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GÖDEL'S LAST WORKS, 1938-1974: THE EMERGING PHILOSOPHY

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Review of

Kurt Gödel, *Collected Works.* Volume II. *Publications 1938-1974.* Editor-in-chief Solomon Feferman. New York and Oxford, Oxford University Press, 1990. xvii + 407.

I

This is the second volume of this impressive series of Gödel's works. Its general characteristics are the same as those of the first volume which appeared in 1986 with the same editorial team. Since I already reviewed those characteristics in my earlier essay-review (THIS JOURNAL, 3, (1992), 58–74), I will concentrate here on the particular content we are offered now. As before, my viewpoint will be that of a philosopher, so I will make comments mostly on the philosophical implications of (or the philosophical theses explicitly maintained in) this set of Gödel's works, as well as on the way in which those implications are taken into consideration in the corresponding introductory notes. The reasons for choosing this procedure, already stated in my former review, can be summed up here: while Gödel's technical works have been well studied and have exerted massive influence in the logico-mathematical development of the second two thirds of this century, his philosophical ideas have been rarely taken into consideration by philosophers. Also, I think that much of Gödel's technical results were obtained mostly in search of logico-mathematical support for his philosophical beliefs. In this connection this second volume of his published works is really fundamental, as it was only in this period that Gödel decided to make public some traits of his philosophical position, which had been maintained at least from the mid twenties, and to which he devoted, as a whole, much more time than to his technical works, in spite of the fact that he persistently refused to publish his purely philosophical writings.[1]

For convenience, the content of this volume can be divided into six parts:

1. The preliminary notes and abstracts plus the definitive work dealing with Gödel's proof of the relative consistency of the axiom of choice (*AC*) and of the generalized continuum hypothesis *(GCH)* with the standard axioms for set theary (1938-40);
2. The article on Russell's mathematical logic (1944);
3. The two versions of the article on Cantor's continuum problem (1947-64);
4. The technical and philosophical writings on relativity theory (1949-52);
5. The two versions of the paper on finitary mathematics, the second printed here for the first time (1958-72);
6. Several remarks and notes.

In the following, I will comment on each of these items, particularly on the respective introductory notes, which is what is really new in this volume, and their treatment of the philosophical questions involved.

II

Gödel's proof of the relative consistency of thc axiom of choice and the generalized continuum hypothesis with the usual axioms of set theory can be seen as the last of his great results, although in this case Gödel was unable to finish the task of proving not only relative consistency, but also independence, which was done by Cohen in 1963. The introductory note to these writings, by R. Solovay, seems to me historically and technically superb. After explaining the historical antecedents of Zermelo's axiom of choice and Cantor's continuum hypothesis, the different axiomatizations of set theory are briefly reviewed in order to point out the model-theoretic work which had been done in set theory before Gödel, that is, to the ideas used by Fraenkel, Skolem, von Neumann and Ackermann on consistency and independence of various axioms.

Then Solovay goes on to a description of Gödel's proof of relative consistency, using constructible sets whose hierarchy was seen by Gödel as a "natural prolongation" to transfinite levels of the ramified theory of types of *Principia Mathematica.* The intuitive idea, i.e. Gödel's "inner model" method, is clearly stated: "he described a certain collection of sets, called the *constructible* sets, and was able to prove (in axiomatic set theory without the

[1] Gödel's most elaborated philosophical unpublished manuscripts, written in the fifties, are going to appear in volume III of this collection, mainly the Gibbs Lecture (Providence, R.I., 1951) and "Is mathematics syntax of language?", versions III and V (which were written in 1953-1959 as Gödel's intended, but never actually a submitted contribution to Schilpp's Carnap volume, finally appeared in 1963). A personal reconstruction and a long historico-philosophical introduction by this reviewer of the Gibbs Lecture, as well as of versions II and VI of the other manuscript mentioned, with a foreword by W.V. Quine, is forthcoming in Spanish (Kurt Gödel, *Ensayos filosóficos inéditos*, Madrid: Grijalbo-Mondadori, in print) and is in preparation in English.

axiom of choice) that each of the axioms of set theory holds in the domain of constructible sets. He also showed that *AC* and *GCH* hold in this domain. From this it follows easily that if the axioms of set theory became inconsistent after adjoining *AC* and *GCH*, then they must already have been inconsistent without these new axioms" (pp. 6—7). Thus, the gist of the proof consists first of defining a constructible universe, through a certain hierarchy of sets using arbitrary ordinals as the levels, then of showing that *AC* and *GCH* hold in this universe, that is, showing that they follow from the "axiom of constructibility" (V = L; i.e. every set is constructible) and that this axiom holds in the constructible universe. Solovay's long last section is devoted to a description of the further work which has been done, both on constructible sets and on the various questions raised by Gödel's work.

