Romulus is self-identical after all." If it is so, their position is equally rediculous. It is not that Quine is unmoved by facts of life; only that he believe that the universe is constituted of a perfect community, and that this was a fact of life.

Concerning van Fraassen's (InS), let us first note that it is to be appended to (SQ) as formulated by Quine, i.e., by treating the axioms standing for their respective universal closures and modifying the rules of inference suitably. We may refer to this as "(SQ)". To extend "(SQ)", van Fraassen adds the following schemata as additional axioms,

1) $(x_i)(x_i = x_i)$, and

2) $(x_i)(x_j)((x_i = x_j) \rightarrow (Ax_i \rightarrow Ax_j)$

where A does not contain individual constants.

The proviso in 2) is necessitated by the fact that the strictly Strawsonian and the not so strictly Fregean semantics of van Fraassen accommodates truth-value gaps. The sufficiency of the proviso for its soundness is due to the fact that in "(SQ)" variables are assumed to have referents. That is to say that van Fraassen's system preserves the standard notion of identity at least for all those contexts when truth-value gaps occur. This moderate aim is understandable in light of his conciliatory gesture that, where truth-value gaps arise, identity ceases to be a logical notion. That identity could be construed, even in this restricted sense, as a logical notion within the framework of "(SQ)" confirms our point that it necessarily requires existence.

"The predicate constant =", writes van Fraassen [84], "is given a definite interpretation: numerical identity. It might therefore be thought that a definite commitment is in order concerning the presuppositions of a statement of the form $t = t'$." Let t and t' be individual variables. The presupposition that t has a referent is within the frame of van Fraassen's system, in effect the same as assuming that this implies that an open formula Ax_i, in order to have a truth-value, will have to be one in which either the terms that occur in it are referring terms or else the terms that occur in it are equivalent to other referring terms, so that the valid counterpart of Universal Instantiation would be $(x_i)(((x_i)Ax_i \mathbin{\&} (\exists x_j)(x_j = x_i)) \rightarrow Ax_j)$.

Though this gambit is an alternative to treating $((x_i)Ax_i \mathbin{\&} \exists! x_i) \rightarrow Ax_i$, or $((x_i)Ax_i \mathbin{\&} x_i = x_i) \rightarrow Ax_i$, as a valid counterpart of UI, it is not at all clear how it is more illuminating: for to say that a term t is a referring one because $\exists!t$ is true is as enlightening as to say that it is true because $(\exists x_i)(x_i = t)$ is true. To say either of these, as Cocchiarella commented, is to "fudge the issue", which is to determine the conditions satisfying which of (1) Plato = Plato, (2) Pegasus = Pegasus, (3) Pegasus = Centaur, and (4) Pegasus = Plato are either true or false.

Other attempts to treat identity as a logical notion do not fare any better. If some of these multiply the "kinds" of quantifiers and "kinds' of the proper subsets of the domain of interpretation, others multiply predicate constants — the identity predicate, to be specific — and thus modes of being. As an example of the former type let us take Routley (InS), (InSR) for short, to be tagged to his version of free logic considered earlier. This system requires a non-empty domain of possible entitites: it does not presuppose the existence of entities, as does (IS), but does assume that entities have "being". It allows instantiation without restrictions and thus permits all things to be self-identical, but does not allow two entities to be mutually substitutable unless they are ontologically fratenal. It is not free from ontological assumptions, but its assumptions are more general than those of (IS). If existence and identity are coextensive and hence are mutually implied in (IS), being and identity are related precisely in the same way in (InSR). This is also the case with (InSC) of Cocchiarella [11].

