Alejandro R. Garciadiego. *Bertrand Russell and the Origins of the Set-theoretic 'Paradoxes'*. Basel/Boston/Berlin, Birkhäuser Verlag, 1992.

Reviewed by

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A central feature in the history of 20th century mathematical logic is Bertrand Russell's discovery of the paradox named for him. It is central for two reasons. One is that the existence of paradoxes in naive set theory is responsible for the form of the two most influential foundational accounts of mathematics in the 20th century, Russell's own type theory and Zermelo's set theory (both published in 1908). The other is that the need for foundational studies of mathematics at all has often been attributed, by both mathematicians and historians, to a crisis in mathematics caused by discovery of the paradoxes. Recent historians have raised doubt about the existence of this "crisis" or its influence on the foundational work that was done.

Certainly in Russell's own foundational work it was a crisis. It is commonplace to say that Russell's communication to Frege that the system of the *Grundgesetze* [Frege *1893*] is infected by Russell's paradox was a devastating blow to Frege. What is less often remarked is that it was also a devastating blow to Russell. His discovery of what he called simply "the contradiction" came in the midst of a three year struggle to formulate his logicism in *Principles of Mathematics* [Russell 1903]. It left him at the end of that struggle forced to "confess" that he had no adequate concept of *class*, one of the "indefinable notions" on which his logicism was to rest. In his "Preface" [Russell *1903*, xv--xvi] he says,

> In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of *class*. And the contradiction discussed in Chapter X proves that something is amiss, but what this is I have hitherto failed to discover.

Russell's ultimate solution was the theory of types. But Russell, himself, recognized that the theory of types was awkward and unsatisfying. His most brilliant followers, the early Wittgenstein and Ramsey, both felt compelled to deal with it. The early Wittgenstein (apparently) thought that type theory was not simply wrong, but also unnecessary. Ramsey distinguished simplified and ramified versions of type theory and argued that only the simplified versions were needed to avoid those contradictions (set-theoretical) which threatened the logicist program. The contradiction is the red thread running through all of Russell's work in logic and foundations.

There are two chief historical issues connected with Russell's discovery of his own paradox, i.e. a set of all sets that are not members of themselves is contradictory, despite the belief of Russell and others that any stateable condition determines a set. These are the questions of when did he discover the paradox and what was the intellectual source of his discovery. The work under review seeks to synthesize recent scholarship aimed at answering these questions. In doing so it aims to reach a mixed audience. This includes students and other non-specialists who are interested in the history of Russell's paradox, but who are not adepts of the journal literature. Dr. Garciadiego is also addressing specialists and making a case for his own views in the scholarly controversies connected with the subject.

Controversy about the date of the discovery arises from Russell's conflicting recollections of when he discovered his paradox. We have a draft manuscript of Part I of *Principles of Mathematics* from May 1901 in which Russell has written out the paradox. This provides an end date. The question then is how long before this was Russell aware of his paradox.

This latter question blends into a second question of the intellectual origins of Russell's Paradox. This is because Russell's discovery of his paradox was not a free-standing intellectual event. Russell's own accounts of the origins of his paradox link it to his consideration of Cantor's diagonal proof for Cantor's Theorem ($\alpha < 2^{\alpha}$, for all cardinal numbers α). Both Russell and Cantor recognized that it was a consequence of this theorem that there is no class of all classes. In *Principles of Mathematics* sec. 346–349 Russell presents (in effect) a rational reconstruction of the thinking that led him to connect Cantor's diagonal proof of his theorem to Russell's own paradox. The challenge for the historian is to tease out the actual intellectual events behind that rational reconstruction.