However, from the philosophical viewpoint nothing is said. It is certainly true that this work by Gödel is perhaps the one whose philosophical implications are most difficult to point out, but I think that at least two questions should be noted as being philosophically relevant. First, to what extent can this *result* by Gödel be seen as providing arguments either for realism or for nominalism (or positivism, as Gödel himself used to say)?; second, can Gödel's *methods* be presented as some sort of illustration of his well-known realistic position?

As for the second question, Solovay has correctly cited (p. 8) the relevant reference to Gödel's letter of 1968 to Hao Wang, in which Gödel said that, as far as the continuum hypothesis *(CH)* is concerned, his proof of relative consistency was impossible to discover for constructivists because the corresponding ramified hierarchy was used therc in a nonconstructivistic way. Solovay points out that the nonconstructivistic element "lies in the use of arbitrary ordinals as the levels in Gödel's extension of the ramified theory" (p. 8). But there is no attempt to develop the way in which this element can be related to Gödel's general realistic attitude. This is particularly unfortunate, because in the letter to Wang, Gödel tried to connect his methods in this work with his general philosophical position. In particular, I think that it is useful here to try to determine the different senses in which Gödel's position can be connected with his methods. In general, I think it can be said, perhaps at the same time, that Gödel's realistic thesis can be seen as: (i) a philosophical consequence of his technical results, (ii) a heuristic principle leading to them; (iii) a philosophical hypothesis to be "verified" through them. Then, it might be said that the methods used in the relative consistency proof can bc seen as proceeding from an attitude made possible by (ii). But this becomes difficult, when we try to do the same thing for (i).

Thus, I come to the first question, i.e. is Gödel's result here to be located on the side of realism? Gödel himself seemed to have great doubts regarding a positive response to this question. In Hao Wang's *Conversations with Gödel*[2] Gödel said that this particular result was rather on the positivistic side, although he does not explain the precise sense in which these words should be interpreted. On the other hand, many would consider it more or less neutral as far as philosophical controversy is concerned. At any rate, it has to be recalled that for Gödel this particular result was to be seen as only the first part of the complete one: the independence of *AC* and *GCH* from standard set theory, as reached by Cohen in 1963. For this complete result we do have Gödel's detailed philosophical reaction, which was

[2] I have been able to read a copy of this unpublished work, which Professor Hao Wang has generously provided me.

added as a supplement to the second version of his Cantor paper (1964). Let us then leave the matter for the corresponding section below.

III

Gödel's essay on Russell from 1944 is, we can say, the kernel of his emerging strongly realistic philosophy which he thought to be true at least from the mid twenties. But it seems he did not dare devote room to it in print until he had finished his most impressive works in mathematical logic and set theory, probably because—among other things—he thought that his results might be used to support that philosophy. The essay is printed here with seven more pages of textual notes which were found in a series of offprints of the original paper, now in Gödel's *Nachlass.* Some of them are interesting inasmuch as they seem to show some of Gödel's doubts on a number of points. The introductory note has been very ably written by C. Parsons, who explains the origin of the paper, and gives a summary of its main contents in several sections, which is very useful as the original essay did not have any separate sections. The more important of these sections are devoted to: the paradoxes and the vicious circle principle; Gödel's realistic position; the ramified theory of types; and Gödel's views on analyticity. However, along with the summary, Parsons adds comments on some of Gödel's views which is also useful, allowing the reader to state some links to certain philosophical problems. As a whole, this introductory note seems to me a good one, although I cannot help missing more historical information about the circumstances that could explain why Russell did not publish any reply to Gödel's paper in his Schilpp volume, as well as additional clarification of some of Gödel's extremely interesting philosophical views. I shall say something on these two points in the following.

