CONTRIBUTIONS OF POLISH LOGICIANS TO DECIDABILITY THEORY*

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1. Introduction.

Research on decidability problems has arisen from Hilbert's program. To save the integrity of classical mathematics and to overcome difficulties disclosed by antinomies and paradoxes of set theory, Hilbert proposed, in a series of lectures and papers in the 1920s, a special program. Its aim was to show that classical mathematics is consistent and that actual infinity, which seemed to generate the difficulties, plays in fact only an auxiliary role and can be eliminated from proofs of theorems talking only about finitary objects. To realize this program Hilbert suggested first of all to formalize mathematics, i.e., to represent its main domains (including classical logic, set theory, arithmetic, analysis, etc.) as a big formal system and investigate the latter (as a system of sequences of symbols transformed according to certain fixed formal rules) by finitary methods.Such formalization should be complete, i.e., axioms should be chosen in such a way that any problem which can be formulated in the language of a given theory can also be solved on the basis of its axioms. Formalization yielded also another problem closely connected with completeness and called the decision problem, decidability problem, or *Entscheidungsproblem*: one could ask if a given formalized theory is decidable, i.e., if there exists a uniform mechanical method which enables us to decide in a finite number of steps whether or not a given formula in the language of the theory under consideration is a theorem. Using notions from recursion theory, one can make this definition more precise. A formal theory T is said to be decidable iff the set of Gödel numbers of) theorems of T is recursive. Otherwise T is called undecidable. If, additionally, every consistent extension of the theory T (formalized in the same language as T) is undecidable, then T is said to be essentially undecidable. We should notice here that these definitions, completely standard today, come from Alfred Tarski and were formulated in his paper, *A general method in proofs of undecidability*, published in the book [Tarski, Mostowski & Robinson *1953*].

In the 1920s and 1930s many results in this direction were obtained. Several general methods of proving completeness and decidability of theories were proposed, and many theories were shown to be complete and decidable.

K. Gödel's results from 1931 on incompleteness (*cf.* [Gödel *1931*]) and A. Church's 1936 results on undecidability (of first-order predicate calculus and of Peano arithmetic and certain of its subtheories (*cf.* [Church *1936*] and [Church *1936a*]), as well as results of J. B. Rosser from the same year on undecidability of consistent extensions of Peano arithmetic (*cf.* [Rosser *1936*]), indicated some difficulties and obstacles in realizations of Hilbert's program. One consequence was the fact that since the end of the 1930s, logicians' and mathematicians' interests concentrated on proofs of undecidability rather than decidability of theories. General methods of proving undecidability were investigated, and particular mathematical theories were shown to be undecidable.

Those general trends and tendencies are reflected also in the history of logic in Poland. That is the subject of the present paper, which has the following structure. Section 2 will be devoted to the contributions of Polish logicians and mathematicians to the study of decidability of theories. In particular, the method of quantifier elimination studied by A. Tarski and his students, together with various applications of it, will be presented. Section 3 deals with investigations of undecidability of theories. Again, we shall discuss some general methods developed by Polish logicians for proving the undecidability of a theory (in particular, Tarski's method of interpretation will be considered), as well as several results on undecidability of particular theories. Section 4 will be devoted to the presentation of (forgotten and underestimated) results of Józef Pepis on mutual reducibility of decision problems for various classes of formulas. Finally, Section 5 will present generalizations and strengthenings of Gödel's incompleteness results due to Polish logicians.

2. Decidability of theories.

2.1. Effective quantifier elimination.

The seminar led by A. Tarski at Warsaw University in 1927-29 was devoted to the method of eliminating quantifiers. This method was initiated by L. Löwenheim in 1915 (*cf.* [Löwenheim *1915*]) and used in fully-developed form by Th. Skolem (*cf.* [Skloem *1919*]) and C. H. Langford (*cf.* [Langford *1927*; *1927a*]). Skolem used it to show the decidability of the theory T_0 of the class of all "full" Boolean algebras, i.e., Boolean algebras of the form $\langle P(A), \subseteq \rangle$, where $P(A) = \{B : B \subseteq A\}$. Langford applied it to establish the decidability of the theory of a dense linear order: (a) without endpoints, (b) with a first element but no last element, and (c) with first and last elements, as well as the decidability (and completeness) of the theory of discrete orders having a first and no last element.

Elimination of quantifiers was studied at the seminar by Tarski and his students. Tarski described it as "a frequently used method, which consists in reducing the sentences to normal form and successively eliminating the quantifiers" (*cf.* [Tarski *1936*]; see also [Tarski *1956a*, 374], which is a revised English translation of [Tarski *1935*] and [Tarski *1936*].)

Generally speaking, this method can be described as follows: Let T be a first-order theory in the language $L(T)$. We are looking for a set Φ of formulas in $L(T)$, called the set of basic formulas for T, such that for every formula φ of $L(T)$ there exists a Boolean combination φ^* of formulas from Φ having the same free variables as φ and such that $T \vdash \varphi \equiv \varphi^*$. Of course, one can always take as the basic set Φ the set of all formulas of the language $L(T)$. But we are looking for "good" basic sets consisting of "simple" formulas. Unfortunately, there is no precise criterion for being "good" or "simple". Usually the following conditions are required: (1) Φ should be reasonably small and irredundant, (2) every formula in Φ should have some straightforward mathematical meaning, (3) there should exist an algorithm for reducing every formula φ of the language $L(T)$ to its corresponding Boolean "representation". From the point of view of decidability problems, the set Φ should have one more property: (4) there should exist an algorithm that tells us, given any basic sentence ψ, either that ψ is a theorem of T or that ψ is refutable from T. If conditions (3) and (4) are met, then we have both a completeness proof and a decision procedure for the theory T. Let us add that a method fulfilling conditions (3) and (4) is sometimes called "effective quantifier elimination" to distinguish it from other types of elimination of quantifiers, in which there are no algorithms.

It seems that Tarski found this method adequate for his purposes, and he did not try to generalize it but (together with his students) simply applied it to the study of various theories. In this way it became in his school *the* method and a paradigm of how a logician should study an axiomatic theory.

What aims did Tarski and his students want to achieve using the method of quantifier elimination?

