SOF'YA ALEKSANDROVNA YANOVSKAYA'S CONTRIBUTIONS TO LOGIC AND HISTORY OF LOGIC

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Modern Logic Publishing
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Abstract. Sof'ya Aleksandrovna Yanovskaya's contributions to the development of the disciplines of mathematical logic and history of logic in the USSR are enumerated. This is followed by a detailed sketch of the work of Soviet logicians of which she gave an historical survey and exposition in her histories of mathematical logic and foundations of mathematics from 1917 to 1947 and from 1948 to 1957 in her two studies of 1948 and 1959. These two studies, taken together, form the basis for a history of mathematical logic in the USSR and reflects the situation formathematical logic in Russia during the the first 40 years of the Soviet period.

0. Introduction and biographical note. January 31, 1996 marks the centenary of the birth of Soviet-Russian historian and philosopher of logic and mathematics Sof'ya Aleksandrovna Yanovskaya. She was born Sof'ya Aleksandrovna Neimark in Pruzhany, in Grodno province, in Russian Poland, now the town of Brest in Belarus'. Her family moved to Odessa in the Ukraine while she was still very young, and before the Russian Revolution, she attended first the gymnasium, the Higher School for Women of Odessa's Novorossiisk University, where she took up serious study of mathematics as a student of noted historian of mathematics Ivan Yure'evich Timchenko (1863 – 1939) and then, beginning in 1915, the university, where she was a student of Samuil Osipovich Shatunovskii (1859 – 1929) until 1917. Shatunovskii's doctoral thesis, which became the algebra textbook *Algebra as the Study of Congruence of Functional Values* [*1917*], introduced the question of the Law of Excluded Middle as a foundational question outside the context of nonclassical, especially paraconsistent and many-valued logics. His students also included Moishe Isai'evich Sheinfinkel' (better known in the West as Moses Schönfinkel; 1889 – 1942). Did Yanovskaya know Schönfinkel at this time? In Moscow in 1987, I asked historian of mathematics Feodr Andreevich Medvedev (1923 – 1994), who had been a student both of Yanovskaya and of Yanovskaya's student, the late Adol'f Pavlovich Yushkevich (1906 – 1994). Medvedev replied that, regretably, no one had thought to ask Yanovskaya that question while she was alive.

Shatunovskii was the most important influence Yanovksya's intellectual development, according to her Odessa girlfriends and classmates Vera Abramovna Gukovskaya [*1982*, 115] and Marya Grigor'evna Shestopal [*1982*, 115], both also mathematicians. Others in Odessa besides Shatunovskii and Timchenko who shaped her scholarly intersts, acccoding to Shestopal [*1982*, 117], were Venyamin Fedorovich Kagan (1869 – 1953) at the Institute of Red Professors, a geometer, leading historian of mathematics and specialist on Lobachevskii, and Evgenii Leonidovich Bunitskii (1847 – 1952). Bunitskii taught mathematical analysis, was interested in number theory, and contributed to application of algebraic logic to number theory. He carried out work, in particular, in "The Number of Elements in a Logical Polynomial" [*1897*], on the logical polynomials previously introduced and investigated by Poretskii; but he also worked on applications of mathematical logic to arithmetic, and in particular to number theory, in such papers as "Some Applications of Mathematical Logic to Arithmetic" [*1896-1898*] and "Some Applications of Mathematical Logic to the Theory of the GCD and LCM" [*1899*]. In this, Bunitskii can be interpreted as very much a forerunner to Yuri L. Matyiasevich, who used recursion-theoretic tools to provide a negative solution to Hilbert's tenth problem.

After a political interlude during the revolution and civil war that followed (marrying revolutionary co-worker Isaak Il'yich Yanovskii in 1918 during her politically active period) she again took up academic work — at the Institute of Red Professors beginning in 1923. Since 1925-1926, she was on the faculty of Moscow State University, serving for a time both in the mathematics and philosophy departments. Beginning in 1925, she conducted a seminar on methodology of mathematics at Moscow State University and began teaching history of mathematics at the University in 1930. She received her doctorate from the faculty of Mechanics-Mathematics of Moscow State University in 1935, but was already a professor in 1931. She helped create the seminar on mathematical logic and served as its director from 1943 Among her closest colleagues and participants in this seminar were Ivan Ivanovich Zhegalkin (1896 – 1947), Andrei Andreevich Markov (1903 – 1979), and Pëtr Sergeevich Novikov (1901 – 1975). On 31 March 1959 she became the first chairperson of the newly created department of mathematical logic at Moscow State University and held the Chair in Mathematical Logic until her death. Aleksandr Aleksandrovich Zinov'ev (b. 1922) — best known to logicians for his work in nonclassical logics and philosophy of logic, especially many-valued logics — wrote [Zinov'ev *1968*, 211] that "in the early sixties there began a noticeable re-organization of the work of the Chair of Logic in the Philosophy Department at Moscow University. A great part of the credit for this belongs of S. A. Janovskaja, who regularly lectured in mathematical logic and its philosophical problems. . . ." A. A. Markov, Aleksandr Sergeevich Kuzichev and Zinaida A. Kuzicheva, writing on "The Work of S. A. Yanovksaya in the Field of Mathematical Logic" (see their [*1982*]), consider in particular her discussion of definition and the role of definition by abstraction in mathematics in general and in mathematical logic in particular, and they also characterize her lecture style.

Sof'ya Aleksandrovna Yanovskaya died on 24 October 1966. (Several biographies of Yanovskaya have been prepared; these include [Anellis *1987*; *1987a*], [Bocheński *1973*], and reminiscences and accounts of various aspects of her scholarly work by several people in the collection *Women-Revolutionaries and Scholars* [Dashevskaya, *et al.*, 1982, 81–124]; additional papers, both in Russian and in English translation, dealing with Yanovskaya's life and work are listed in [Anellis *1987*; *1987a*]).

1. Work in history and philosophy of logic and history and philosophy of mathematics. Yanovskaya's primary interests were in history and philosophy of mathematics and logic, and her polemical writings were crucial in gaining respectability for mathematical logic in the Soviet Union at a time when the subject, as developed by formalists such as Hilbert and logicists such as Russell, was held to be idealistic and inimical to dialectical logic. Her paper "From the History of Axiomatics" [Yanovskaya *1958*] belongs to this genre and can be seen as a reply to those papers, such the [*1947*] paper "On the Problem of the Axiomatization of Logic" of Levan Petrovich Gokieli (1901 – 1975), which had as its principal targets Alexander Vladimirovich Kuznetsov (1926 – 1984) and Yanovskaya. In this paper, which presents arguments against "formalism" he was largely concerned with the problem of the algebraization and axiomatization of logic. He nevertheless managed in this paper to continue accept Hilbert's conception of formalism, according to which logic is an empty formalism, based on symbols and their manipulation (*Rechnen*); moreover, his chief criticism of logic as formalism is very much akin to that discussed by logicians and philosophers of mathematics in the 1920's, namely that it leads to a metalogical infinite regress. Gokieli, a set-theorist of some repute and otherwise a serious mathematician, found it incumbent upon him at one point to write a paper in which the concept of function is redefined in a distorted way so as to make it conform to dialectical-materialist notions as set forth in Marx's mathematical manuscripts (see Gokieli's [*1937*] "On the Concept of Fuction").