Parsons says that Gödel sent in a first manuscript to Schilpp in May, and the final one in September, after some revision. He adds that Russell meanwhile had completed his replies to the other papers, so he decided not to reply to Gödel's but only to add a brief note at the end of the replies (which is transcribed here too) in which Russell says that the paper arrived when he "had no leisure to work on it". The problem is that one might think that Russell actually did see the May version (as suggested by Hao Wang in his *Reflections on Gödel*), and then decided not to reply, perhaps—the reader may conclude—for reasons other than the one he states in the note.

The study of the correspondence between Russell and Schilpp (now in the Russell Archives, McMaster University) decides the question completely: Russell was unable to see the May version because Gödel explicitly prohibited Schilpp to show it to him. Besides, it shows that Russell was, during a period of some four months, eagerly waiting for some version of Gödel's paper to write the reply, even after he had finished with the rest of the replies. Finally, Russell was by then frantically busy: among other things preparing his departure for England. A further sign of his wanting to publish some reply is that in the 1971 edition of his Schilpp volume he published a little known, but very interesting short "addendum" to his replies which was almost completely devoted to Gödel. True, the addendum is no reply to Gödel's criticisms of 1944, but I think it can be regarded as some sort of compliment to Gödel and, to my knowledge, it is the only important place in print in which Russell somehow gives us his view of Gödel's celebrated results.

The places where I miss more clarification of Gödel's philosophical ideas are the ones related to his well-known, although difficult to understand, realistic view, and to his thesis that mathematical propositions are analytic. As for Gödel's realism, Parsons makes useful comments on Gödel's main argument for realism: that the assumption of the objective existence of sets and concepts is as legitimate as the ordinary assumption of physical bodies to obtain a satisfactory theory of our sense perceptions (p. 106). It seems to me that when Gödel refers here to "physical bodies" he is also thinking of the "theoretical concepts" which physicists admit as long as they are convenient assumptions to obtain more satisfactory physical theories (the "inferences" in Russell's sense, as opposed to the "logical constructions"). But if so, to say that we admit certain objects to obtain a "satisfactory" system, either mathematical or physical, seems to me dangerously close to some sort of holistic position, for the role of an entity in a theory may in part depend on the relation of this theory to other auxiliary hypotheses or theories in a whole system of them. Thus, as holism is one of the main enemies of platonistic realism, a certain tension would appear at this point.

On the other hand, the expression "a satisfactory theory" seems to have some pragmatic connotations, especially when we realize that for Gödel the acceptance of certain axioms may depend on their "fruitful" consequences more than on their intrinsic truth, which is the only one to which our supposed faculty of mathematical intuition may lead us. If these unexpected connections are justified, then some sort of review of Gödel's concept of science, perhaps too dependant an the classical scheme of hypothesis-verification, would be necessary.

Regarding Gödel's view of analyticity, Parsons correctly analyzes the two senses of the word for Gödel (p. 116): mathematics cannot be tautologically analytic because it is undecidable, but it is analytic in the sense that its propositions are true in virtue of the meaning of their concepts. Besides, Parsons makes the important remark that this notion of analyticity is very close to the one which was maintained for the Vienna Circle, although he does not point out that Carnap's objection to Wittgenstein's notion of tautological analyticity in *Logical Syntax is* also similar to the one that Gödel made in his 1944 essay. However, no attempt is made at further analyzing Gòdel's notion of meaning, nor his underlying conviction that only by showing that mathematics is analytic can a realistic position be saved from empiricism. Gödel's unpublished manuscripts, in particular his Gibbs Lecture (1951) and his several versions of "Is mathematics syntax of language?" (1953-59), can be used to throw some light on these problems, in particular on Gödel's analogy between mathematics and physics and on his thesis of analyticity. Parsons mentions the second of these manuscripts in a footnote (p. 117), but only mentions a "very brief examination", which is very unfortunate, for one of the more fruitful and legitimate purposes of the study of the unpublished material by an author is precisely to help in our understanding of her/his published writings.[3]

[3] Parsons mentions some very interesting references where Gödel's philosophical ideas have been discussed, but there is nothing about the literature in which Gödel's philosophical lines are explored. In particular, nothing is said about P. Maddy's writings on mathematical realism, many of them already published before 1990. For a later treatment of that development Maddy's recent book is indispensable: *Realism in mathematics* (Oxford, Clarendon Press, 1990).