First, they used it to characterize definability. Namely, suppose that a structure $\mathcal{M}$ and a set of first-order sentences T that are true in $\mathcal{M}$ are given. Suppose also that one has found a basic set Φ for the theory T. Then the relations on $\mathcal{M}$ that are definable by first-order formulas (with or without parameters) are exactly the Boolean combinations of the relations that are defined by formulas in Φ. It seems that the first who made this point was Tarski in [Tarski *1941*]. This paper introduces the decision procedure for the theory of reals, but it does so by way of describing the first-order definable relations on the reals. Tarski was greatly interested in the notion of a definable relation in a structure, but his papers on decidability or completeness rarely mentioned this aspect of the results.

The second reason Tarski and his collaborators used the method of quantifier elimination was the fact that it could contribute to the study of decidability problems. One should notice here that in his papers published before World War II, Tarski seldom mentioned effective decidability. In describing his results he put emphasis on completeness proofs rather than on algorithmic decidability. On the other hand, it was clear for him by 1930 that effective quantifier elimination could give:

> a mechanical method which enables us to decide in each particular case whether a given sentence (of order 1) is provable or disprovable

(*cf.* [Tarski *1956a*, 134], the English translation of [Tarski *1941*]). In [Tarski *1939*] he refers to:

> The "effective" character of all positive proofs of completeness so far given — not only the problem of completeness but also the decision problem is solved in the positive sense for all the deductive systems mentioned above.

One can observe a very noticeable change of emphasis in Tarski's writings after the war. Now he stresses the importance of decidability results and gives them priority. On the first page of [Tarski *1953*] he describes the decision problem as "one of the central problems of contemporary metamathematics". Both of his abstracts [Tarski *1949*] and [Tarski *1949a*] are opened by saying that he has found a decision procedure — and only later is it added that various consequences follow from the procedure. In his writings before World War II it was just the reverse — the results on completeness were stressed first, and only at the end was it added that the proofs involve solving the decision problem.

In the monograph [Tarski *1967*] written in 1939 (and published in 1967) he says:

> It should be emphasized that the proofs sketched below have (like all

> proofs of completeness hitherto published) an "effective" character in the following sense: it is not merely shown that every statement of a given theory is, so to speak, in principle provable or disprovable, but at the same time a procedure is given which permits every such statement actually to be proved or disproved by means of proof of the theory. By the aid of such a proof not only the problem of completeness but also the *decision problem* is solved for the given system in a positive sense[11]. In other words, our results show that it is possible to construct a machine which would provide the solution of every problem in elementary algebra and geometry (to the extent described above).

And in Note 11 of [Tarski *1967*] we find the following words:

> It is possible to defend the standpoint that in all cases in which a theory is tested with respect to its completeness the essence of the problem is not in the mere proof of completeness but in giving a decision procedure (or in the demonstration that it is impossible to give such a procedure).

What were the reasons for this change? The first reason is the fact that until the mid-1930s there was no precise definition of the term "algorithm", and mathematicians doubted if there ever would be one. The works of A. Church, A. Turing, S. C. Kleene, J. B. Rosser, and others and the development of recursion theory changed the situation. Tarski recognized this change immediately and began to apply recursion theory in his papers.

The second source of Tarski's new perspective on his work on completeness/decidability problems was that he had become interested in proofs of undecidability, publishing many results during the years 1949–1953. (We discuss these in the next section.)

Coming back to the aims Tarski and his students wanted to achieve using the method of quantifier elimination, one should add one thing more. Namely, they saw not only that this method yields much information on completeness and decidability, but also that it can be used to describe and classify all complete extensions of a given first-order theory T. In connection with this, Tarski introduced in his seminar some key notions of model theory, in particular the notion of elementary equivalence.

To finish the general discussion of the method of quantifier elimination, we want to consider the connection between the completeness of a theory and its decidability. (This was already mentioned above). We mean here the theorem stating that if a first-order theory T based on a recursive set of axioms (in fact, "recursive" can be replaced by "recursively enumerable") is consistent and complete, then T is decidable. The explicit formulation of this can be found in the paper by Antoni Janiczak [Janiczak *1950*] from 1950. He proved there the following:

Theorem 2.1. *A complete, consistent theory T satisfying the conditions (a′) – (d′) is decidable.*

The conditions (a′) – (d′) say, respectively, that the set of (Gödel numbers of) sentences of the language $L(T)$ of the theory T should be recursive, that the set of (Gödel numbers of) axioms of T should be recursively enumerable, that the arithmetized counterpart of the (metamathematical) relation "a formula χ results from formulas φ and ψ by the deduction rule $\mathfrak{R}$" is recursive, and finally that there exists a recursive function *Neg* such that if x is a Gödel number of a formula φ, then $Neg(x)$ is a Gödel number of the negation of φ.

The key fact used in the proof of this theorem is the negation theorem, which states that a relation is recursive iff it and its complement are recursively enumerable. One finds this result in Kleene's paper [Kleene *1943*] (Theorem V) and in the [*1955*] paper by Janiczak (a posthumous work prepared by A. Grzegorczyk from the notes left by the author, who died prematurely in Warsaw on 5 July 1951).

At the end of Janiczak's paper [Janiczak *1950*] it is written: "It is worth remarking, in connection with our theorem, that each decidable and consistent theory can be enlarged to a decidable, complete and consistent theory by the method of Lindenbaum . . . . This result is due to Tarski." And Janiczak refers here to Theorem II of Tarski's abstract *On essential undecidability* (*cf.* [Tarski *1949b*]).

A survey of Tarski's work on decidable theories is given by Doner and Hodges (*cf.* [Doner & Hodges 1988]).

2.2. Applications: decidability of particular theories.

The first decidability results of Tarski came from 1926–1928 (*cf.* [Tarski *1935*] and [Tarski *1936*]). They were based on the work of Langford mentioned at the beginning of this section. More exactly, Tarski proved the following theorems.

Theorem 2.2. *If T is the theory of dense linear orders, then for any first-order sentence φ in the language of T we can compute a Boolean combination of the sentences "There is a first element" and "There is a last element" that is equivalent to φ modulo T.*

Theorem 2.3. *Let T be the theory of linear orders in which every element except the first element has an immediate predecessor, and every element except the last element has an immediate successor. Then for any first-order sentence φ in the language of T we can compute a Boolean combination of the sentences "There is a first element," "There is a last element," and "There are at most n elements" (for positive integers n) that is equivalent to φ modulo T.*

In the Appendix to his [*1936*] Tarski listed all the complete theories of dense linear orders; they are the theories of the orders η, $1 + \eta$, $\eta + 1$, and $1 + \eta + 1$, where η is the natural ordering of the rational numbers. The complete theories of linear orders are the theories of finite orders and the theories of the orders ω, ω^*, $\omega^* + \omega$, and $\omega + \omega^*$, where ω is the natural ordering of the natural numbers and ω^* is its inverse. Those classifications were obtained with the help of the above-quoted theorems.