The chief portions of this book consist of a long "Preface", which provides a précis of the book as a whole. There is a chapter on "antecedents" which is aimed at those who are unfamiliar with the history of Cantorian set theory. This is followed by a chapter that expounds the features of what the author believes is a "standard interpretation" of the history of the paradoxes in the secondary literature. This "standard interpretation" holds that an atmosphere of "crisis" developed in foundations of mathematics as a result of the discovery of various paradoxes around 1900, especially "Burali-Forti's Paradox" and "Cantor's Paradox". This led to the eventual development of the three foundational schools of platonism, logicism, and intuitionism. The author points out that at least in the cases of Burali-Forti and Cantor the discoverers of these "paradoxes" did not view them as such. In the case of Burali-Forti he thought of himself as presenting a *reductio ad absurdum* argument against Cantor's belief in trichotomy for the ordinal numbers. In Cantor's case he thought that consideration of the set of all sets showed that one must distinguish between "consistent" multiplicities; which were in the realm of his *Mengenlehre* and "inconsistent" multiplicities which transcended it. The author suggests that he will take this insight into the divergent views of the mathematicians who first considered the set-theoretic "paradoxes" and use it to propound a new interpretation to replace the "standard" one. Other than to lay out some of the complexity of actual intellectual history, which is a contribution, it cannot be said that the author succeeds in replacing the standard two-paragraph textbook account of the history of the set-theoretic paradoxes with something equally graspable.

A third chapter is devoted specifically to the historical background of the Russell of *Principles of Mathematics*. This includes some material about his birth, childhood, upbringing, family situation, etc. More directly relevant to the theme of this book is an examination of Russell's early attempts as a Cambridge fellow to write books on the foundations of mathematics. These early writings were guided by a neo-Hegelian belief in the inconsistency of any science, mathematical or otherwise, that is short of the final absolute knowledge of reality. This period and theme in Russell's intellectual history has been thoroughly explored in [Griffin *1991*]. Neither in the present book nor in Griffin's is any definite connection established between Russell's neo-Hegelian belief in the ultimate contradictoriness of mathematics and his concern with paradoxes after his conversion to the platonic realism of G.E. Moore. Russell's ability to abruptly and completely change his outlook is legendary. It is perhaps best epitomized in the famous story of how he suddenly decided that he no longer loved his first wife, Alys. But it was not limited to personal matters. Once he abandoned neo-Hegelianism, Russell never looked back.

The intellectual heart of the book is chapter four. This is a detailed analysis of the stages of the writing of *The Principles of Mathematics*. The analysis is based on the author's own research in the Russell Archives at McMaster University. (Much of the relevant material has now been published as [Moore *1993*].) In broad outlines the author's analysis of the stages of writing *Principles of Mathematics* conforms to that of others who have worked with the manuscript material (*cf.* [Moore *1993*]). In particular, the author dates Russell's "discovery" of his own paradox to May 1901. He explains why certain evidence pointing to a date as early as January 1901, or to a later date in June 1901 is probably misleading. He is particularly sensitive to Russell's unwillingness to immediately credit his own paradox as something fundamental and not an elementary mistake in reasoning. He, sensibly, attributes to this Russell's delay in communicating his paradox to others, in particular to Frege to whom he wrote a year later in June 1902.

The final chapter, although it is entitled "The 'Semantic Paradoxes'" gives the history of only two of these, "Berry's paradox" and "Richard's paradox". Mixed in with this is a history of the König-Zermelo debate about the Axiom of Choice and the Well-ordering Theorem and debates about this in the London Mathematical Society in which Russell was involved. This latter material is justified because Russell used some of this debate as the basis for one of the "contradictions" he listed in [Russell *1908*].

An appendix contains an extensive, but not comprehensive selection of 27 letters to Russell from such figures as C. Burali-Forti, A. N. Whitehead, G. G. Berry, and G. H. Hardy, on matters related to the book's topic. There is also an extensive and quite comprehensive bibliography of literature related to the themes of the book.

The general conception of the book is very worthwhile; it gathers in one place the available information on the history of Russell's paradox and its relation to other mathematical paradoxes. On this level the book succeeds. A student can find here an orientation to the history and issues surrounding Russell's paradox which could otherwise only be achieved by reading a diverse collection of journal articles.