Routley invokes domains of possibilia and tries to characterize those Meinongian domains in an extended standard syntax. One can, however, accomplish this by resorting to a syntax which contains modal operators. In this sense Routley's systems are to be treated on a par with the free modal logic of Prior. Prior's [51] contains a suggestion, and his [52] contains a partial development of, a free modal logic. Ruzsa [70] has shown that this system fails to meet what is claimed for it, namely that $(\exists)(x = x)$ can be denied in it. As Prior's reading of the existential quantifier is normal, his system requires that the domain of interpretation be non-empty, clinching his claim that it is free from existential assumptions. Ruzsa [70] developed a reformulation of Prior's system so as to meet his intentions; in this development, individual constants (rigid designators) and individual variables (non-rigid designators) are strongly differentiated in the sense that they satisfy different rules of identity. Further in (InSRu) $(\exists x)$ is stronger than it is in (IS) because $(\exists x) \sim \equiv\sim (x) \sim$ does not hold. Similarly, even as a modal system it is non-standard: $M \equiv\sim L \sim$ does not hold in it. In (InS)Ru $\exists x)(x = x)$ becomes unstable when the domain of interpretation is empty; thus it allows for truth-value gaps. All the theorems of (SQ=) come out as theorems when the standard interdefinability of quantifiers is not used in their proofs, that is when $\sim (x) \sim$ is not replaced by $(\exists x)$. Ruzsa thinks that sacrificing the standard laws of quantifier distribution is the inevitable cost of our ability to use non-referring terms.

(InS)G of Grice [18] and (InS)K of Kearns [31], which is the same as the identity theory of Lejewski [44], belong to the category of identity theories which multiply identity predicates. The inclusion of Grice's system in this category may be contested on the ground that only one predicated constant, namely =, figures in its vocabulary. Nevertheless, it in effect multiplies what Grice himself calls ''types of identity". Its syntactical specification is carried out in such a way that two predicate constant symbols are not needed since two different senses of identity can be expressed in it by using the same predicate constant. Thus we can say that if (InS)K explicitly multiplies predicate constants, (InS)G does it implicitly. Despite the difference in their interpretations, they have one more thing in common, namely that the concept of identity as captured by (IS) is definable in them, again with the difference that it is definable in a second-order duplication of (InS)G and in the first-order (InS)K. All that is needed to note that (InS)G multiplies modes of existence is to look at the domain of interpretation for which it is sound and complete. It constitutes "a non-empty domain within which two subdomains are to be distinguished: the special subdomain (which may be empty) the elements of which will be each a unit set in D whose element is also in D; special subdomain". This means that Grice's attempt to bestow logical status to (InS)G needs not only the existence of entities but also their heterogeneity. His system, while exploding the "myth" of self-identity being a measure of existence (by allowing validity to $t = t$ irrespective of whether t has a referent in the residual subdomain or not) perpetuates another myth, namely that all terms not having referents in the special subdomains may be identical. You may smell a Fregean rat here. To avoid that, the only way left open is to multiply null sets, as Bunge [3] suggests (see [46] and [78]). But that would clinch the extensionality thesis which is central to standard set theory.

Kearns does not multiply modes of being in (InS)K, but being a "nominalist", he multiplies modes of naming. (In this context nominalism and Platonism are to understood in an extremely limited sense; one who opts for referential interpretations is a Platonist, and one who opts for substitutional interpretations is a nominalist.) As a nominalist, he might even hold that identity is a relation that holds between names and not between entities. But how does logic dictate that 'Pegasus' is not the same as 'Pegasus' while 'Plato' is the same as 'Plato'? Kearns holds that 'Pegasus = Pegasus' is false while 'Plato = Plato' is true. Why not allow all names to be self-identical? (To be priggish, why not allow all tokens of the same type be identical?) Kearns would hold that it could not be allowed because $=$ is not a universal predicate; nor is it a univocal predicate. (IS) is defective because it treats $=$ as a universal and univocal predicate, Kearns argues. (InS)K, on the other hand, takes into account each of its senses that are univocal and treats it as a universal predicate in that sense. And corresponding to each of those senses there is a distinct identity predicate. But these distinct univocal predicates are not unrelated; they are interdefinable. Kearns holds that it is legitimate to bestow logicality on the theory which captures the formal properties of these interrelated univocal predicates. In (InS)K he tries to capture the formal properties of at least two such univocal predicates of identity, namely 'is existentially identical with' and 'is non-existentially identical with', $\overset{e}{=}$ and $=$ respectively. As these are not all the senses of identity, his theory is partial. On the strength of his theory, it is true that 'Pegasus $=$ Pegasus', but it is false that 'Pegasus $\overset{e}{=}$ Pegasus'; but both 'Plato $=$ Plato' and 'Plato $\overset{e}{=}$ Plato' are true. In (InS)K, $\overset{e}{=}$ and $\exists!$ are related in the same way as $=$ and $\exists!$ are related in (IS). So all the objections raised against (IS) on that score, can also be raised against (InS)k.