Having solved the problem of completeness and decidability of the theory of dense linear orders and of the theory of discrete orders, one should ask about the theory of well-orderings. The study of this problem has a long and interesting history.

In the late 1930s A. Tarski and his student Andrzej Mostowski developed the outline of a proof of the decidability of this theory, described semantically as the set of sentences true in all well-ordered structures. They used the method of quantifier elimination, and by the summer of 1939 they reached a clear idea of the basic formulas. Their main aim was to prove the adequacy of certain axiom systems and to classify the complete extensions of the theory. (Decidability would then be a by-product.) Many technical details remained to be worked out. The work was interrupted by World War II. Tarski escaped the German occupation of Poland and stayed in the USA; Mostowski spent the wartime in Poland. Each began to work out the technical details. Unfortunately, Mostowski's notes were destroyed in Warsaw in 1944, and Tarski's were lost in the course of his many moves. Hence the work had to be started over from the very beginning. They published the abstract [Mostowski & Tarski *1949*] and made plans to reconstruct the proof. This was not realized, and nothing was done beyond writing down the specifications of the basic formulas.

In 1964 Tarski assigned to his student John Doner the task of working out some of the details. This was done in a rough form, and again nothing happened until 1975. In this year Mostowski and Tarski met and made plans to finish the paper. Unfortunately, Mostowski's unexpected death in August 1975 upset the plans. In this situation Tarski invited Doner to take up the work. The final result was the joint paper of Doner, Mostowski, and Tarski, *The elementary theory of well-ordering. A metamathematical study*, published in 1978 (*cf.* [Doner, Mostowski & Tarski *1978*]). One finds there a number of results not mentioned in [Mostowski & Tarski *1949*].

The next theory studied by Tarski by the method of quantifier elimination was the theory of Boolean algebras. Tarski never published either the decision procedure or a proof of the classification of complete first-order theories of Boolean algebras. One can reconstruct the essential elements of these from his abstract [Tarski *1949*]. Tarski announced there that he had a decision procedure for the theory of Boolean algebras, and he used it to classify all the complete first-order theories of Boolean algebras in terms of countably many algebraic invariants. Each invariant was expressible as a single first-order sentence. In the monograph [Tarski *1948*] he mentions on page 1 that he found this decision procedure in 1940.

In connection with these results one should note that already in the paper [Tarski *1936*] Tarski published the analogue of Theorem 2.3 for atomic Boolean algebras. He even claimed that if we drop the assumption of atomicity, then there are just countably many completions of the axioms of "the algebra of logic". He seems to mean here that there are just countably many complete first-order theories of Boolean algebras. But it is not clear how he could have proved this result without having the full classification.

One should also stress that, unlike Tarski's proof, the model-theoretic proofs of the classification we have today, e.g., those by Yu. L. Ershov and H. J. Keisler, do not give a primitive recursive decision procedure (though they imply the decidability of the theory of Boolean algebras).

The problem of the decidability of the theory of Boolean algebras was also treated by Stanisław Jaśkowski [Jaśkowski *1949*].

The method of quantifier elimination was also applied by Tarski to the study of geometry. M. Presburger writes in the paper [Presburger *1930*, footnote on p. 95] that in 1927–1928 Tarski proved the completeness of a set of axioms for the concept of *betweenness* ("*b* lies between a and c") and that of *equidistance* ("a is as far from b as c is from d"). The method used by him was quantifier elimination. Tarski himself saw this result as "a partial result tending in the same direction" as his later theorem on real-closed fields (*cf.* [Tarski *1948*, footnote 4]).

In conjunction with Tarski's results on decidability of geometry, one should also mention his paper [Tarski *1959*]. Using results on real-closed fields (*cf.* [Tarski *1948*]) that we shall discuss later, he studied various theories of elementary geometry, defining the latter with the words: "we regard as elementary that part of Euclidean geometry which can be formulated and established without the help of any set-theoretical devices" (p. 16). In particular, he considered there the system $\mathfrak{E}_2$ in the language containing predicates for the betweenness and equidistance relations and based on the following axioms: identity, transitivity, and connectivity for betweenness, reflexivity, identity, and transitivity for equidistance, Pasch's axiom, Euclid's axiom, five-segment axiom, axiom of segment construction, lower and upper dimension axioms, and elementary continuity axiom. It is proved that the theory $\mathfrak{E}_2$ is complete and decidable and is not finitely axiomatizable. It is also shown that a variant $\mathfrak{E}''_2$ of $\mathfrak{E}_2$ (obtained from $\mathfrak{E}_2$ by replacing the elementary continuity axiom, which is in fact a scheme of axioms, by a weaker single axiom) is decidable with respect to the set of its universal sentences.

Another domain in which Tarski applied the method of quantifier elimination was that of fields. In his [*1931*] paper Tarski wrote that he had a complete set of axioms for the first-order theory of the reals in the language with primitive nonlogical notions 1, $\leq$, and +. The paper contains a sketch of the quantifier-elimination procedure for this theory and a description of those relations on the reals that are first-order definable in the language with the indicated primitive notions.

The most important and most famous result of Tarski for fields concerns the theory in the richer language with 0, 1, +, $\cdot$, $\leq$ as nonlogical primitive notions. We mean here his fundamental theorem:

Theorem 2.4. *To any formula $\varphi(x_1, \ldots, x_m)$ in the language with* 0, 1, +, $\cdot$, $\leq$ *one can effectively associate: (1) a quantifier-free formula $\varphi^*(x_1, \ldots, x_m)$ in the same language and (2) a proof of the equivalence $\varphi \equiv \varphi^*$ that uses only the axioms for real-closed fields.*

The first announcement of this result can be found in Tarski's [*1930*] abstract, where he wrote:

> In order that a set of numbers A be arithmetically definable it is necessary and sufficient that A be a union of finitely many (open or closed) intervals with algebraic endpoints

(English translation by L. van den Dries, *cf.* [van den Dries *1988*]). This follows immediately from Theorem 2.4 for $m = 1$. No information on the proof can be found in the abstract. One should notice that the emphasis was put on definability (*cf.* our earlier remarks).