Yanovskaya found it necessary from time to time to take part in self-criticisms in order to survive the attacks of overzealous dialectician-ideologues, but her work in the late 1950's on the history of axiomatics led to the decline of attacks on formalism and logic by dialectical philsophers and coincided with a general improvement of scientific life in the USSR. She maintained her dialectical "credentials" by her work, primarily during the 1930's, but also later, on Marx's mathematical manuscripts. (For example, the Marxist view of the infinitesimal calculus, as portrayed by Yanovskaya (see, e.g., [Yanovskaya *1983*]; quotation, p. xv), was that real analysis is simply reducible to algebra in the sense of Lagrange; it is understandable as the algebra of motion or the "mathematics of a variable quantity [which] must be of an essentially dialectical character" which does not require the limit concept. Yanovskaya and her colleagues in the USSR, of course, did not have the luxury afforded to Jean van Heijenoort, who before becoming an historian of logic had served as Trotsky's secretary and bodyguard, to belittle the Marxist interpretation of mathematics. Moreover, van Heijenoort was attacking, not Marx directly, but Engels, who, van Heijenoort said, was a mathematical dunce even compared with Marx; see [van Heijenoort *1985*, especially pp. 150–151]. For a complete philosophical and historical account of the battle between the dialecticians and the formal logicians, see [Cavaliere *1990*]; for a brief summary of the main points of contention in this debate, see [Anellis *1994*].)

In the mid-1930's, it was Yanovskaya and her colleagues who took the lead in defense of mathematical logic. Her book *Collection of*

Papers in Philosophy of Mathematics [Yanovskaya *1936*] included papers such as Glivenko's [*1936*] "The Crisis in the Foundations of Mathematics in the Current State of Its Development", Kolmogorov's [*1936*] "Contemporary Mathematics", P. S. Aleksandrov's [*1936*] "On New Trends in Mathematical Thought Arising in Connection with Set Theory", A. G. Kurosh's [*1936*] "Contemporary Algebraic Views", all of whose primary aim was to present an accurate depiction of the true state of contemporary mathematics, indepenmdetly of polemical or ideological confusions, as well as some more philosophical papers by the editor and by Molodshii, her severest critic. The papers were largely expository, and they were not merely intended to survey the state of a specialty, but also to acquaint their readers with the thoughts and attitudes of workers in the field.

It is hardly surprising, then, that Zinov'ev [*1968*, 212] should have written of Yanovskaya that she was "the pioneer of the discussion of the philosophical problems of modern logic," including ". . . the relationship between constructive and non-constructive methods [*1959*], the introduction and removal of abstractions of higher orders, the application of the criteria of practice to logic [*1960*], and others." She carried on this struggle through her teaching, writing, editing, and translating. She edited the translation into Russian of such texts as Hilbert and Ackermann's *Grundzüge der theoretischen Logik* [*1947*] and Tarski's *Introduction to the Logic and Methodology of Deductive Sciences* [*1948*]. This helped open the way for a vast translation program, much of which Yanovskaya carried out herself; her translations included works of Alonzo Church and of Stephen Cole Kleene.

Yanovskaya's work opened the way to a renewed achievement by Soviet logicians in problems of proof theory and on Gödel incompleteness. In history of mathematics her students included Nikolai Ivanovich Styazhkin (1932 – 1986), who specialized in history of logic, Medvedev, and Sergei S. Demidov. Yushkevich recalled [Yushkevich *1982*, 110] that in the 1940's she was "occupied almost exclusively with mathematical logic", whereas he was especially interested in history of mathematics; although he wanted to study with her, she did not take him on as her graduate student, but she did sit on his dissertation committeein 1940.

Yanovskaya also wrote numerous short studies surveying the original technical results of such western logicians as Hilbert, Ackermann, and Tarski, among many others. Some of these essays belong to the prefaces and introductions to the works which she translated. She also did much to popularize a number of technical developments through her contributions to the *Great Soviet Encyclopedia*, for example on such topics as "Formalism" [*1936b*; *1956a*], "Logistics" [*1938*], and "Mathematical Paradoxes" [*1939*], and in her paper "On the so-called 'Definition by Abstraction'" [*1936a*], in which, for example, she explained the concept of equinumerity in simple, straightforward terms (see also [Markov, Kuzichev & Kuzicheva *1982*]). Beyond that, she contributed to the discussion on particular aspects of the history of mathematical logic and foundations of mathematics. Her [*1958*] paper "From the History of Axiomatics" and its abstracted [*1956*] forerunner, "From the History of Axiomatic Method", should be understood as a contribution to the discussion of the foundational "crisis" which non-Eucldean geometry had for stimulating work in axiomatics and the distinction between an axiom system and a formal deductive system.

Yanovskaya did not ignore even the most trivial and insignificant writings on logic or its history that came to her attention. One example is brought to our attention by Bocheński [*1973*, 4-5], who wrote:

> How vast her erudition was can be seen from the following anecdote, involving the present author. He published, in 1954, a small article under the title 'Spitzfindigkeit'. It was intended as a sort of witty paper on the classical polenics against formal logic and was, surely, not to be considered as an important contribution to philosophy. Moreover, it was published in a volume called *Festgabe an die Schweitzer Katholiken* — a book offered to Catholic supporters of the University of Fribourg on the occasion of its anniversary. It does not seem that anybody — except the stern reviewer in the *Journal of Symbolic Logic* — ever studied it.
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> But Janovskaja not only read it thoroughly. She felt in deep disagreement with the author and wrote several pages in order to refute his opinion concerning Descartes' views on logic ... The only thing to be stressed here is the enormity — it seems the word is not an exaggeration — of the erudition possessed by the Soviet logician and her truly insatiable reading.

This, quite naturally, does not necessarily mean that Yanovksya, did not occasionally make errors of fact or judgment, as we shall have occasion to see in the following pages. The "several pages" to which Bocheński referred are her paper "On the Role of Mathematical Rigor in the History of the Creative Development of Mathematics and Especially Descartes' 'Geometry'" [Yanovskaya *1962*].

We shall merely mention without comment her numerous studies on infinitesimal analysis, on Rolle, on Descartes, on Lobachevskii, for example, or on the algebraization of geometry, on ancient Egyptian work on fractions, on Marx's mathematical manuscripts, and on numerous other aspects of the history or philosophy of mathematics; a glance at her 69-item vitæ bibliography will show the range of her scholarship (see, e.g. "Section A. Selected works by Janovskaja" in the bibliography in [Anellis *1987*, 54–55]).

3. Yanovskaya's surveys of contemporary Soviet research in mathematical logic and foundations of mathematics. Yanovskaya's most important surveys of Soviet work in logic were "The Foundations of Mathematics, and Mathematical Logic" of [*1948*] and "Mathematical Logic and Foundations of Mathematics" of [*1959*], both of which include bibliographies of immense value. (Both of these papers are merely mentioned in [Markov, Kuzichev & Kuzicheva *1982*], but not analyzed there. I shall do so momentarily.) These two surveys, taken together and studied in retrospect, provide an excellent starting point for any study of the history of Soviet research in mathematical logic and foundations of mathematics and serve as a critically important introduction to the history of mathematical logic in the USSR.