IV

The two versions of Gödel's essay on Cantor's continuum problem, published in 1947 and 1964, are reproduced in this volume, with an introductory note written by G. H. Moore, the historian of mathematical logic and set theory. The note is a model of clarity and accuracy from the historical and technical points of view. After explaining the historical background of the continuum problem from Cantor to Gödel's work before 1947, as well as the origin of Gödel's paper of 1947, Moore goes on to a description of its main content. In this section there is only one page devoted to—mostly a couple of quotations of—Gödel's Platonistic viewpoint, according to which Cantor's conjecture must be either true or false in itself, for its undecidability from the usual axioms of set theory is only a sign of the lack of further axioms which would describe more fully an objective, underlying reality, which has to be supposed in the same way as physical objects are assumed for physical theories. Then Moore adds three more sections, dealing respectively with the 1964 version, the later technical research affecting the problem (the more spectacular part of which is, of course, Cohen's results showing the independence of the continuum hypothesis from the standard axioms of set theory), and a very interesting survey of Gödel's several versions and drafts of an unpublished new paper on the problem, together with several highly relevant quotations from the correspondence of this time.

However, the philosophical treatment of the 1964 version is very limited, in spite of the fact that Moore himself says in the introduction to his note that this paper by Gödel contains no new technical results but "gives considerable insight into his philosophical views on set theory" (p. 154). That version contained a supplement mostly devoted to further developments of Gödel's philosophical objectivism for the existence of mathematical objects, but Moore give us in the corresponding place of his introductory note only a quotation and only seven lines of comments. I have already pointed out the extreme importance of Gödel's philosophical position for Gödel himself, as well as some of the interesting problems underlying his analogy between mathematics and physics, so I think that this note would have been a good place to delve deeper into some of these problems, especially to try to establish some links to Gödel's technical arguments.

But even forgetting these problems, the five last paragraphs of Gödel's supplement (pp. 267—269) are philosophically so rich, so laden with epistemological and ontological implications, that there should have been at least a description of each of them, together with some comments connecting them to Gödel's other known arguments in print, as well as to some of Gödel's arguments in the unpublished manuscripts written between the two versions of the Cantor paper (specifically the Gibbs Lecture and the several versions of the unsubmitted essay on Carnap). In the end, if some resort to unpublished technical manuscripts is regarded as necessary to understand the implications of thc published technical arguments, why not apply the same reasoning to the unpublished philosophical writings in connection with the published philosophical arguments? Unfortunately, the problem remains the same as I have repeatedly noted: Gödel's philosophical position cannot (should not) be put aside and be taken into consideration only in a secondary way in relation to his mathematical results and arguments: for him both were inextricably tied, and this in such a way that one can even maintain—as I do—that for him his philosophical beliefs were, by far, the more important, while his mathematical work was, somehow, a search for arguments which might be used in support of those deep beliefs.

VI

I come now to Gödel's writings on relativistic cosmology and its philosophical implications. The reader will find here the two technical papers on Gödel's new solutions of the equations of Einstein's general theory of relativity of 1949 and 1952, in which Gödel introduced "rotating universes", as well as the philosophical one he published in the Schilpp Einstein volume of 1949. This paper, which also contains some footnotes that were added by Gödel to the German edition of 1955, was devoted to a discussion of the main consequences of his technical arguments: that the reality of time and change is rather illusory, as can be illustrated by the possibility of time travel which is allowed by the rotating universes. The introductory note to the technical papers is due to the physicist S.W. Hawking. In one page Hawking gives the reader some background to Gödel's solutions, some discussion of their relationship to Mach's principle, and extremely brief descriptions of their content, but no philosophical discussion is even attempted. However, a final note, added by the editors, mentions some detailed discussion of Gödel's cosmological ideas which can be found in the literature.

H. Stein wrote the introductory note to Gödel's contribution to the Einstein volume which contains some useful clarification of the kind of idealism that Gödel seems to hold in the paper, i.e. that change is not objectively real, although that does not necessarily imply the stronger thesis that time is mere appearance. Also, Stein discusses the possible relationships of Gödel's position to Kant's transcendental idealism, as well as the apparent discrepancy between the paper and an unpublished manuscript which can be found in Gödel's *Nachlass* dealing with similar questions: "Some observations about the relationship between theory of relativity and Kantian philosophy", from which some ideas are summarized.