A precise formulation of the fundamental theorem and a clear outline of its proof were given in the 1967 monograph, *The Completeness of Elementary Algebra and Geometry* (*cf.* [Tarski *1967*]). This work was written in 1939, but the war made its publication impossible then. (The paper reached the stage of page proofs, but publication was interrupted by wartime developments.) Its title suggests a change of emphasis from definability to problems of completeness.

A full and detailed proof of Theorem 2.4 finally appeared in Tarski's [*1948*] work, *A Decision Method for Elementary Algebra and Geometry*, prepared for publication by J. C. C. McKinsey. Its title reveals a second change of emphasis — this time from completeness to decidability. In the "Preface" to the second edition (from 1951) Tarski wrote:

> As was to be expected it reflected the specific interests which the RAND corporation found in the results. The decision method . . . was presented in a systematic and detailed way, thus bringing to the fore the possibility of constructing an actual decision machine. Other, more theoretical aspects of the problems discussed were treated less thoroughly, and only in notes.

Note that the results in [van den Dries *1988*] were formulated in terms of the field of real numbers, but they hold generally for real-closed fields — the latter are mentioned only in footnotes.

Real-closed fields and algebraically closed fields were discussed by Tarski in his abstract from 1949 [Tarski *1949a*]. One finds there a description (in an algebraic language) of each class of the form: all algebraically closed (or real-closed) fields that are models of some complete first-order theory. Tarski remarks also that he found a decision procedure for the theory of algebraically closed fields and says that his classification "follow" from this procedure. A decision procedure for real-closed fields, based on an extension of Sturm's theorem "to arbitrary systems of algebraic equations and inequalities in many unknowns," is also mentioned. It implies that the theory of real-closed fields is "consistent and complete" and that any two models of this theory are elementarily equivalent.

As we mentioned earlier, the method of effective quantifier elimination was used not only by Tarski, but also by his students. Among results obtained by the latter, one should mention here Mojżesz Presburger's result on decidability of the arithmetic of addition and Wanda Szmielew's result on decidability of the theory of abelian groups.

In the school year 1927/28 A. Tarski gave a course of lectures on first-order theories. During this course he presented a set of axioms for the theory of addition of natural numbers. It is formalized in a first-order language with 0, S, and + as the only nonlogical primitive notions. (Hence there is no multiplication.) Today this is called Presburger arithmetic. Tarski formulated the problem of showing that the axioms are complete. This was solved by Presburger in May 1928 and published in 1930 (*cf.* [Presburger *1930*]). The result was also presented as his master's thesis.

The method used by Presburger was effective quantifier elimination, of course. It was shown that one can take as a set of basic formulas the set consisting of formulas of the form $\alpha x + a = b$, $\alpha x + a \leq b$, $b < \alpha x + a$, $\alpha x + a \equiv_n b$, where a and b are terms in which the variable x does not occur freely, α is a natural number, the symbol αx is an abbreviation for $\underbrace{x + \ldots + x}_{\alpha \text{ times}}$, the relations $<$ and $\leq$ are defined in the usual way, and the relation $\equiv_n$ is defined as follows:

$$x \equiv_n y \equiv \exists z\, (x = y + (\underbrace{z + \ldots + z}_{n}) \vee y = x + (\underbrace{z + \ldots + z}_{n}))\,.$$

In this way one obtains the completeness and the decidability of the theory under consideration. (Notice that Presburger, like Tarski at that time, formulated his result in terms of completeness and did not mention decidability.)

One should also add here that similar results for the theory of the successor operation (in the first-order language with 0 and S the only primitive nonlogical notions) and for the theory of multiplication (in the first-order language with only 0, S, and $\cdot$) of natural numbers were obtained by J. Herbrand (1928) and Th. Skolem (1930), respectively. They used the method of quantifier elimination as well.

The second important result obtained in Tarski's school by the method of effective quantifier elimination was the result of Wanda Szmielew on decidability of the theory of abelian groups (*cf.* [Szmielew *1949*, *1949a*, *1955*]). She gave a classification of all complete first-order theories of abelian groups. This was done by describing a set of algebraic invariants which were expressible by first-order sentences. Those results have various consequences (noted by Szmielew, e.g., in [Szmielew *1955*]). Using them, one can obtain many examples of non-elementarily-definable (in her terminology, non-arithmetical) classes, e.g., the class of all finite groups, the class of all simple groups, the class of all torsion groups, and that of all torsion-free groups. It can also be shown that there exist two infinite groups of the same power (for instance two denumerable groups) that are elementarily equivalent but non-isomorphic.

3. Undecidability of theories.

As we mentioned above, the results of K. Gödel, A. Church, and J. B. Rosser on undecidability and essential undecidability shifted the interests of logicians towards undecidability of theories. Polish logicians followed this shift and contributed to undecidability research as well. This section is devoted that activity. First, the work of Tarski on general methods of establishing the (essential) undecidability of first-order theories will be discussed. (A survey of Tarski's work on undecidable theories is given by McNulty; *cf.* [McNulty *1986*].) Then some applications, due to Polish logicians, of those methods will be indicated.

3.1. General methods of proving undecidability.

The most important work in this respect is the paper by A. Tarski, *A general method in proofs of undecidability*, published as Part One of the work [Tarski, Mostowski & Robinson *1953*]; *cf.* [Tarski *1953*]). It contains his results obtained during 1938–1939 (and summarized in the abstract [Tarski *1949b*]).

Tarski distinguishes there two types of methods for proving undecidability: the direct method and the indirect one. The former was applied by Gödel, Church, and Rosser (in the above-quoted papers). It uses the notion of recursivity of functions and relations and is based on the condition that all recursive functions and relations are strongly representable in the theory being considered. Hence this method may be applied only to theories in which a sufficient number-theoretical apparatus can be developed.

The second method, the indirect one, "consists in reducing the decision problem for a theory T_1 to the decision problem for some other theory T_2 for which the problem has previously been solved" (*cf.* [Tarski *1953*, 4]). This reduction can take place in two ways: to establish the undecidability of a theory T_1, one can try to show either that (1) T_1 can be obtained from some undecidable theory T_2 by deleting finitely many axioms (and not changing the language) or that (2) some essentially undecidable theory T_2 is interpretable in T_1.