Attention should be called to the change in title of these two surveys, since the change in title reflects the shift of emphasis over the decade 1947-1957 from justification of mathematical logic as a discipline to an increasing tempo of active work in mathematical logic by eminent Soviet logicians, some of whom were Yanovskaya's students. This change is likewise reflected in the number of pages which Yanovskaya is able to write about Soviet contributions to mathematical logic within the broader context of Soviet work in all of mathematics; 41 pages in 1948 compared with 107 pages in 1959. Yanovskaya herself was responsible in no small degree for this growth. The period 1917-1957 which Yanovskaya surveys in her two main histories of Soviet research on mathematical logic and foundations of mathematics show a maturation in three significant ways: (1) the growth of interest in the history of logic, exemplified by the abundance of histories of mathematical logic; (2) the increasing number of technical studies of past and contemporary Western research in mathematical logic, exemplified by a flood of translations of classical texts as well as of current specialized studies, to many of which she herself contributed when she was not directly responsible for their creation; and (3) the self-confident achievement in technical work, manifested by the growth of strong centers of original research, from the 1920's to the present, by Soviet logicians whose work was totally integrated into the contemporary logic scene. The first two of these, in which Yanovskaya was the foremost leader, contributed greatly to the third.

3.1. Yanovskaya's survey of Soviet research in foundations of mathematics and mathematical logic, 1917-1947. Yanovakaya's [*1948*] was the first significant study of work in logic during the early Soviet period 1917-1947 and sets Soviet contributions within the broader context of contemporary research in mathematical logic. (Here, of course, we can report only the more significant and interesting results dealt with by Yanovskaya's survey.)

The survey begins with an obligatory discussion of the contributions of Marx and Lenin and the question of formal and dialectical logic and deals with the contributions to thinking on philosophical problems of mathematics of such politicians as Lenin, Stalin, Zhdanov, and philosophical defenders of dialectics as Gokieli, the infamous Arnošt Kol'man (b. 1892 – 1979) and Vladimir Nikolaevich Molodshii (1906 or 1911 (?) – 1985 (?)). In this essay, she declared that Soviet mathematicians reject the view that (mathematical) propositions say nothing about reality. An example of this is Andrei Nikolaevich Kolmogorov (1903 – 1987), whose intuitionistic mathematics shares with dialectical logic the rejection of the the Law of Excluded Middle. Having thus justified herself and her field against the polemics of the dialecticians, she proceeded to survey the history of Soviet work in logic.

Soviet work in logic, she noted, has its roots in the Boole-Schröder school of algebraic logic, which culminated in the work of the Kazan school in the later years of the nineteenth century, including especially the work of Platon Sergeevich Poretskii (1846 – 1907) and Lobachevskii, among others. Yanovskaya saw Poretskii's [*1884*] paper "On Methods of Solution of Logical Equations and the Inverse Method of Mathematical Logic" as the first attempt at a complete theory of "qualitative inference", that is, of a monadic predicate calculus, understanding 'quality' as 'monadic predicate'.

Her next concern in this paper was with the Law of Excluded Middle (LEM) and constructive logic. She argued that there is no 'crisis' in mathematical logic; rather, there is just the problem of extending the laws of logic of finite domains to infinite domains. Here, the chief question is whether LEM can be applicable in infinite domains. There are also to be contended with the logical paradoxes. The response to these problems is development of constructive logic. The task for Soviet logic is to axiomatically develop constructive logic while discarding the idealistic philosophy which Brouwer provides for it and which distinguishes the Dutch (Brouwerian) intuitionistic logic from the Soviet school of constructivism. She then described key results that led to the development of the Soviet school of constructivism, including a discussion of LEM in Shatunovskii's [*1917*] algebra text, and the work of Andrei Nikolaevich Kolmogorov (1903 – 1987) and Valerii Ivanovich Glivenko (1896 (o.s.)/ 1897 – 1940). Shatunovskii helped make explicit the modern concept of the unification of algebraic logic and function theory as quantification theory in his [*1917*] textbook *Algebra as the Study of the Congruence of Functional Values.* According to Yanovskaya, the introduction to Shatunovskii's textbook gives one of the first treatments of LEM. (But we know, of course that Brouwer raised the issue in his writings beginning in 1908, that Charles Pierce toyed with the conception of constructing various new and different logical theories by altering or eliminating one or another logical principle, such as the Law of Non-contradiction and LEM, as early as 1895, that hints of this were published by Paul Carus in *The Monist* of 1910, and that Nikolai Aleksandrovich Vasil'ev (1880 – 1940) took up these ideas at least as early as 1910, if not earlier; *q.v.*, e.g. [Bazhanov *1992*, 48–50].) Kolmogorov and Glivenko together were the founders of the Soviet constructivist school. Markov also began his work during this period.*

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* The origin of Kolmogorov's idea of developing a non-classical logic seems to be problematic. It had at one time been suggested that Kolmogorov was inspired by Vasil'ev's paraconsistent logic, about which Kolmogorov would have learned from Luzin's review of Vasil'ev's work, in which Brouwer is also mentioned. I myself have been a purveyor of this viewpoint. However, it is equally probable that Kolmogorov could have been influenced by Yanovskaya's discussions of Shatunovskii's work, or by some combination of the work of both Vasil'ev *and* Shatunovskii, keeping in mind that Kolmogorov was in close contact with both Luzin and Yanovskaya through the seminar which she began conducting at Moscow State University in 1925. That, however, still makes the timing of Kolmogorov's work problematic, since his first work on intuitionistic logic was published in 1925 (see [Kolmogorov *1925*]) and obviously then had to have been written some time in early 1925 at the latest, while Luzin's manuscript on Vasil'ev was written later, since it mentions the years 1922-26 and 1924-26. It is probably safe to conclude that, whatever the influences of Vasil'ev and Shatunovskii on Kolmogorov may have been, either through their own work or conveyed by Yanovskaya and Luzin, Kolmogorov doubtlessly already knew something of Brouwer's work while he was working on his own [*1925*] paper.

Besides the philosophical differences between Brouwerian intuitionistic logic and the "Markovian" constuctive logic, whcih we shall turn to momentarily, there are two principal technical differences which we may discuss here in the context of quantification theory. One is that Heyting's formalization $\mathcal{H}$ of 1930 contains as an axiom of formula $\mathfrak{k}$:

$$\mathfrak{k} : A \rightarrow (\bar{A} \vee B)$$

while Kolmogorov's system $\mathcal{B}$ of [*1925*] does not. As a consequence, $\mathcal{B}$ is known as the *minimal calculus*. But as Kolmogorov noted, $\mathfrak{k}$, an axiom of $\mathcal{H}$, is an expression of LEM in Hilbert's system. According to Kolmogorov, $\mathfrak{k}$ merely is a symbolic representation of the logic of judgments, and hence Brouwer's criticisms of LEM fail to apply to it. The axiom remains, nevertheless, a matter of dispute. Moreover, Glivenko, in his [*1928*] paper "Sur la logique de M. Brouwer", working with a full knowledge of Heyting's ongoing research for [*1930*] and [*1930a*], was able to derive another form of LEM, namely $\neg\neg(p \vee \neg p)$, in an incomplete fragment of propositional calculus. Moreover, in his [*1929*] paper "Sur quelques points de la logique de M. Brouwer", Glivenko proved that Heyting's system with LEM adjoined gives classical logic. (This has led Beeson [*1985*, 433], we may add, to assert that Glivenko in [*1929*] was the first to note that intuitionistic logic with LEM adjoined is equivalent to classical logic. But as we have already noted, Kolmogorov in [*1925*] had already indicated that classical mathematics can be translated intuitionistically, and hence that system $\mathcal{B}$, with LEM and Double Negation adjoined, will yield $\mathcal{H}$. (There is evidence, we may add, that Glivenko was unaware of the work of Kolmogorov and this time, since he fails to refer to Kolmogorov's work. He did, however, have clear and ready access to Heyting's work in progress, and was able to use Heyting's results, prior to their publication in 1930, in his own work of 1929.)