We may briefly note some other alternatives, in particular (InS)Sc of Scales [71], intended to be tagged to his attributional logic which not only shares with (InS)K the invalidity of the unconditional self-identity and the coextensiveness of existence with restricted self-identity, but also has a common semantic feature with (InS)Sct of Scott [75]. Both these systems are sound and complete only with reference to models which include both domains and outer domains. What these outer domains should consist of is not a matter that can be settled by logic, insofar as what logic itself should consist of is determined in terms of models which include them. This objection, however, has no bearing on Scott's views on identity, since for him identity is an extra-logical notion. In [75] he argued that it is so basic a concept that its properties cannot be thought of as a part of logic.

14 Further Groupings

"It should be noted", writes Schock [74], that "if identity is understood as a predicate rather than some special symbol and interpreted accordingly, then the validity of self-identity in systems of Hailperin-Leblanc, and Hintikka requires that individual constants denote (although some or all of them may denote fictions ...) In other words, these logics do not really allow for empty terms". As an alternative, Schock designed a system which provides rubrics for empty terms and yet has all the salient features of the systems mentioned by him, specifically existence and self-identity are coextensive, and the set of substituends and the set of referring terms are the same. The identity part of this system (InS)SK is shown to be consistent and complete when either

1. all of the individual variables are interpreted to be empty terms, or else

2. no individual variable is interpreted to be an empty term.

If (1) all of its individual constants may be taken as empty terms, and if (2) some of its individual constant may be empty and some may be non-empty, then (InS)SK is a free logic in a very peculiar sense. Its interpretations are not carried over a possibly empty domain D of individuals, but over the set D^* where all and only the $x \geq 0$ membered subsets of D are member of D^*. Thus, even if D is empty, D^* will not be, since the null set will be a member of it. But what legitimises the "existence' of the null set? This is important because if D^* is empty, Schock's interpretation collapses. Only on the basis of these assumptions could Schock provide a justification for the inclusion of identity in logic. But through these assumptions a considerable amount of ontology creeps into logic, for "in set theory there is not", as Goe [17] noted, "one reality to be discovered, but alternative ones depending upon the choice of axioms." The implications of this point are important as they have some bearing on what we have been trying to suggest. So the point needs to be spelled out a little more explicitly. But before that we shall take into account one more identity theory, as it too, will provide support to what we are suggesting about identity.

All the identity theories which we considered so far treat = as a predicate constant. Skyrms [76] thinks that the troubles that arose due to that could be overcome by treating it as a connective instead. He sketches a theory of non-standard identity (InS)SS in which = is considered as a connective but without saying, unfortunately, what sort of connective it is. Though it does not appear to be a diadic truth-functional connective, as it always occurs between terms only, it behaves like a truth-functional connective because of the linkage between the truth-values of, say $t_1 = t_2$, which is an atomic formula, and the truth-functional compound $At_i \Rightarrow\!\!> At_j$. In the semantics of (InS)SS, $t_2 = t_2$ will have as its value only truth, whether t_i has a referent in the domain of interpretation or not. $t_i = t_j$ will also have truth as its value for one assignment of denotation to t_j (say by invoking the null set), and falsity for other assignments, that is when t_2 has a referent and t_j does not have one. Skyrms defines validity *à la* van Fraassen in terms of supervaluations.