Only the second way was elaborated by Tarski. Before describing Tarski's results, let us note here that the notion of interpretability, i.e., of definability of the fundamental notions of one theory in another theory, is a key ingredient in Tarski's investigations. Recall what has been said above on Tarski's motivations behind his work on effective quantifier elimination. It should be added that the results on undecidability of theories were very often only one kind of consequence of Tarski's results on definability and interpretability.

In his [*1953*] paper Tarski treats interpretability and relative interpretability, dealing with each of them in two forms: full (i.e., unqualified) and weak.

Let T_1 and T_2 be two first-order theories. Assume that they have no nonlogical constants in common. (By an appropriate change of symbols, this assumption can always be fulfilled.) The theory T_2 is said to be interpretable in the theory T_1 iff we can extend T_1 by adding to its axioms some definitions of nonlogical primitive notions of T_2 in such a way that the extension turns out to be an extension of T_2 as well. The theory T_2 is weakly interpretable in the theory T_1 iff T_2 is interpretable in some consistent extension of T_1 formalized in the same language $L(T_1)$. The main theorem concerning undecidability is now the following one (*cf.* Tarski *1953*, Theorem 8]):

Theorem 3.1. *Let T_1 and T_2 be two theories such that T_2 is weakly interpretable in T_1 or in some inessential extension of T_1. If T_2 is essentially undecidable and finitely axiomatizable, then:*

(i) *T_1 is undecidable, and so is every subtheory of T_1 that has the same constants as T_1;*

(ii) *there exists a finite extension of T_1 that has the same constants as T_1 and is essentially undecidable.*

Recall that an extension S of a theory T is called inessential iff every constant of S that does not occur in T is an individual constant and every theorem of S is provable in S on the basis of theorems of T.

Weak interpretability and the above Theorem 3.1 widen in a considerable way the range of applications of the method of interpretation in proving the undecidability of theories. A further widening is provided by the notions of relative interpretability and weak relative interpretability. They are defined in the following way: A theory T_2 is said to be relatively interpretable (weakly relatively interpretable) in a theory T_1 iff there exists a unary predicate P that does not occur in T_2 and such that the theory $T_2^{(P)}$ (obtained by relativizing T_2 to the predicate P) is interpretable (weakly interpretable) in T_1 in the sense explained above. It can be shown that for any theory T and any unary predicate P that is not a constant of T, the theory $T^{(P)}$ is essentially undecidable iff the theory T is essentially undecidable.

It happens very often that one can easily show that a certain theory T_2, known to be essentially undecidable, is relatively interpretable (or weakly relatively interpretable) in a given theory T_1, while the proof of interpretability of T_2 in T_1 is either much more difficult or impossible. Hence the combination of (weak) relative interpretability and Theorem 3.1 is a proper strengthening of the method of interpretation.

3.2. Applications: undecidability of particular theories.

To apply the method of interpretation described above, one has to have a finitely axiomatizable, essentially undecidable theory that is weak enough to be interpreted even in theories quite distant from it. Peano arithmetic and its extensions, as well as various versions of set theory that were known in the late 1930s to be essentially undecidable, could not play this role. Peano arithmetic is not finitely axiomatizable and set theory is too rich.

The problem was solved by A. Mostowski, A. Tarski, and Raphael M. Robinson. In 1939 Mostowski and Tarski constructed a finitely axiomatizable and essentially undecidable subtheory $\overline{Q}$ of the arithmetic of natural numbers. It was closely related to the theory of non-densely ordered rings (*cf.* [Mostowski & Tarski *1949a*] where an analogous theory is described). Around 1949/50 R. M. Robinson and A. Tarski streamlined this system, and finally a simple, finitely axiomatizable, and essentially undecidable theory Q arose. It was described in the paper by A. Mostowski, R. M. Robinson and A. Tarski, Undecidability and essential undecidability in arithmetic (*cf.* [Mostowski, Robinson & Tarski *1953*]). Q is the first-order theory in the language with S, 0, +, and $\cdot$ as nonlogical primitive notions and based on the following axioms:

$$Sx = Sy \rightarrow x = y,$$

$$0 \neq Sx,$$

$$x \neq 0 \rightarrow \exists y\,(x = Sy),$$

$$x + 0 = x,$$

$$x + Sy = S(x + y),$$

$$x \cdot 0 = 0,$$

$$x \cdot Sy = (x \cdot y) + x.$$

It is shown in [Mostowski, Robinson & Tarski *1953*] that Q is essentially undecidable and that no axiomatic subtheory of Q obtained by removing any one of its axioms is essentially undecidable. Essential undecidability of Q was proved by a direct method, based on ideas found in [Tarski *1933*] and [Tarski *1935a*] as well as in [Gødel *1931*]. A result of Tarski, stating that the diagonal function and the set of Gödel numbers of theorems of Q cannot both be strongly represented in Q, was used. (In fact, this result holds for any consistent and axiomatizable extension of a certain fragment of Q, denoted in [Mostowski, Robinson & Tarski *1953*] by R.) Since the diagonal function is recursive and hence strongly representable in Q, it follows that Q is essentially undecidable.

It is worth noticing here that the key step in the proof of the above theorem is to show that all recursive functions and sets are strongly representable. This is based on a characterization of recursive functions found by Julia Robinson in 1950 (*cf.* [Robinson *1950*]). And again the construction of the theory Q arose by a keen insight into the semantical notion of definability. (Strong representability is a kind of definability.)

Another example of a theory with properties similar to those of Q, i.e., finitely axiomatizable and essentially undecidable, was constructed by Andrzej Grzegorczyk in his [*1962*] paper. In fact, two theories F and F^* are given there. The theory F is a first-order theory formalized in a language with two individual constants 0 and S and one binary function symbol $|$. Its nonlogical axioms are the following:

(F1) $\forall x\ (0 \neq S \mid x)$,

(F2) $\forall x\ \forall y\ (S \mid x = S \mid y \rightarrow x = y)$,

(F3) $\forall x\ (x = 0 \vee \exists y\ (x = S \mid y))$,

(F4) $\forall x\ \forall y\ \exists z\ \{z \mid 0 = x\ \&\ \forall u\ [z \mid (S \mid u) = y \mid (z \mid u)]\}$.

The theory F^* is formalized in the same language as F, and the set of its nonlogical axioms consists of (F1)–(F3) plus the following axioms:

(F*4) $\forall f\ \forall a\ \exists g\ \{g \mid 0 = a\ \&\ \forall x\ (g \mid (S \mid x) = f \mid x)\}$,

(F*5) $\exists g \forall x\ (g \mid x = 0)$.

