J. Chapman and F. Rowbottom *Relative Category Theory and Geometric Morphisms: A Logical Approach* Oxford Logic Guides 16, Oxford, Clarendon Press, 1992

Reviewed by

Colin McLarty
Department of Philosophy, Case Western Reserve University,
Cleveland, OH USA 44106

This book aims to make topos logic an adequate tool for all topos theory by extending it to handle category theory in toposes, or "relative category theory", with the relative Giraud theorem as test case. The problem and the test were remarked as early as Johnstone [*1977*, xviii].: "the formal language approach breaks down when confronted with the relative Giraud theorem; whilst [it] is a very powerful tool in proofs within a single topos, it is not well adapted to proofs in which we have to pass back and forth between two toposes by a geometric morphism."

It is well known that each topos S has an *internal languaqe* called L_S, a multi-sorted constructive set theory interpreted in S. For any topos S, Heyting's intuitionistic predicate logic is sound in L_S, but classical logic generally is not. Of course classical logic ls sound in some toposes, such as the topos of classical sets, **Set**. But in general the law of excluded middle and the axiom of choice fail.

There are different views on the internal language. Most category theorists are not logicians and many dislike the syntactic details needed to make it a rigorous tool. Barr and Wells [*1985*] avoid it almost entirely. On the other hand Bell [*1988*] introduces toposes almost entirely in terms of it. Chapman and Rowbottom justly say their book "is essentially self-contained, except for basic category theory, which may be found in Mac Lane [*1971*] or Barr and Wells [*1985*], Chapter 1. However, it forms a natural sequel to Bell's book [*1988*]" (p.7).

Their task falls into two parts: treating small categories in the internal language, and treating certain large ones. A *small category* in a topos is with an object of objects and an object of arrows. In the topos **Set** then, it is a category with a set of objects and a set of arrows. A large category is one too big to be small. In **Set** it is a category with a proper class of objects and of arrows. The theory of small categories in **Set** has always been largely constructive and so works in any topos. But expressing it in the interval language has been surprisingly thorny.

The composite of arrows f and g is defined if the codomain of f is the domain of g. Logicians usually formalize partial functions by relations. Instead of a term gf for "the composite of f and g" we use a relation $C(f, g; k)$ read "k is the composite of f and g". The definability condition for composites is then stated

$$(\exists k)\, C(f, g; k) \leftrightarrow \mathrm{Dom}(g) = \mathrm{Cod}(f)$$

with the functions Dom and Cod for domain and codomain. But simple formulas become nearly unreadable in this notation. The associativity equation $f(gh) = (fg)h$ becomes:

$$C(f, g; k) \;\&\; C(k, h; i) \;\&\; C(g, h; j) \;\&\; C(f, j; n) \;\rightarrow C\; i = n$$

No one works with this notation.

So when Johnstone [*1977*] had to do category theory in a topos, he avoided the internal language he used elsewhere very elegantly. On the other hand McLarty [*1992*] does category theory in the internal language with fg as an abuse of notation made rigorous in an exercise.

Chapman and Rowbottom go much farther, and this is the central technique of the book. They extend the internal language L_S to a theory of partial functions L'_S. This L'_S has the same sorts as L_S and includes the same relation and function symbols. But let $F(x, y)$ be any partially functional relation in L_S, i.e. any relation such that

$$F(x, y) \;\&\; F(x, Z) \;\rightarrow\; y = z$$

Then L'_S has a function symbol f_F such that

$$f_F(x) = y \;\leftrightarrow\; F(x, y)$$

with $f_F(y)$ undefined otherwise. Chapman and Rowbotton use a different notation but this will do to explain the point. So L'_S has function symbols for composition for every category in L_S although composites are not always defined. The language L'_S is close to Scott's [*1979*] using partial description operators. The authors use L'_S to describe small categories, small cartesian closed categories, and small toposes in S.

As to foundations, to speak of a topos S the authors assume a metatheory dealing with collections of objects and arrows of S. It could be some set theory or some topos. By *meta-class* they mean whatever kind of collections the metatheory uses.

They handle the large categories they need by means of a new definition of a *category in S* for any topos S. A category C in S has a meta-class $|C|$, called the *indexing meta-class* of C, and for each A in $|C|$ an S object C_A and for each pair A,B of members of $|C|$ an S object C_{AB}. Think of each C_A as a collection of objects of C and each C_{AB} as the collection of all arrows of C from any object in C_A to any one in C_B. We need S arrows $\mathrm{dom}_{AB}: C_{AB} \rightarrow C_A$ and

$$\mathrm{cod}_{AB}: C_{AB} \rightarrow C_B \text{ for every } A \text{ and } B \text{ in } |C|,$$

thought of as taking each arrow in C_{AB} to its domain and codomain respectively. And for all A, B, and C in $|C|$ we need a partial function of composition from the product $C_{AB} \times C_{BC}$ to C_{AC}. The composite of a pair $<f,g>$ must be defined if $\mathrm{cod}_{AB}(f) = \mathrm{dom}_{BC}(g)$ and must satisfy the usual axioms.

In the special case where $|C|$ is a singleton C is a small category in S. In the special case where each composition function is total C is a category enriched in S in the usual sense.