Nevertheless, the student must be warned that the book is often an unreliable guide. The most significant fault is that the reader would easily get the impression that in *Principles of Mathematics* Russell talks of both "Burali-Forti's Paradox" and "Cantor's Paradox". Russell does write in *Principles of Mathematics* of a "contradiction" discovered by Burali-Forti (section 301) and in subsequent publications in which he gives a roster of "contradictions" ([Russell *1908*], Whitehead & Russell *1910*]), he does list "Burali-Forti's contradiction". But Russell does not, at least through the period of *Principles of Mathematics*, refer to "Cantor's paradox" or "Cantor's contradiction". He does write of "difficulties" caused by Cantor's theorem in connection with Russell's assumption, in 1903, that there must be a class of all classes. But he makes clear that it was his analysis of these "difficulties" which led him to what he takes to be "the contradiction".

These points are not terminological subtleties because it is one of the author's fundamental historical contentions that it was Russell who gave the work of Burali-Forti and Cantor currency as paradoxes. While the reviewer agrees with the author's view about Russell as a source in the literature for "Burali-Forti's contradiction", the textual evidence is not there for the claim that "Russell created the 'paradox' of the greatest cardinal number" (p. xix).

This work would have been greatly improved and would have provided a better guide to the history of the subject if there was a discussion of the terminological confusion related to the words "paradox" and "contradiction". The need for such clarification was pointed out in reviews of earlier work by the author ([Corcoran *1987*; Corcoran *1988]*). There it was suggested that "paradox" is a "participant-relative" term, the applicability of which is tied to the status of the participant.

"Contradiction" or "inconsistency" can be a term of objective logical appraisal, i.e. a recognition that two or more sentences cannot be simultaneously true. Whether this is viewed as problematic depends on the intellectual values that the participant brings to the situation. Sometimes a participant is interested in exhibiting an objective contradiction in a theory which is disbelieved on separate grounds, in order to publicly disprove the theory. Poincaré's relation to Cantorian set theory is of this sort. In other cases, the participant has the positive goal of proving one statement by showing that the negation of that statement is objectively inconsistent with accepted proposition. This is proof by contradiction.

We could reserve the word "paradox" for situations where participants recognize an objective contradiction in their own beliefs without having a prior plan for interpreting its meaning. We could say that a "paradox" has both an objective and a subjective component. On this account, "Russell's paradox" would be a true paradox, because of Russell's response of puzzlement to the inconsistency. "Cantor's

Paradox" is, on the other hand, none at all, because Cantor's discovery of "inconsistent sets" proved for him something he believed all along, that the infinity of the Alephs is something distinct from the infinity of God. In many cases, such as those of Cantor and Burali-Forti, the author is sensitive in practice to these sorts of distinctions. But the reader would be better served by a systematic discussion of the author's basis for treating some arguments as paradoxical and others not.

In other matters the reader must also be on guard. For instance, on p. 87 Russell is said to need to show "that the part obtained by subtracting a single individual from the totality was isomorphic with the totality itself," when what is at issue is not isomorphism but simple one-to-one correspondence. On p. 39 the false suggestion is made that using his distinction between consistent and inconsistent classes Cantor should have seen that "[the set of all finite cardinal numbers] was inconsistent and could not have a cardinal number associated with it."

The book is also marred by a remarkable number of typographical errors and mangled sentence constructions which are at the least annoying and at worst make it difficult to construe what is being said. A great amount of the blame for this should go to the publisher which, while producing a physically attractive book, seems to have devoted no effort at all to copy-editing.

This book will be of interest to the specialist as a compendium of the literature on Russell's paradox and for specific arguments on aspects of its history. But it cannot be safely handed to the student as a comprehensive and reliable guide to this important episode in intellectual history. That book is yet to be written.

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Bart Kosko, *Fuzzy Thinking: The New Science of Fuzzy Logic* (New York, Hyperion, 1993), xviii +318pp., and Daniel McNeill and Paul Freiberger, *Fuzzy Logic* (New York/London/Toronto/Sydney/Tokyo/Singapore, Touchstone Book, Simon & Schuster, 1993), 319pp.

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Fuzzy logic has achieved much noteriety recently, attracting the critical attention not only of logicians and engineers, but has also captured the popular imagination because of its brilliant applications in the tools of everyday life, from cameras to washing machines to high-speed railway systems. The goal of the two books under review is to satisfy the curiosity of those who seek an explanation of the new science of fuzzy logic and to appeal to its noteriety. Kosko's book has received acclaim

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