The theories F and F^* have common extensions, but they are independent, e.g., (F*5) is not a theorem of F and (F4) is not a theorem of F^*. Now using the fact that all recursive functions are strongly representable in F it is proved that F is essentially undecidable. (Hence the direct method is applied here.) The essential undecidability of F^* is proved by showing that a weak essentially undecidable set theory of R. L. Vaught can be interpreted in it. (The theory of Vaught is obtained by simplifying the theory considered by Szmielew and Tarski in [Szmielew & Tarski *1952*].)

After the publication of the paper [Mostowski, Robinson, & Tarski *1953*], research activity concerning undecidable theories increased sharply. There were at least three centers of research: Berkeley (under the leadership of Tarski), Princeton (where Gödel was a member of the Institute for Advanced Study and where Church, Rosser, and Kleene had done their pioneering work in the theory of recursive functions), and Novosibirsk (under the leadership of A. I. Mal'cev). The reason for this increase of interest was the fact that the method of interpretation, together with the finitely axiomatizable and essentially undecidable system Q, opened the way for various applications. We shall indicate here some of these applications, restricting ourselves — according to the subject of this paper — to those due to Polish logicians.

Some applications were mentioned already in the paper by

Mostowski, Robinson, and Tarski (their [*1953*]). In particular, it was shown there that certain theories of integers, as well as various algebraic theories, are undecidable. More precisely, it was proved (Theorem 12) that the arithmetic J of arbitrary integers (in the language with + and · only) and all its subtheories (in the same language) are undecidable and that there are finitely axiomatizable subtheories of J that are essentially undecidable. The same holds for the theory $J^<$ of integers formalized in the language with +, ·, and <. Both theories J and $J^<$ are defined semantically as the set of all sentences (in the indicated languages) true in the structure $\langle I, +, \cdot \rangle$ (or, respectively, $\langle I, <, +, \cdot \rangle$), where I is the set of all integers and the functions and relation have their usual meanings.

Another group of results in [Mostowski, Robinson & Tarski *1953*] concerns the undecidability of various algebraic theories. In particular, it is proved that the elementary theories of rings, commutative rings, integral domains, ordered rings, and ordered commutative rings, with or without unit, are undecidable (Corollary 13) and that the elementary theories of non-densely ordered rings and non-densely ordered commutative rings, with or without unit, are essentially undecidable (Theorem 14).

In 1946 A. Tarski proved the undecidability of the elementary theory of groups — this result was announced in [Tarski *1949c*] and expounded fully in the paper *Undecidability of the elementary theory of groups*, published in [Tarski, Mostowski & Robinson *1953*] (*cf.* [Tarski *1953a*]). Using (relative) interpretability of various systems of integer arithmetic, it is proved that the theory G of groups (formalized in the language with · as the only nonlogical constant) and every subtheory of G (in the same language as G) are undecidable and that there exists a finitely axiomatizable extension of G that has the same nonlogical constant as G and is essentially undecidable.

It should be noted here that a weaker result in the same direction was announced by S. Jaśkowski in his [*1948*].

A. Tarski proved (*cf.* [Tarski *1949d*]) the undecidability of the following theories: the theory of modular lattices, the theory of arbitrary lattices, the theory of complemented modular lattices, and the theory of abstract projective geometries. Again, the method of interpretation was used. It was also noticed that the indicated theories are not essentially undecidable, since the theories of Boolean algebras and of real projective geometry are decidable (*cf.* the previous section).

A. Tarski and W. Szmielew considered the undecidability of various weak fragments of set theory (*cf.* [Szmielew & Tarski *1952*] and [Tarski

1953]). In particular, they proved (by interpreting the system Q) that a small fragment S of set theory is essentially undecidable.The theory S is formalized in the language with two nonlogical constants: E (= being a set) and the membership relation $\in$, and based on the set of axioms stating that: (i) any two sets with the same elements are identical, (ii) there is a set with no elements, and (iii) for any two sets a and b there is a set c consisting of those and only those elements that are elements of a or are identical with b. From this theorem of Tarski and Szmielew it follows that every consistent theory that is an extension of S is essentially undecidable, hence all axiomatic systems of set theory (with E and $\in$ as nonlogical constants) that are known from the literature are essentially undecidable. The result can be extended to systems of set theory formulated in the language with $\in$ alone.

A. Tarski and Lesław W. Szczerba considered the undecidability of various geometrical theories. In particular, in Tarski's [*1959*] paper it is shown that the system $\mathfrak{G}'_2$ of geometry, obtained from the system $\mathfrak{G}_2$ described in the previous section by supplementing it with a small fragment of set theory, is essentially undecidable. In thier [*1979*] Szczerba and Tarski studied the undecidability of various systems of affine geometry.

Using Tarski's general method of interpretability together with results of Mostowski, Robinson, and Tarski on the theory Q, A. Grzegorczyk considered in his [*1951*] the undecidability of some topological theories. In particular, he proved:

(1) There exists an elementary theory T of closure algebras that is undecidable and finitely axiomatizable and such that each theory of closure algebras consistent with T is undecidable. (The undecidability of the closure algebra was also proved by another method by S. Jaśkowski in 1939, *cf.* [Jaśkowski *1948*].)

(2) There exists an elementary theory T of the algebra of closed sets such that T is essentially undecidable and finitely axiomatizable and such that every Brouwerian algebra consistent with T is undecidable. (One consequence of this theorem is the undecidability of the abstract algebra of projective geometry and of general lattice theory — those results were obtained by another method by Tarski, *cf.* [Tarski *1949d*].)

(3) The algebra of bodies is undecidable.

(4) Every algebra of convexity true in a Euclidean space E_n, for $n \geq$

2, is undecidable.

(5) Every semi-projective algebra true in a Euclidean space E_n, $n \geq 2$, is undecidable.

(I am here using the terminonogy of Grzegorczyk, *cf.* his [*1951*].) By an algebra of convexity (similarly for a semi-projective algebra) true in a Euclidean space, a (syntactically given) theory is meant whose axioms are true in a Euclidean space.)

The main idea of the proofs of the above theorems is that the arithmetic Q can be interpreted as an arithmetic of finite sets.