For Glivenko, the translatability of classical logic into intuitionistic logic, already proven by Kolmogorov in [*1925*], for an important fragment of classical logic, namely for the classical propositional calculus, hinged on a theorem of Brouwer's of 1923 that asserted that "the absurdity of the absurdity of the absurdity of a proposition $\mathcal{P}$ is equivalent to the absurdity of $\mathcal{P}$", i.e. $\neg\neg\neg\mathcal{P} \leftrightarrow \neg\mathcal{P}$. In his paper on [*1929*], Glivenko proved that if $\mathcal{P}$ is classically provable, then its noncontradictoriness (i.e. the absurdity of its absurdity) is intuitionistically provable. In addition, he proved that if the absurdity of $\mathcal{P}$ is classically provable, then it is also intuitionistically provable. Whatever the claims to be made for Glivenko, Yanovskaya opines that Kolmogorov's [*1925*] anticipated Gödel's *Dialectica*-interpretation, according to which "*die intuitionistische Arithmetic und Zahlentheorie nur scheinbar enger ist als die klassische.*"

In [*1932*] Kolmogorov gave a summary of the work on axiomatization of Brouwer's intuitionistic logic in his paper "Zur Deutung der intuitionistischen Logik", taking into account the work in particular of Heyting's [*1930*] and of Glivenko. He wrote there ([Kolmogorov *1932*, 58]) that

> Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben . . . systematisieren. . . . Man kann eine entsprechenden Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlössungs-schemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z.B. intuitionistischen Voraussetzungen.

The second major technical difference between Dutch intuitionism and Soviet constructivism has already been alluded to. It is the Markov algorithm.

Yanovskaya also noted that Soviet logicians also responded to the logical paradoxes specifically by developing multi-valued logics. Here, D. A. Bochvar (b. 1903) was the leading figure in the early Soviet period.

Also receiving attention here is the work of Novikov. Almost immediately following the announcement of Gödel's incompleteness results in 1931, Soviet mathematicians, most notably Novikov, began work on the question of the decidability of the propositional calculus. In the USSR, the general questions of the consistency of logical calculi were initially raised (outside the context of the philosophical challenge to formal logic by dialectical logicians) by Novikov, in connection with the extended propositional calculus which he had developed.

In his [*1943*] paper "On the Consistency of a Three-valued Logical Calculus", Bochvar analyzed the Russell paradox in terms of $\sum^{-}$; its aim was to show that the Russell paradox was unformulable in $\sum^{-}$, and thus prove that $\sum^{-}$ is consistent.

Next Bochvar moved from classical first-order to higher-order functional calculi, in the [*1944*] paper "On the Question of Paradoxes of

Mathematical Logic and Set Theory". If the higher-order functional calculi are typeless, we add to the classical first-order functional calculus existential axioms of unexplicit formulation. This adjunction gives Bochvar's extended system $\sum$. The first-order functional calculus is consistent; but Bochvar's extended system $\sum$, first-order functional calculus with adjoined existential axioms and without types, is inconsistent, since it admits nonexplicitly-formulated existential axioms. Therefore, we obtain the antinomies through the existential equations that are not admissible in logic. On the other hand, if we deny inexistence or existence in the extended calculus K_0, (i.e., $\sum = K_0$), we find that K_0 is then consistent, since K_0 then becomes first-order quantification theory with identity.

Bochvar explains that the Russell paradox is based on the function F, defined as

$$F(\varphi) =_{\text{df}} \overline{\varphi(\varphi)}$$

In K_0, we get the corresponding formula

$$(\exists\psi)[\varphi(\psi(\varphi)) \sim\varphi(\varphi)]$$

which contradicts the existential axiom implicitly assumed in the definition of F.

Bochvar's next paper, "Some Logical Theorems on Normal Sets and Predicates" ([*1945*]) is an explicit comparison of $\sum$ with set theory. $\sum$ is considered as being unpropertied, a system of mere formulæ of typeless logic. Soon thereafter, Novikov undertook in his [*1947*] paper "On Logical Paradoxes" to compare his system N with Bochvar's typeless system $\sum$. What propositions of N, Novikov asked, can be stated in $\sum$ without contradictions? All, said Novikov, because axioms of the form

$$(\exists p)(x_1,\ldots, x_n)\ (p\,(x_1,\ldots, x_n))$$

of $\sum$ are equivalent to axioms of the form

$$G\,(x_1,\ldots, x_n).$$

In his paper of [*1939*], "Sur quelques théorèms d'existence", Novikov extended the classical propositional calculus by allowing denumerably infinite conjunction and disjunction. In this paper, Novikov gave a definition of *regularity* for formulae and proved that a formula is decidable if and only if it is regular. Specifically, a formula is *regular* if it is reducible to disjunctive normal form. Given the result that a formula is decidable if and only if it is regular, Novikov concluded that (1) the extended propositional calculus is consistent, and (2) that an infinite disjunction is deducible if each of its terms is a finite formula i and, if such an infinite disjunction is decidable, then there exists a number N such that the disjunction of N-many terms is deducible. (To Yanovskaya's comments we may add that, in his [*1940*] review of this paper, Alonzo Church argued that it is pointless to apply the second result to existence. Moreover, the number of typographical errors in the paper make understanding and checking of Novikov's results impossible.)

To satisfy ourselves of the validity of Novikov's results, we must turn to Bochvar's [*1940*] paper "Über einen Aussagenkalkül mit abzahlbaren logischen Summen und Produkten", which gave a new and simpler proof of Novikov's [*1939*] results. To prove that Novikov's extended calculus of infinite sums and products (N) is complete, Bochvar proved the equivalent statement, that '*every tautology of N is demonstrable*' is equivalent to '*every formula of N being proven to be satisfiable or refutable*'. Bochvar also showed that N is consistent. Each formal axiom of N is a tautology and each rule of inference of N is a tautology; thus, if the premises are tautologies, then so are the conclusions. Bochvar's proof is nonconstructive, and he attempted to give it an intuitionistic characterization.

Novikov's next attempt to prove the consistency of his system N made use of the work of Bochvar. In the [*1943*] paper "On the Consistency of Certain Logical Calculi", Novikov presented a modification of his earlier system N. Here the construction of formulae of N are made subject to conditions of intuitionistic logic: it was said that we require recursive construction in order to obtain infinite formulæ.

In the new system, a reduced formula has no implication connective, and bar negation is applicable only to single letters. In order to obtain a reduced formula from a regular formula, we apply De Morgan's Laws and the equivalence $(p \rightarrow q) \equiv ({\sim} p \vee q)$. Regular formulæ have reduced formulæ which are themselves regular. Now Novikov was ready to argue that if a formula is regular, then it is provable. Novikov will extend his techniques for consideration of the first-order functional calculus in a paper of 1949.