The example they use most is s , called the category of internal sets and functions of *S*, or the self-image of *S*. Here the indexing meta-class is Obj(*S*), the meta-class of all *S* objects. For each *S* object Ɨ the object $\mathsf{s}_{\text{Ɨ}}$ is the power object of Ɨ. For any *S* objects Ɨ and Ɉ the object $\mathsf{s}_{\text{ƗɈ}}$ contains all triples <I,f,J> with I a "subset" of Ɨ and J of Ɉ, and f a partial function from Ɨ to Ɉ with I as domain of definition and all its values contained in J. The composite of any <I,f,J> with any <J,g,K> is <I,fg,K>.

The key fact is that *S* sees s as well-pointed, because any object in any topos is internally the union of its singleton "subsets". We can transfer questions about toposes over *S* to toposes over s , and then neglect any explicit indexing or use of generalized elements.

This approach to large categories is interestingly simple and natural in the logical presentation. Its reliance on partial functions may be less natural categorically than the known alternatives: indexed categories (see Johnstone and Paré [*1978*]) and fibered categories (see Bénabou [*1985*]. It is less general as it applies only to categories in a topos. And it is perhaps less systematic just because it is less general.

One knows the indexed or fibered category of groups over any category *S* with finite limits. At any index Ɨ it is the category of groups in the slice *S*/Ɨ, which is the category of Ɨ-indexed families of groups. But even if *S* is a topos, what is its category of groups in Chapman and Rowbottom's sense? The category with Obj(*S*) as indexing meta-class and assigning to each Ɨ the internal "set" of all group structures on "subsets" of Ɨ might work well but it is not clear. The authors never mention such things but keep their examples close to the self-images of toposes.

Giraud proved, in effect, that any elementary topos defined over **Set** with a few nice properties is a Grothendieck topos, i.e., it is representable as the category of all sheaves on some site in **Set**. Altogether this says a lot about the topos. For applications see Mac Lane and Moerdijk [*1992*].

The Giraud theorem, and much more, was relativized to any elementary topos *S* in place of **Set** by Diaconescu [*1975*]. Any geometric morphism $S' \rightarrow S$ of elementary toposes with a few nice properties makes *S'* representable as the category of sheaves over some site in *S*. Johnstone [*1977*] follows Diaconescu's proof. Anyone familiar with the internal language will see its workings behind the proof but it is not used and it is not clear how it could be when you get right down to it. That is the point of the present book. Barr and Wells [*1985*] give a more algebraic proof.

Chapman and Rowbottom's proof of the relative Giraud theorem is not short. Whether it is clearer than the others will depend on how much you prefer logical apparatus to category theory. At any rate, it is crucial to their extension of topos logic towards a comprehensive framework for topos theory.

The authors say "it seems incoherent to make stronger assumptions (set theoretic or logical) at the 'meta' level than at the level of the topos. For this (and other) reasons, work at the 'meta' level will be constructive" (p. 9). The general claim is already debatable. Indeed the metatheory must make some stronger assumptions to define semantics for the object theory. More important here is to square it with the puzzling remark that "we wish to convince the reader that local set theories *can* be constructed in a topos [i.e. it can be done constructively—CM]. However most of the material is disposable in the sense that it will not have to be referred to in practice" (p. 100). A *local set theory* is a theory which can be the internal language of a topos.

The Giraud theorem for an arbitrary topos S involves toposes in S. Results on those toposes were proved using their internal languages with L'_S as metatheory. Thus it requires constructive metatheory. But basic model theory, as opposed to many methods for obtaining particular models, is nearly constructive anyway. Syntax deals with finite strings; basic semantics uses explicit constructions.

There is only one trick to a constructive metatheory of topos logic. The authors describe it but without clearly saying it is the only trick. We must restrict ourselves to expressions "with free variables that are 'indicially distinct'—equal precisely when their indices are—so that comparison of free variables reduces to comparison of indices (and avoids comparison of types) and so is decidable" (p. 9). This is merely a matter of labelling and "can be ignored in practice" (p. 9), which seems to explain the above quote from p. 100.

Finally, categories in S are not flexible enough to give the adjoint functor theorems. So the book closes with an appendix on generalizing categories in S to categories in S with parameters. These give adjoint functor theorems although, according to the closing remark on p. 254, it is an open question whether these work so smoothly as in the classical case and the expected answer is no. Category theoretically, a category with parameters in S is just a category in a slice S/I. The authors prove that fact but decline to use it (p. 222) as they prefer a thoroughly logic oriented account.

REFERENCES

BARR, M. and WELLS, C. 1985. *Toposes, Triples and Theories.* Berlin, Springer-Verlag.

BELL, J. L. 1988. *Toposes and Local Set Theories.* Oxford, Clarendon Press.

BÉNABOU, J. 1985. *Fibered categories and the founcations of naive category theory* Journal of Symbolic Logic **50**, 10–37.

DIACONESCU, R. 1975. *Change of base for toposes with generators* Journal of Pure and Applied Algebra **6**, 191–218.

JOHNSTONE, P. T. 1977. *Topos Theory* London, Academic Press.

JOHNSTONE, P. T. and PARÉ, R., eds. 1978. *Indexed Categories and Their Applications,* Springer Lecture Notes in Mathematics No. 661.

MAC LANE, S. 1971. *Categories for the Working Mathematician.* Berlin, Springer-Verlag.

MAC LANE, S. and Moerdijk, I. 1992. *Sheaves in Geometry and Logic.* Berlin, Springer-Verlag.

MCLARTY, C. 1992. *Elementary Categories, Elementary Toposes.* Oxford, Clarendon Press.

SCOTT, D. S. 1979. "Identity and existence in intuitionistic logic," in *Applications of Sheaves.* Springer Lecture Notes in Mathematics No. 753, 660–96.