(InS)SS was thought of as a total Fregean alternative to the quasi-Fregean system of van Fraassen. Skyrms is of the opinion that "if one is to be a Fregean at all, he will have to be so with reference to identity. For a strict Fregean, $A \vee \sim A$ must lack a truth-value if A contains a non-designating singular term. With the aid of supervaluations, van Fraassen adds an Aristotelian redemption to the notion of *Sinn* to restore to it the lost status of being a logical truth. This notion of redemption is that if every way of filling holes in its reference makes it true, then it is a logical truth. Van Fraassen applies this only to structures containing sentential connectives". Skyrms is extending it to identity. The upshot of (InS)SS is that existence is not a property of individuals, let alone a property coextensive with self-identity, or with being identical with another individual. In other words 'exists' is not a first-order predicate, and hence its definability or indefinability in terms of other first-order predicates does not arise at all. 'Exists' is a second-order predicate, and existence is a semantic property, perhaps like truth. In (InS)SS, 'x_i exists' and "$(\exists x_i)(x_i = t_i)$' is true' are in effect the same. This however is not a very satisfying result; and as we are not in van Fraassen's system, argument completeness and statement completeness do not coincide. This is to say that $(A_1 \ \& \ A_2) \rightarrow A_3$ is semantically valid if and only if $A_1, A_2 \ \vdash A_3$ is a valid argument form does not hold in either of these systems. If van Fraassen's system is statement-complete but not argument-complete, (InS)SS is argument-complete but not statement-complete. In it $((x_i)Ax_i \ \& \ (\exists x_i)(x_i = t_i)) \rightarrow At_i$ will not be a logical truth though $(x_i)Ax_i \ \& \ (\exists)(x_i = t_i)) \rightarrow At_i$ will come out as va valid argument form. It is the convergence or the divergence of these two kinds of completeness with reference to a system which shows whether it is logical or extra-logical (see my [59]). Extra-logical assumptions will have entered into a system at the point where this convergence breaks down. This is why identity will have to be treated as an extra-logical notion. The conditions under which entities could be identical cannot be imposed on them, as it is the nature of the entities could be identitical cannot be imposed on them, as it is the nature of the entities that determines the conditions of their identity.

15 The Upshot

The foregoing outcome should not undermine the importance of the theory of identity, however. Without identity conditions, no "informative discourse" is possible, since identity and existence are mutually implied. It is true that, as Kearns [33] says, free logic "does not possess the resources for distinguishing what exists from what does not exist ... (and) any language that is suitable for informative discourse about individuals must possess a device for making that distinction." Now 'informative discourse' is a count-noun. A theory is the result of a unification of an informative discourse' is a count-noun. A theory is the result of a unification of an informative discourse which states the conditions for being informative and the interrelations between the different constitient parts of the language in which that discourse has been carried out, so that the information contained in that discourse is only true information about the entities of which it is supposed to be a discourse. Further, to state the conditions about which a discourse is informative is to specify the domain of its reference with its constituents such that all the referring expressions of the language in which that discourse is carried out either have referents in that domain or else have a link with that domain derivatively by being connected with the expressions of that language which have referents in that domain (depending upon the theoretician's semantic predilections and ontological predispositions). A statement of those connections is a proper part of that discourse although it might share those connections with other discourses. Logic has no role in contesting these connections, much less in stating them. One may call a statement of such connections with reference to a discourse its "meaning-postulates", in the fashion of Carnap. If logic enters into the picture, it is only at the stage of unifying such postulates and the hitherto discrete bits of information. Russell, more than anyone else, ought to have realized this in connection with his strategy for incorporating non-denoting terms into theories. On the contrary, his strategy, his theory of descriptions that is, resulted not only in a "paradigm of philosophical enquiry" carrying with it the myth that "logic" can solve unresolved metaphysical and ontological problems, but also in a paradigm of confusion which has ever since blocked a demarcation of the boundaries of the concept of logic. Logic does not so much provide the conditions for a unification of knowledge as much as it foils to provide new knowledge. This is to say that the meaning postulates of a theory (in which are to be included the axioms of identity allowing that theory to provide the conditions of identity and uniqueness of the entities of which it is supposed to be a theory) cannot be had from logic. So logic, even free logic, "does not possess the resources for distinguishing what exists from what does not exist" because it need not possess these. Kearns remarked that "the general logical framework of talk about individuals does not require the concept of existence", and parity of reasoning it does not require the concept of identity either.

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