Complementary to the development of constructive logic by Shatunovskii, and especially by Kolmogorov, Glivenko and Markov, is the development of combinatory logic by Schönfinkel. There are some who hold that Schönfinkel is included by as Soviet more for the sake of justifying the work of Soviet logicians than for a legitimate description of his nationality or the venue of his work in logic. But Schönfinkel was born in Odessa, was — as we have noted — a student in Odessa of Shatunovskii, and, although he did most of his work at Göttingen, he returned to the USSR at the beginning of the German fascist reign, and died in Moscow.) The value of his work cannot, however, be overestimated, for he provided the "building blocks" for the development of combinatory logic; the work in this field done in the 1930's by H. B. Curry could not have been done without the foundations lain by Schönfinkel. Yanovskaya, in characterizing Schönfinkel's theory, draws a analogous connection with the theories operators to the Sheffer stroke, but considers (p. 33) Church's calculus λ-conversion, erroneously as *merely* a ". . . 'formalization' of Schönfinkel's ideas."

Together, Kolmogorov and Schönfinkel gave us viable alternatives to the classical quantification theory developed by Frege, Peano, Whitehead, and Russell. The critic of Yanovskaya is undoubtedly correct, however, in saying thast she overemphasizes the significance of Schönfinkel's work in another of its aspects: it is true, as George Kline [*1951*; 47] declares, that the priority for the concept of the function as an abstract object different from its values (*Wertverlauf*) belongs to Frege, not to Schönfinkel. Yanovskaya also argued here the importance of Schönfinkel's work with Paul Bernays [Schönfinkel & Bernays *1929*] on the *Entscheidungsproblem*. (Schönfinkel's work is of course well known in the West, and we therefore will otherwise summarize Yanovskaya's characterization and analysis of it.)

Another, closely related, development in mathematical logic is the arithmetization of this concept of truth. In one of the earliest papers on propositional logic in the Soviet period, "On the Calculus of Propositions in Symbolic Logic" ([Zhegalkin *1927*]). Zhegalkin, a professor at Moscow State University, sought to provide a mechanical rule for the calculus which would allow determination of the truth or falsity of an arbitrary elementary, i.e. atomic, proposition. The procedure which Zhegalkin devised and described in his [*1927*] boils down to a preliminary version of what we have come to know as truth tables. Given any two atomic propositions p and q, we can determine whether their product pq or sum $p+q$ are true or false by computing the value of pq and $p+q$ in accordance with simple arithmetic rules, where a proposition is false if its arithmetic value is *zero*, and true if its arithmetic value is *one*. There arithmetic here is Boolean, where we begin with the equalities $p + p = 0$ and $pp = p$. But Zhegalkin then extended the Boolean apparatus to propositions of first-order functional calculus, and used his machinery to prove the truth of a number of propositions of Whitehead and Russell's *Principia Mathematica*. In his next paper, on "The Arithmetization of Symbolic Logic" ([*1928-1929*]), Zhegalkin carried out the next step in his project. Here, he applied his mechanical procedure for determining the truth values of molecular propositions directly to propositions of first-order logic. Again, he was concerned with sums and products of propositions, for which he constructed actual truth-tables, not only for quantified sentences, for which he borrowed the Peirce-Schröder definition of universal quantification and existential quantification respectively as sum and product, but for each of the familiar logical connectives as well. Unlike Wittgenstein, whose *Tractatus logico-philosophicus* of 1921 presented truth-value computations in the familiar tabular form, Zhegalkin presented his truth-value computations as a Boolean-valued list, with 1 = **true** and 0 = **false**. But to Zhegalkin was assigned the honor of bringing together the Boolean procedure of arithmetization of quantification theory with the development of a truth-value semantic.

The leading Soviet worker in applications of metamathematics was Anatolii Ivanovich Mal'tsev (1909 – 1967). Much of this work on the border between algebra and logic properly belongs to the specialized subfield of of mathematical logic called model theory, and some overlap into the field of universal algebra. Here we must mention Mal'tsev's [*1936*] paper *Untersuchungen aus dem Gebiete der mathematischen Logik*, which is concerned with a generalization of Gödel's [*1930*] completeness results for first-order functional calculus and with an infinitary generalization of Skolem's [*1934*] proof of the uncompleteability of the characterization of the natural numbers. Mal'tsev's approach was model-theoretic, but we can consider it as a metamathematical work.

Mal'tsev gave a truth-theoretic definition of models. Specifically, he defined "every assignment of truth-values to the elementary propositions," or atomic propositions, of propositional calculus as a model. What is significant here is the intimate role which the concept of *logical truth* plays in metamathematical investigations of mathematical systems.

There is, of course, an important distinction between the validity of a proof and the validity or truth of a formula. No proof is true or false, but only either valid or invalid. However, if we define a proof as an *extended formula*, then, like any formula, its validity is determined by its truth-conditions; that is, a formula is *valid* if and only if, for *every* assignment of truth-values to its constituents, it is true, so that if it is valid, then it is *truth-constant* or *t-definite*. (This fact coincides quite nicely with Tauts' [*1967*; *1968*] definitions of truth-values as formulæ and of logic as a classification of formulæ according to their truth-values. In fact, it is likely that Tauts' work is a deliberate generalization of Mal'tsev's [*1936*] results, although there is no direct reference to Mal'tsev in Tauts' work.)

As formulated by Mal'tsev, Gödel's theorem says that for a countable system of formulae of propositional calculus to be consistent, it is sufficient that every finite part of that system be consistent.

To generalize this result to systems of any power, not merely to those which are countable, Mal'tsev began by defining a generally infinite set of propositions $\mathfrak{S}$ of propositional calculus. Next, he defined $\mathfrak{S}$ to be consistent if and only if we can assign to every elementary formula of $\mathfrak{S}$ (from which all remaining formulæ of $\mathfrak{S}$ are built) the values T (true) or F (false) in such a way that every formula of $\mathfrak{S}$ is true. That is, if $\{e_i\}$ is the set of elementary formulæ of $\mathfrak{S}$ and $\{f_i\}$ is the set of all formulae of $\mathfrak{S}$ such that for each e_i, the value of the assignment α of truth-values is either true or false, that is, if $\mathrm{v}[\alpha: e_i]$ = t/f, and each $f_i = (e_0 * e_1 * \ldots *e_n)$, ($n \in \mathbb{N}$, for $e_0, e_1, \ldots, e_n \in \{e_i\}$ and * an arbitrary logical connective of the propositional calculus, we have then $\mathrm{v}[\alpha': f_i\,] = \mathrm{t}$ for the related assignment α' of truth-values.

Providing a model-theoretic proof and using these definitions, Mal'tsev arrived at the theorem generalizing Gödel's theorem to an arbitrary infinite power:

Theroem: *In order that a system $\mathfrak{S}$ of formulae propositional calculus be consistent, it is necessary and sufficient that every finite subsystem of $\mathfrak{S}$ be consistent.*

For an arbitrary ordinal ω_α, Mal'tsev considers a system $\mathfrak{S}$ of power $\aleph_\alpha$. By a well- ordering argument and the use of transfinite induction, Mal'tsev was able to show that if the theorem holds for all systems of power less than $\aleph_\alpha$, then it holds for systems of power $\aleph_\alpha$. The finite case is trivial. Therefore, Mal'tsev had only to prove his results for the cases where ω_α is either a limit ordinal or a successor ordinal. This generalizes Gödel's proof of the completeness of first-order predicate calculus.