Applying the method of interpretability and using the undecidability of the theory of non-densely ordered rings (see above), Antoni Janiczak proved the undecidability of some simple theories of relations and functions. These results were contained in his master's thesis, which was submitted, shortly before his unexpected death in July 1951, to the Faculty of Mathematics of the University of Warsaw. The results were published in the paper [Janiczak *1953*], prepared for print by A. Mostowski with the assistance of A. Grzegorczyk. Janiczak proved the undecidability of the theory of two equivalence relations, of the theory of two equivalence relations whose intersection is the identity relation, of the theory of one equivalence relation and one bijection, and of the theory of one 1–1 relation and one function (many-one relation). It is also mentioned in the paper [Janiczak *1953*] that the theory of one equivalence relation is decidable. (This can be shown by the method of quantifier elimination.)

4. Reducibility results.

Another approach to the decidability problem (*Entscheidungsproblem*) was represented by Józef Pepis (a mathematician active at the University of Lvov, killed by the Gestapo in August 1941). He explicitly distinguishes three versions of it (*cf.* [Pepis *1937*]): the tautology decision problem (*Allgemeingültigkeitsproblem*), the satisfiability decision problem (*Erfüllbarkeitsproblem*), and the deducibility decision problem (*beweistheoretisches Entscheidungsproblem*). The first problem consists of finding a uniform mechanical method — or proving that there is no such method — that would enable us to decide in a finite number of steps if a given formula is a tautology. In the second case one asks if there exists a uniform and mechanical method of deciding in a finite number of steps if a given formula can be realized, i.e., if it has a model. Hence this version has a semantic character. The last case concerns syntax — one asks here if there exists a method with the indicated properties that would enable us to decide in a finite number of steps if a given formula is a theorem of a given theory, i.e., if the formula can be deduced from the theory's axioms.

Observe that so far we have been interested in and discussed mainly the third version. Note also that all these versions are equivalent, i.e., a positive (or negative) solution to one of them yields a positive (or negative) solution to the others. Hence one can simply speak here about the decidability problem. Pepis says in his [*1937*] that the most convenient approach to the *Entscheidungsproblem* is the second one, i.e., the decidability problem for satisfiability — and therefore he concentrates on this approach.

On the other hand, one can distinguish between a direct approach and an indirect one. The former consists of solving the decision problem for a particular given theory, the latter of reducing a given general decision problem to some particular cases of the decision problem. Pepis was only intere sted in the second approach, and all his papers are devoted to the study of various reducibility procedures.

He published four papers on decidability and reducibility. In all of them the first-order predicate calculus, formalized in a language with propositional variables and identity relation, (*enger logischer Funktionenkalkül*, as Pepis used to call it) was studied from the point of view of the reducibility of the satisfiability decision problem for one class of formulas to the satisfiability decision problem for another class of formulas.

The first paper was published in 1936 (*cf.* [Pepis *1936*]). Results contained in it were generalized in Pepis' 1937 doctoral dissertation [Pepis *1937*], submitted to the Jan Kazimierz University in Lvov. The third paper [Pepis *1938*], from 1938, contains new results on reducibility, which generalize the results of W. Ackermann [*1936*] and L. Kalmár [*1936*]. The fourth paper [Pepis *1938a*], also from 1938, is devoted to the introduction and discussion of a certain new, simple, and general reduction procedure.

It is impossible to quote here all the results of Pepis. We shall only indicate some examples. The statement: "In considering the satisfiability decision problem for first-order predicate calculus, we can restrict ourselves with out loss of generality to formulas with the given property E" means that any formula of first-order predicate calculus is equivalent — from the point of view of satisfiability — to a formula with the property E.

Pepis proved that in considering the satisfiability decision problem for first-order predicate calculus, we can restrict ourselves without loss of generality to the following formulas:

1. formulas in prenex normal form in which a unique 3-ary predicate occurs and that possess a Skolem prefix of the form

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 \exists y_1 \exists y_2 \dots \exists y_n ,$$

2. formulas in prenex normal form with prefix

$$\forall x \forall y \exists z \forall x_1 \dots \forall x_n$$

and such that the matrix contains only two (one unary and one 3-ary) predicates,

3. formulas in prenex normal form with prefix

$$\forall x_1 \forall x_2 \dots \forall x_n \exists y$$

and such that the matrix contains only two (one unary and one 3-ary) predicates,

4. formulas in prenex normal form with prefix

$$\exists x \forall y \exists z \forall u_1 \forall u_2 \dots \forall x_n$$

and such that the matrix contains only one unary, one binary, and one 3-ary predicate,

5. formulas of the form

$$\forall y \forall z \exists x \Phi(x, y, z) \ \& \ \forall x_1 \forall x_2 \dots \forall x_n \ \mathscr{A} (x_1, x_2, \dots, x_n)$$

where Φ is a predicate and the formula $\mathscr{A}$ contains, besides Φ, one further (unary) predicate,

6. formulas of the form

$$\forall y \, \forall z \, \exists x \, (R_1(x, y) \,\&\, R_2(x, y)) \,\&\, \forall x_1 \forall x_2 \ldots \forall x_n \, \mathscr{A}(x_1, x_2 \ldots x_n)$$

where R_1 and R_2 are predicates, the formula $\mathscr{A}$ contains, besides R_1 and R_2, one further (unary) predicate, and the symbols R_1 and R_2 occur in $\mathscr{A}$ only negatively.

Pepis' results partially lost their meaning in light of the theorem of Church (*cf.* [Church *1936*]) that showed that first-order predicate calculus is undecidable and in this way gave a negative solution to its decision problem. Nevertheless, their value consists of the indication of reducibility properties for various classes of formulas. On the other hand, it seems that Pepis did not accept Church's Thesis, and he did not share the opinion that the recursive functions comprise all effective methods. (See his paper [Pepis *1937*, 169–170].) Hence he was convinced that the results of Church did not solve the problem definitively, and he treated the decidability problem for predicate calculus as still open.

Reducibility methods similar to those used by J. Pepis were applied by Stanisław Jaśkowski to study the decision problem for various mathematical theories. In particular, [Jaśkowski *1948*] announces certain reductions of the decision problem for first-order predicate calculus to decision problems for various topological and group-theoretical expressions. In the paper [Jaśkowski *1948*] Jaśkowski proved that the decision problem for predicate calculus is equivalent to the problem of whether or not for every parameter Θ of the interval $(-1, 1)$, a system of ten ordinary differential equations given by him explicitly possesses a real solution over the interval $(-1, 1)$, satisfying the particular initial condition. At the end of the paper it is stated that the negative solution to the former (which follows from the theorem of Church) yields a negative solution to the latter.