I have only two criticisms to make. For one thing it seems to me a mistake not to have published here the manuscript just mentioned, together with a comparative study of it with the first one. The argument that these first two volumes are exclusively devoted to Gödel's *publications* might be acceptable, but—fortunately—that policy has not been consistently maintained: the first volume included Gödel's unpublished doctoral dissertation, and in this second volume the reader finds a second version of the *Dialectica* paper on finitary mathematics which had not been previously published. This was obviously done to shed light on the corresponding published material, so it seems to me that the same should have been done in this case. The second critical point concerns Stein's comments. In them there is nothing regarding the possible relationship between the technical results and their "philosophical implications", so the reader probably will come to believe that Gödel suddenly, and neutrally, "discovered" his new solutions, *then* realized that some important philosophical consequences could be drawn from them. However, another possibility might be more credible: it seems to me that we have here another case of Gödel's usual procedure of trying to demonstrate the truth of certain philosophical theses previously maintained: to explicitly look for scientific, hence undeniable, technical support.[4]

[4] Two recent books are very relevant to the themes of Gödel's papers on relativistic cosmology and its philosophical implications: P. Horwich, *Asymmetries in time* (Cambridge, Mass.: The MIT Press, 1987), in which, among many other interesting things, Gödel's ideas are explicitly discussed, and the actual possibility of time travel is defended; P. Yourgrau, *The disappearance of time: Kurt Gödel and the idealistic tradition in*

VII

I congratulate the editorial team for including in this volume the improved 1972 version of the *Dialectica* paper of 1958 on finitary mathematics, in spite of the fact that Gödel never decided to publish this second version. A. Troelstra has been in charge of the introductory note, which runs as follows: a history of both papers (2 pages); a statement of Gödel's main aim (2 pages); a technical description and explanation of Gödel's methods and results (14 pages); a study of further technical related research (5 pages). As we can see from the number of pages devoted to each item, Gödel's aims in writing this paper have been considered only to a very limited extent, and only from a rather technical point of view, despite the explicit recognition by Troelstra that, although Gödel himself said that in the 1972 version he was more concerned about the philosophical implications of these results, had already presented the 1958 paper "as a foundational contribution", that is, as one which is interesting mostly because of a philosophical gain. But philosophical gains are supposed to be justified from the viewpoint of some general philosophical program.

Troelstra clearly describes Gödel's main aim as the attempt to justify classical systems as intuitively as possible, but without accepting Hilbert's restrictions to finitary mathematics, that is, to purely combinatorial mathematics admitting only finite, discrete and visualizable objects. Thus, although Gödel's incompleteness results force certain abstract notions to be admitted in consistency proofs, Gödel thought that Heyting's abstract notion of constructive proof had to be replaced by something more finitistic; hence his new system **T**, based on a certain notion of functional, which can technically replace Heyting's arithmetic, while preserving a more concrete, intuitive character. However, Troelstra says nothing about the philosophical gain, apart from the reference to intuitiveness in general. It seems to me that this was important for Gödel, because from his realistic viewpoint it was essential to be able to show that we do possess an intuition which has some direct access to abstract notions, so the attempt to show this for the particular case of certain abstract notions needed in consistency proofs can be regarded as a partial development of his philosophical program.

From a more detailed point of view, Gödel's papers, especially the second one, contain many rich remarks, sometimes devoted to further elaboration of his notion of intuition which was so important in connection with his realistic program. Among them, the first two paragraphs of both papers, as well as notes **b** and **c** (p. 272) seem to me essential to understand Gödel's notion of intuition, in particular the way in which it has to be distinguished from the Hilbertian and the Kantian notions, which, in the end, depend upon a very restricted concept of finitary, discrete mathematics, which was unacceptable for Gödel's view

philosophy (New York: Cambridge Univ. Press: 1991), which contains a useful study of Gödel's unpublished manuscript mentioned in the text, as well as an attempt to relate Gödel's strange views concerning time and change to the rest of the philosophical theses he seems to have maintained. This second book appeared after 1991, so it could not appear in the bibliography, but the first one appeared in 1987, so it clearly should have been included, and even explicitly mentioned, especially because it contains a rich bibliography listing other papers which are relevant to a study of Gödel's cosmological ideas. I think this is another consequence of the unfair treatment of Gödel's philosophical ideas: while it is clearly understood that "further work" along these lines has to be explicitly taken into account, and even discussed, this is only done for his technical work.