Mal'tsev next considers Skolem's [*1934*] result that it is not possible to construct a countable system of formulae of first-order predicate logic that completely characterizes the structure of natural numbers. Working with formulae in first-order predicate logic which are in Skolem normal form (prenex normal form), Mal'tsev defines a *configuration* as a subset $\mathfrak{B}$ of a universal set of predicate constants, and defines a configuration to be *complete* if and only if it contains at least one term of each of a pair of opposed elements of the universal set, i.e. if for each $\mathfrak{b}_i$ and $\mathfrak{b}_j$ of the universal set, the configuration contains one of $\mathfrak{b}_i, \mathfrak{b}_j$ provided $\mathfrak{b}_j$ is a complement of $\mathfrak{b}_i$. A configuration is *consistent* if for any pair $\mathfrak{b}_i$ and its complement, the configuration contains only one member of the pair.

With these basic definitions in hand, we are ready to introduce a new relation of *equality*. Given the set $\mathfrak{B}$ and any other set $\mathfrak{B}'$, we establish a correspondence between $\mathfrak{B}$ and $\mathfrak{B}'$ where (i) each element of $\mathfrak{B}$ corresponds to one and only one element of $\mathfrak{B}'$; and (ii) two different elements $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{B}$ correspond to the same element of $\mathfrak{B}'$ if and only if the term $\mathfrak{a} \approx \mathfrak{b}$ belongs to $\mathfrak{R}$, where $\mathfrak{R}$ is the configuration with equality. After replacing each element of $\mathfrak{B}$ in every term of $\mathfrak{R}$ with the corresponding element of $\mathfrak{B}'$, we obtain the configuration $\mathfrak{R}'$. Now the configuration $\mathfrak{R}$ is consistent if and only if the corresponding configuration $\mathfrak{R}'$ is consistent in the sense, already presented for the propositional calculus, that a configuration is consistent if and only if it contains no opposing elements. If a configuration is consistent with respect to the consistency of its corresponding configuration only, it is *relatively consistent*; and if it is consistent in the sense of correspondence of equality, then it is *absolutely consistent*. Thus, we obtain the lemma according to which *if the configuration* $\mathfrak{R}$ *is consistent with respect to relative equality and satisfies the conditions,* $\mathfrak{I}$, *of identity (reflexivity), symmetry, and transitivity of* $\approx$ *and of of induction on the predicates of the system* $\mathfrak{S}$, *of which* $\mathfrak{R}$ *is a configuration, so that* $(x)(y)(P(x)\ \&\ x \approx y \rightarrow P(y))$, *then* $\mathfrak{R}$ *is absolutely consistent.*

This lemma implies the theorem that, *given a system* $\mathfrak{S}$ *of first - order predicate logic with equality, we can obtain a new binary predicate* $E(x,y)$ *to replace equality* $x \approx y$ *in the system* $\mathfrak{S} \cup \mathfrak{I}$ *in such a way that we obtain a new system* $\mathfrak{S}^E$ *in which equality does not occur, but which nevertheless is equivalent to* $\mathfrak{S}$.

Next, for every system $\mathfrak{S}$ of first-order logic, Mal'tsev constructs a system of formulae of propositional calculus equivalent to $\mathfrak{S}$ with respect to satisfiability. A formula is *satisfiable* is there is at least one assignment of truth-values to its constituents (elementary formulæ) such that the formula is true. Thus, if a formula $\mathcal{F}$ is consistent, then every finite subset of formulæ of $\mathcal{F}$ is satisfiable, up to isomorphism. Hence, if $\mathcal{F}$ is a configuration in a model $\mathfrak{U}_{\omega}$ of the propositional calculus, then if $\mathcal{F}$ is a configuration, it is ω-satisfiable. In this way, Mal'tsev obtains for the first-order predicate logic the generalization of Löwenheim's [*1915*] theorem, that *every domain for an infinite system* $\mathfrak{S}$ *of first-order predicate logic includes a subdomain whose power does not exceed the power of* $\mathfrak{S}$.

Yanovskaya next considered the application of Boolean algebra to the analysis and construction of electrical relay-contact circuitry in [*1910*] by Paul Ehrenfest (1880 – 1933) in his review of the Russian edition [*1909*] of Louis Couturat's *L'algèbre de la logique*. Work on the details of this application was begun, according to Yanovskaya, in 1934-1935 by Glivenko's student Viktor Ivanovich Shestakov (b. 1907). Shestakov, said Yanovskaya, wrote up his results in January 1935, but his paper, not published until 1941, so that Claude Shannon, whose [*1939*] publication "A Symbolic Analysis of Relay and Switching Circuits" appeared before Shestakov's [*1941*] "Algebra of Two-terminal Circuits, Constructed Exclusively from Two-terminal Components (Algebra of A-circuits)", thereby received the credit for the results and the claim to priority. This work was continued by Shestakov and then also taken up by Mikhail Aleksandrovich Gavrilov (1903 – 1979).

Finally, Yanovskaya examines Markov's work on recursive functions and the work of A. A. Zykov (b. 1922), Novikov's student, on the problem of the consistency and completeness of the lower functional calculus.

Yanovskaya did not consider advanced work in set theory. She did take up descriptive set theory in her follow-up survey of [*1959*], meanwhile, Aleksei Andreevich Lyapunov (b. 1911) and Novikov surveyed the field in their [*1948*] article on "Descriptive Set Theory", which appeared in the same volume as Yanovskaya's paper.

3.2. Yanovskaya's survey of Soviet research in mathematical logic and foundations of mathematics, 1948-1957. Yanovskaya resumed the narrative in her [*1959*] paper "Mathematical Logic and Foundations of Mathematics", in which she concentrated on the period 1948-1957. (This immense survey of over 100 pages is abstracted by E. J. Cogan [*1962*] in less than one page.) In this paper, she began by considering the personal and professional relations between logicians, discussing their training, the seminars in which they participated together, and their common experiences. She then related their work to the broad international field of work in logic.

Yanovskaya begins and ends her survey with the warning (pp. 13, 115) that this survey is not complete, "does not pretend to be complete" (p. 115), noting (p. 13) that, even though she is dealing with only a ten-year period, "the cadre of soviet scholars producing work on the problems of mathematical logic has grown so rapidly [in this period] that it is already scarely possible to give a short sketch of the state of the abundance of information on the entire body of work and the results obtained." Indeed, it proved to be more than one person alone could possibly handle, and she had the help in writing the survey of Sergei Ivanovich Adyan (b. 1931), Z. I. Kozlova (b. 1914), A. V. Kuznetsov, A. A. Lyapunov, and Vladimir Andreevich Uspenskii (b. 1930). (Here, because of the vast amount of material covered even though it is itself already not complete, we will confine ourselves to presenting outlines of the contents of the chapters of the survey, supplementing these with a more detailed consideration only in order to continue a sketch of those lines of research that we reported as we traced the work dealt with in the [*1948*] survey and to pick up as well the most notable new lines of work reported in the [*1959*] survey.

Chapter One of Yanovskaya's survey contains a discussion of "Some Problems in Set Theory, and Yanovskaya mentioned important results in particular of members of the Luzin group. The first section is devoted to axiomatic set theory and gives results especially of Luzin, of Novikov, and of Aleksandr Sergeevich Esenin-Vol'pin (b. 1924), along with consideration of the work of B. S. Sodnomov (b. 1922), Mikhail Yakovlevich Suslin, (1894 – 1919), Uspenskii, and B. A. Trakhtenbrot (b. 1921), and continues the discussion of Bochvar's work on K_0.