In the paper [Jaśkowski *1956*] some generalizations of Pepis's results to algebraic structures can be found. A short proof of one of the reducibility theorems of Pepis is also given. The algebraic structures considered by Jaśkowski are free groupoids. It is shown, in particular, that in the case of a free groupoid $\mathscr{F}$, (1) the satisfiability problem for the class of elementary sentences is reducible to the decision problem for the class of first-order sentences with prefix

$$\exists E \, \forall x_1 \ldots \forall x_n,$$

in $\mathscr{F}$, where E is a unary predicate, and (2) the tautology problem for the class of elementary sentences is reducible to the decision problem for the class of first-order sentences with prefix

$$\forall E\, \exists x_1 \ldots \exists x_n$$

in $\mathcal{F}$.

5. New proofs of the incompleteness theorem.

In discussing the contributions of Polish logicians to decidability problems, one should also mention works devoted to generalizations and strengthenings of Gödel's classical theorems on incompleteness, as well as scientific articles presenting results on decidability to the general public.

We start with a paper by A. Tarski [Tarski *1939a*] from 1939. An enlarged system of logic is considered there, the enlargement being obtained by adding rules of inference of a "non-finitary" ("non-constructive") character. The existence of undecidable statements in such systems is shown. The author emphasizes the part played by the concept of truth in relation to problems of this nature. One should also note a certain kinship between these results and the results of Rosser [Rosser *1936*].

Another author whose contribution to the classical incompleteness theorems should be mentioned here is A. Mostowski. In his paper [Mostowski *1949*] one finds an interesting construction of a new undecidable sentence. The main properties of this sentence are that it is set-theoretical in nature, is stronger than Gödel's sentence, and is not effective. Nevertheless, its content is distinctly mathematical and intuitive. His construction does not use the arithmetization of syntax and the diagonal process, as was the case with Gödel and other authors. Instead, Mostowski uses some set-theoretical lemmas and the Skolem-Löwenheim theorem. Mostowski's undecidable sentence is stronger than Gödel's in the sense that the latter ceases to be undecidable if one adds the infinite ω-rule to the system. The former does not have this property — there is no "reasonable" rule of inference that, when added to the system being considered, would decide it. The undecidability proof is non-finitary — it rests on the axioms of Zermelo-Fraenkel set theory, including the axiom of choice (which, in fact, can be eliminated from the proof) and an axiom ensuring the existence of at least one inaccessible cardinal. Mostowski's undecidable statement expresses a fact concerning real numbers; more precisely, it states that an $\mathscr{A}$-set is not empty.

Another contribution of Mostowski to the domain under discussion here is his paper [Mostowski *1961*] from 1961. The notion of free formula is introduced there. If φ is a formula with one free numerical variable, then φ is said to be free for a system S if for every natural number n, the formulas $\varphi(\mathbf{0}) \ldots \varphi(\mathbf{n})$ are completely independent, i.e., every conjunction formed of some of those formulas and of the negations of the remaining ones is consistent with S. (Here $\mathbf{n}$ denotes the n^{th} numeral, i.e., $\mathbf{0}$ is the term 0 and $\mathbf{n} + \mathbf{1}$ is the term $S\mathbf{n}$). It is proved that free formulas exist for certain systems S and some of their extensions. An even more general result is obtained: given a family of extensions of S satisfying certain very general assumptions, there exists a formula that is free for every extension of this family. It should be noted here that the method of the proof applies not only to systems based on usual finitary rules of inference, but also to systems with the infinitary ω–rule.

Mostowski also wrote two important popular works devoted to the incompleteness results. One was published in Polish in 1946 (*cf.* [Mostowski *1946*]), the other in English in 1952 (*cf.* [Mostowski *1952*]). Both enjoyed considerable popularity. The aim was to present "as clearly and as rigorously as possible the famous theory of undecidable sentences created by Kurt Gödel in 1931" (*cf.* Preface, [Mostowski *1952*]). Though based on classical material, they introduced some new ideas. In particular, in the book [Mostowski *1952*] the theory of $\mathfrak{R}$-definability was developed. It presents a simultaneous generalization of the theory of definability and that of the general recursivity of functions and relations. This theory proves to be a very convenient tool — one can express in it, in a clear way, the assumptions that are the common source of the various proofs of Gödel's incompleteness theorem.

To finish this section we wish to mention a paper by Andrzej Ehrenfeucht from 1961 (*cf.* [Ehrenfeucht *1961*]). The notion of a separable theory is studied there, and some interrelations between separability and essential undecidability of theories are established. To be more precise, a theory T is said to be separable iff there exists a recursive set X of formulas such that (1) if φ is a theorem of T, then $\varphi \in X$ and (2) if $\neg\varphi$ is a theorem of T, then $\varphi \notin X$. It can be seen easily that every inseparable theory is essentially undecidable, but not vice versa. In fact, an essentially undecidable but separable theory is constructed in the paper. It is also shown that an axiomatizable theory T is inseparable iff for any recursive family $\{T_i\}$ of axiomatizable consistent extensions of T, there is a closed formula φ undecidable in each T_i. This result establishes a relation between a theorem of Grzegorczyk, Mostowski, and Ryll-Nardzewski (*cf.* [Grzegorczyk, Mostowski & Ryll-Nardzewski *1958*]), stating the inseparability of Peano arithmetic, and the result of Mostowski (*cf.* [Mostowski *1961*]) that shows the existence of an undecidable sentence for any recursively enumerable family of extensions of Peano arithmetic.

6. Conclusions.

With this we come to the end of our survey of results due to Polish logicians and devoted to decidability theory. One can easily see that Polish mathematicians and logicians were, from the very beginning, in the mainstream of investigations devoted to the *Entscheidungsproblem*. What is more, they contributed to the development of this field in a significant way.

Especially, Tarski and his students (and later, students of his students) were active here. They not only solved the problem of decidability in the case of many particular theories by establishing their decidability or undecidability, but also developed general methods of such proofs, which became classical and standard.

Though decidability problems also have a philosophical character and research in this field can be described in terms of the study of the "cognitive power" pertaining to logical means of proof, it seems that such a philosophical motivation was not the main factor stimulating the activity of Polish logicians. Tarski and his students were adherents of the separation of logical research from philosophical study. For them, logic and foundations of mathematics constituted a separate field having its own problems and methods, a field developing independently of other branches of mathematics and philosophy.

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