of intuition, one being able to lead us to abstract notions, or, as he also says here, to meanings, structures or thought contents. But no attempt is made by Troelstra to take into consideration such passages, which are not even explicitly pointed out to the reader. Thus, while those mainly interested in Gödel's technical ideas and methods doubtless will be satisfied with Troelstra's note, those interested in Gödel's philosophical ideas, although glad to have the 1972 version now available, will continue to miss a philosophical study of its philosophical content, together with a comparison to the 1958 version.[5]

VIII

There are other remarks and notes which are also important from the philosophical viewpoint. Let us at least point them out. Gödel's remarks for the Princeton bicentennial of 1946 on absolute demonstrability and definability can be usefully regarded as "instances of the application to concrete problems of Gödel's realistic point of view", as Parsons has written in his introductory note to them (p. 149). Also, Gödel's "Some remarks on the undecidability results" (1972) contains rich comments on the philosophical side. The second of those remarks, although devoted to a description of another version of the first undecidability theorem, contains the only known definition of "analytic"—apart from the one included in the essay on Russell of 1944—which is very usefully discussed at length by S. Feferman and R. Solovay in the corresponding introductory note (pp. 289 *ff.*). As for the third remark, although it comprises only 22 printed lines, it is so rich that J.C. Webb needed almost 12 pages to analyze its content, which deals mainly with Turing's notion of mechanical procedures in connection with the nature of the human mind.

IX

As a whole, the book is absolutely indispensable for anyone interested in Gödel's ideas, or generally on the history and philosophy of logic and mathematics. As I have indicated, philosophers more interested in the philosophical implications of Gödel's ideas may be somewhat disappointed by the treatment that these ideas have received in some of the introductory notes, but the book is, at any rate, indispensable for them.

As for the third volume of this fascinating series which is being finished at the time I write this (March 1993), it will contain Gödel's most important unpublished manuscripts, most of them given as lectures or intended for publication, although never actually submitted. Regarding the editorial team, some changes have taken place. Unfortunately, Jean van Heijenoort died in 1986, although the second volume was mostly completed by then. The

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[5] Not even other philosophically relevant literature is mentioned. Here I am thinking of Hao Wang's *Reflections on Gödel* (Cambridge, Mass.: The MIT Press, 1987), which contains a brief but useful study of the philosophical implications of Gödel's paper, as well as of other of his papers with philosophical interest. It is really surprising that Wang's book has not been taken into consideration or even mentioned by any of the authors of the introductory notes. But what is really astounding to me is the fact that the book has not even been included in the bibliography.

editors, however, have written: "His spirit will continue to animate all our work, and we have taken his standards as our own" (p. vi). Due to his work on Russell's *Collected papers,* Gregory H. Moore declined to continue after this second volume of the series, which is regrettable, given his ability. On the philosophical side, Charles Parsons and Warren Goldfarb have been added to the team and are in charge of the edition of some of Gödel's philosophically most important manuscripts. This is good news, so that readers should no doubt look forward to reading the forthcoming volume.[6]

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[6] My thanks are due to Francine Abeles and Roberto Torretti for improving the English of this paper.

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Raymond M. Smullyan, *Recursion Theory for Metamathematics*, New York, New York, Oxford University Press, 1993. xiv + 163 pp.

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This book is a sequel to *Gödel's Incompleteness Theorems* (reviewed in *Modern Logic,* Vol. 4, no. 1). Smullyan's latest could likewise have been called, with some justice, *Representation and Separation Theorems of Ehrenfeucht-Feferman, Putnam-Smullyan, and Shepherdson.* Although that would not be so catchy a title, those theorems lie at the heart of much of the book. So first, a brief description of these theorems.

In a 1960 paper, Ehrenfeucht and Feferman showed that for certain consistent axiomatizable systems, every recursively enumerable set can be represented in the system by some formula. That same year, a paper of Putnam and Smullyan proved that for such systems, a stronger separation conclusion holds: given disjoint r.e. sets A and B, there is a formula which represents A and whose negation represents B. Shepherdson in 1961 used a different kind of argument to obtain representation and separation under different assumptions about the system.

Now the purpose of R*ecursion Theory for Metamathematics* is not merely to present these results. Indeed, they already appeared, proofs and all, in *Gödel's Incompleteness Theorems*. Rather, here the theorems are examined from several angles, given multiple proofs, strengthened and generalized in various ways, and used as springboards to other techniques