Section 2, on descriptive set theory, was written with the assistance of Kozlova and Lyapunov and considers the work of Luzin, Suslin, Pavel Samuilovich Urysohn (1898 – 1924), Pavel Sergeevich Aleksandrov (1996 – 1982), Kozlova, and Lyapunov.

The second chapter, on "Theory of Algorithms and Computable Functions and Operations", discusses recursion theory and the theory of algorithms, including in particular consideration of of work in computable functions and operations. This chapter, "On Representations of Recursive Functions. Functions of Large Oscillation", is devoted to results on representations of recursive functions, the general theory of algorithms, effective enumerability and separability of sets, the reducibility of mass problems, and degrees of unsolvability, Post's reducibility problem, and the descriptive properties of arithmetic sets. The sections of this chapter are 3, "On Representations of Recursive Functions. Functions of Large Oscillation" discussing work of Kuznetsov, Markov, and Aleksandr Yakovlevich Khinchin (1894 – 1959); 4, on the "Definition of Algorithm. The General Theory of Algorithm" considering in particular Markov's work, as well as that of his student Nikolai Makarovich Nagornyi (b. 1928). Results of Kolmogorov and Uspenskii were also discussed. In section 5, on "Enumerable Sets and Countable operations on Sets. General Concepts of Enumeration and Programs", Uspenskii's work is the focus. Section 6, "Definition of Mass Problems and Algorithmic Components of Mass Problems. Structure of Degrees of Solvability", the work of Markov, Kolmogorov, Yu. T. Medvedev (b. 1929), Uspenskii, and A. A. Muchnik (b. 1934) is considered. Section 7 considers results of Markov, Novikov, Adyan, G. S. Tseitin (b. 1936), Kuznetsov, Medvedev, Uspenskii, Muchnik, and especially Trahktenbrot on "Post's Problem and Problems Connected with It". Section 8 on "Descriptive Properties of Arithmetic Sets. Problems of Classification of Sets, Functions, and Other Objects", written with the assistance of Kuznetsov, surveys results by Novikov, Uspenskii, and especially Trakhtenbrot and Muchnik.

Closely related to the second chapter is the third, "Mathematical Applications of Theory of Algorithms". Section 9, on "Algorithmic Problems of Algebra", was written with Adyan, and concerns applications of the theory of algorithms to modern algebra (for example Mal'tsev's work on the word problem for groups and for associative algebras, and the Boone-Novikov theorem), although the main figures whose work is considered here are Markov, Adyan, and Tseitin, and especially Markov. Section 10, "The Constructive Interpretation of Mathematical Expressions. Constructive Mathematical Analysis", focusses on the work in particular of Markov and N. I. Shanin (b. 1919), along with work of Tseitin, and deals with constructive proofs of mathematical theorems and with recursive analysis.

The fourth and final chapter is devoted to a survey of results relating "Logical and Logico-mathematical Calculi", and includes sections dealing with constructive calculus from both the classical and constructive points of view (section 11, "Constructive Calculi from the Classical and Constructive Viewpoint"), focussing again on the work of Glivenko and Kolmogorov; on logical calculi and their models dealing with questions of decidability and completeness and consistency of logical theories (section 12) which considers work of Traktenbrot especially, along with the work of Zykov, Muchnik, Uspenskii, Kolmogorov, Novikov, B. A. Falevich, and Mal'tsev, and concludes with a section (§13), written by Kuznetsov and edited by Yanovskaya, on algebra of logic and its generalizations dealing with the work on algebraic logic of Novikov, Markov, Kuznetsov, Sergei V. Yablonskii (b. 1924), and by Gellius N. Povarov (b. 1928)**, V. I. Shestakov, Yablonskii, and Trakhtenbrot on its applications, in particular to computer programming. (Biryukov [*1982*, 87, 94] reported that cybernetics was one of Yanovskaya's particular interests.)

The conclusion of this survey includes information on miscellaneous results of Soviet researchers. including history and philosophy of logic, which do not otherwise fit into the organization of the survey. It includes, for example, a reference to Aleksandra Denisovna Getmanova's criticisms of logicism, in particular of Russell, and to the work of Boris Vladimirovich Biryukov (b. 1922)† on Frege, as well as to Styazhkin's historical studies of logic in pre-revolutionary Russia.

Having outlined the main contents of Yanovskaya's [*1959*] survey, let us pick up the main threads of the topics which were examined in her [*1948*] survey.

3.2.1. On K_0, Bochvar's system $\sum$ and Novikov's system N, and the consistency of theories. In the next major paper with a new result on K_0, the [*1949*] paper "On Classes of Regularities", Novikov was ready to extend his techniques for consideration of the first-order functional calculus. He began by giving a definition of *regularity of classes* in terms of regular formulae. Formulæ of first-order functional calculus are reduced to normal form, according to which they have the form

$$(\mathcal{A}_1 \vee \ldots \vee \mathcal{A}_p) \,\&\, (\mathcal{B}_1 \vee \ldots \vee \mathcal{B}_q) \,\&\, \ldots$$

with negation applying only to the elementary (i.e. atomic) expressions $\mathcal{A}_1, \ldots, \mathcal{A}_p, \mathcal{B}_1, \ldots, \mathcal{B}_q, \ldots$.

Now consider the operations A, B, C on such expressions, with the operations performed under the following conditions:

(A) one of the expressions $\mathcal{A}_i, \ldots$ begins with $\forall$;

** G. N. Povarov was a member of the *Modern Logic* editorial board.

† Boris Biryukov is a member of *Modern Logic*'s advisory board.

(B) one of the expressions $\mathcal{A}_i, \ldots$ begins with $\exists$;

(C) one of the expressions $\mathcal{A}_i \mathcal{B}_i, \ldots$ is a logical product

To perform A, drop $\forall$ and replace it with a free variable; to perform B, replace the bound existential variable with a proper mutant (substitution instance); to perform C, apply the distributive rule over the disjunction. For the class K_0 of logical sums, where K_0 is closed, write the reduced class K as $P_0 \& \ldots \& P_n$. By the operations A, B, and C, the class K_0 is a *regularity class* closed with respect to the rules of product multiplication. Now that we have obtained regularity classes from our classes of first-order functional calculus, we can prove the consistency of that theory.

In his [*1949a*] paper "On the Axiom of Complete Induction", Novikov used the results on regularity classes to consider the question of whether arithmetic theorems are provable without the use in their proofs of mathematical induction. This work is rather technical and falls mainly in the province of proof theory, although it touches also on issues of model theory and set theory as well.

In the 1940's and 1950's, Bochvar sought to apply three-valued logic towards a solution of the logical and set-theoretical paradoxes. This work began in [*1938*; *1981*] with Bochvar's development of "A Three-valued Calculus and Its Application to the Analysis of the Paradoxes of Extended Functional Calculus". Bochvar's system $\sum$ of three-valued logic is equivalent to the first-order quantification theory of Hilbert and Ackermann, without theory of types. The three values are true (T), false (F), and nonsense (N, or $\downarrow$). In two-valued logic of first-order, 'Q', we are led to the Grelling and Russell paradoxes, according to Bochvar, by means of the formulation $Q \supset \subset \sim Q$, i.e. $Q \leftrightarrow \sim Q$. In the system $\sum^{-}$, we have $Q \equiv \sim Q$ being demonstrable; but it leads to the result $\downarrow Q$ instead of leading to a paradox, that is, it leads to nonsense.

Ten years after Novikov undertook in his [*1947*] paper "On Logical Paradoxes" to compare his system N with Bochvar's typeless system $\sum$, Bochvar in 1957 renewed his work on the extended calculus. In [*1957*], he was concerned with the "Question of the Paradoxes and the Problem of the Extended Predicate Calculus" and referring to his old paper of [*1944*]. Here, he focussed on the axiom of extensionality.

3.2.2. On constructive mathematics and Markov on theory of algorithms. Markov began his work on algorithmic theory in the years immediately after the end of World War II. The work began in 1947 in connection with his work on the decidability of algorithms for distinguishing certain properties of associative algebraic systems. This led to a need to clarify the concept of *algorithm*. Markov's first published discussion on the general theory of algorithms is found in [*1951*] and in English and German translations; its first presentation in an international forum took place in Budapest in 1950, at the First Hungarian Congress of Mathematicians, where Markov delivered a talk on "Theory of Algorithms" which was subsequently published in [*1952*] in the conference proceedings. This work was expanded and more fully developed in a series of works on "Theory of Algorithms". The original Budapest paper was also published in [*1951*] in the journal *Works of the Steklov Mathematics Institute*. In this paper, Markov found from a survey of the literature by Church, Kleene, and Turing that these researchers helped to make the idea of algorithm precise, but they did not clarify the idea of *algorithm* itself; that is, the algorithm as a mathematical tool was sharpened by the studies of Church, Kleene, and Turing, but the understanding of the concept of *algorithm* still needed sharpening. The aim of Markov's work in this paper and in succeeding works was to give a clear and precise definition of *algorithm*. Much of the technical aspect of Markov's work along these lines belongs to a discussion of Soviet work in recursion theory, since Markov's concept of algorithms is close to that of Post, except that Post algorithms correspond to general recursive functions, while Markov algorithms correspond to primitive recursive functions. By far the most detailed development of the Markov algorithm is given by Markov's book-length paper of [*1954*] on "Theory of Algorithms" (and in the subsequently published book *Theory of Algorithms* [*1984*] coauthored with Markov's student Nikolai Makarovich Nagornyi and completed by Nagornyi after Markov's death.)

What we call the Markov algorithm, what Markov called the "normal algorithm", is defined constructively, that is, by a fixed alphabet and some auxiliary symbols which, together, gives one the "scheme" of the algorithm. The algorithm is then understood as a combinatory rule for operations of transformation of words of the alphabet, that is, as a prescription for permutations of (strings of) letters in the alphabet. Let $\mathbb{P}$ be a word in an alphabet $\mathbb{A}$. Then our algorithm defines the successive discrete steps from which, starting with $\mathbb{P}$, we obtain some new word in the alphabet $\mathbb{A}$. Let $\mathscr{A}$ be our algorithm. $\mathscr{A}$ transforms $\mathbb{P}$ into $\mathbb{Q}$, where $\mathbb{Q}$ is the last word obtained from $\mathbb{P}$ by $\mathscr{A}$. Then the equation $\mathscr{A}(\mathbb{P}) = \mathbb{Q}$ is the *proof* of $\mathbb{Q}$ and $\mathbb{Q}$ is a *theorem* of $\mathbb{P}$ in alphabet $\mathbb{A}$. (Later, in his two papers "On Alphabet Coding" ([*1960*;

1961]), Markov showed how to code words in a finite alphabet $\mathbb{A} = \{a_0, \ldots, a_m\}$. By substituting for each letter in one alphabet some word of another alphabet $\mathbb{A}'$, we obtain an isomorphic coding. This gives us an effective test for deciding whether the two alphabets are isomorphic. If there is a one-to-one correspondence from the letters of $\mathbb{A} = \{a_0, \ldots, a_m\}$ unto each word of $\mathbb{A}' = \{u_0, \ldots, u_m\}$, then the proofs of theorems in A are computable. (In the follow-up paper ([*1961*]), Markov extended this encoding result to variable-length binary codes. Thus, we can reduce the coding to occur within an alphabet $\mathbb{A} = \{0, 1\}$. Then if $\mathbb{A}(a_i) = 0$, our proof has yielded a contradiction, so that we have in fact $\neg a_i$ being true; and if $\mathbb{A}(a_i) = 1$, we have a_i being true. This obviously has immediate uses for machine computation.)

The introduction of the concept of *algorithm* into the formal system created by Kolmogorov by A.A. Markov in the 1950's and 1960's provided a foundation for such a metaphysics-free computational constructive system.

In "Mathematical Logic and Computational Mathematics" [*1957*] Markov considered the connection of mathematical logic with computational mathematics. Written for the general scientific reader, this paper explains how logic, in particular constructive logic as a theory of algorithms, is applicable to the solution of simple arithmetic problems. Markov gave the Soviet constructivist school its characteristic computational procedures and its characteristic focus. Markov's constructive calculus depends upon normal algorithms which provide rules of combinatory permutations on mathematical objects, called words, built up from a finite alphabet which allow us to obtain a new word (theorem $\mathbb{Q}$ from a work (axiom) $\mathbb{P}$ such that, by the application of the algorithm $\mathcal{A}$, we obtain $\mathcal{A}(\mathbb{P}) = \mathbb{Q}$. The Markov constructive logic can be understood as the formalization of intuitionistic first-order functional logic, together with the adjoined Markov Principle (MP):

$$\text{MP:}\ \forall x \in \mathbb{R}\ (\neg\, x \leq 0 \rightarrow x > 0)$$

which, as understood and used by Markov himself, is what has been called the *primitive-recursive Markov Principle* (MP_{PR}), i.e.

$$\text{MP}_{\text{PR}}\text{:}\ \neg A\ \neg A(n, m) \rightarrow \exists n\, A(n, m)$$

where A is a property of the constructive objects in the language FIPC (formalized intuitionist predicate calculus — IPC for short). Much of this work was carried out in the 1960's and 1970's, however, and is beyond the scope of Yanovskaya's survey.

2. Concluding comments. Sof'ya Aleksandrovna Yanovksaya's influence was exerted more through her teaching and personal contacts that through her research. The first half of her academic career was given over in large measure to the ideological defence of mathematical logic, and the remainder of her energies were divided between aiding her Soviet colleagues in keeping abreast of the work of their western colleagues, defending and increasing the gains made in the ideological struggle between the formal logicians and the dialectical logicians, and inculcating and preserving among her students and colleagues a knowledge of the history of mathematics in general and of mathematical logic in particular, including, through the [*1948*] and [*1959*] surveys the accomplishments in mathematical logic that they themselves had authored. The influence which she exercised is of course diminishing as her students die out. But her greatest accomplishment, the formation and shaping of a professional cadre of historians of mathematics and historians of logic is a lasting and fitting monument and tribute to her work. As Zinov'ev wrote [*1968*, 209]: "S.A. Janovskaja in particular has performed a truly titanic task in preparing specialist logicians," and Biryukov [*1982*, 96] wrote that, next to his own mother, he considered her to be the woman who most "played a decisive role in my life." When the philosopher Igor Sergeevich Narskii (b. 1920) was able to declare, in his crucial policy-making paper [*1966*] setting forth the new ground-rules for the role that logic should play in university education in the USSR, that "traditional logic no longer exists, that formal logic now *is* mathematical logic," much of the credit for that must specifically and emphatically be assigned to Sof'ya Aleksandrovna Yanovskaya.

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