ℵ Modern Logic ω

A Survey of Nonstandard Set Theory

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Abstract. This survey paper presents an up-to-date account of Nonstandard Set Theory (**NST**). The introduction presents a brief historical perspective on motivation and the techniques exploited for the realisation of alternative models for the system, first systematically advanced in Robinson's work. It also elaborates on the need for an axiomatic foundation for nonstandard analysis as against an ultrapower enlargement of type-structure or cumulative structure over the systen of real numbers or the system of natural numbers, in particular. In section 1, a systematisation of the axiomatic foundation for nonstandard analysis is presented. The axioms of Extension, Transfer, Saturation and the principle of Internality together with their consequences are discussed. Section 2 presents a systematization of **NST** as a conservative or nonconservative extension of the Zermelo-Fraenkel set theory with the axiom of choice. It elaborates on Nelson's internal set theory in Section 2.1, on Hrbáček's axiomatics in Section 2.2, on Kawai's system, along with Kinoshita's refinements, in section 2.3, and on Fletcher's stratified nonstandard set theory in section 2.4. Finally, the paper indicates the possibility of relating **NST** to Alternative Set Theory (**AST**) in so far as the latter is concerned with constructing ultrafilters using various types of automosphisms and endomorphisms. It also notes that introducing some restricted form of infinitesimal analysis not dependent on the explicit use of the transfer principle, as opposed to pursuing approaches heavily depending on set-theoretic sophistication, might yield good results.

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§0. Introduction. Though the history of infinitesimals and infinity is long and tortuous, nonstandard analysis, as a canonical formulation of the method of infinitesimals, is only about thirty years old. Hence, definitive answers for many of its methodological issues are yet to be found.

The real number system $\mathbb{R}$ can be considered as the cornerstone of virtually all of mathematical enterprises, and of much other scientific knowledge. This is so because it provides a natural model for the kernel of almost all basic operations of the human mind. Owing to its great significance, particularly from the point of view of measurement of time and of linear spaces, a number of studies dealing with the determination of the actual structure of $\mathbb{R}$, under the banner of different philosophies of mathematics, are on hand.

A formalist would argue that there are alternative ways to construct the system of real numbers, i.e., different non-isomorphic models of the same system are possible and that consistency is the only requirement they need to satisfy. On the other hand, a Platonist and, more strongly, a Kantian, would champion the view that there is a unique representation of $\mathbb{R}$, asking which of the different competing models represent the true real number system. Apart from the discovery of different geometries in the early part of the nineteenth century, which shocked the advocates of the Platonic and Kantian view that the axioms of mathematics should be self-evident truths, the discovery of Skolem's construction of unorthodox arithmetic (1934), Gödel's incompleteness theorems (1931), and Cohen's remarkable construction of a model in which Cantor's famous continuum hypothesis was false (1963), paved the way for the legitimisation of the theory of alternative models for the same mathematical system, and of $\mathbb{R}$ in particular. The classical isomorphism theorem for the system of real numbers does not hold good in these extended systems. Today, in the wake of model theory and the theory of formal languages, unlike a hundred years ago when the objects constituting the subject matter of mathematics (e.g. the objects of physics) were (albeit irritatingly) supposed to be factually known whether they existed or not, it is generally accepted that ontological commitments no longer remain important. The same is true of infinitesimals. The question of material existence of infinitesimals, unlike the case of the Weierstrassian picture of the (Archimedean) continuum, does not need to be addressed for the extended (non-Archimedean) continuum. All one is required to demonstrate is that a proof using infinitesimals (Robinson's theory) is at least in no way weaker than one without it) Weierstrass's theory). The success of this thesis, systematically advanced in [Robinson *1966*] for the first time, led to a proliferation in the development of the nonstandard mathematical systems in general, and nonstandard analysis in particular.

In fact, prior to 1960, the concept of infinitesimals remained as one born out of necessity only. Incidentally, all those foundational vindications led to a virtual assault on the logical foundations of the real line continuum itself. The problem of irrational numbers, which was altogether abandoned by the Greeks after Pythagoras, who declared it to be hopeless, got reenvigourated and the inducing force was the newly developing field of symbolic logic, specially the discovery of the theory of formal languege. In 1960, Abraham Robinson, exploiting the power of the theory of formal language reinvented the method of infinitesimals, which he called nonstandard analysis because it used nonstandard models of analysis.

The basic feature of this process lies in the fact that the sentences about the nonstandard objects that correspond to standard objects are true (in the nonstandard interpretation) only if the same sentences are true with reference to the standard objects (in the standard interpretation). That is, we prove results about standard objects by reasoning about nonstandard objects. What one actually needs to prove a theorem involving only standard objects is to "embed the standard objects in the nonstandard enlargement." The theorem then is actually proved with reference to the nonstandard interpretation of its object symbols and relation symbols. The principle involved is known as the *transfer principle*; it says that the same assertions of the formal language are true in the standard universe as are true in the nonstandard universe. The outcome is a more elegant and shorter proof. The embedding of algebraic integers in ideals, the construction of complex numbers and the introduction of a point at infinity in projective geometry are a few such examples.

One may view nonstandard objects as imaginary or ideal elements. The theory of certain mathematical objects (standard ones) are simplified and made considerably more expressive by assuming the existence of additional ideal objects: infiniteslmals formalising the concept of being infinitly close, and infinitly large objects formalizing the concept of being infinitly distant. The basic assumption is that the nonstandard objects are supposed to posses formally the same properties as do the standard objects.

For a naive example, one can consider ficticious elements, such as phoenixes, as nonstandard objects in the set of all birds having eagles and sparrows among its standard objects. The clue here is that the phoenixes possess all the properties of the standard birds. Analogically speaking, the infinitesimals have the same properties as do ordinary number, yet the fact remains that they also have the property of being positive and smaller than any arbitrary positive number. Robinson, utilising the expressive power of a formal language, resolved the aforesaid paradox by constructing a system, containing infinitesimals in addition to all ordinary real numbers, "identical" with the system, containing only ordinary real numbers, "in regard to only all those properties expressible in that formal language." The property of being infinitesimal cannot be so expressed and hence the paradox vanishes. This fundamental. discovery became the core of further researches in nonstandard analysis.

It is well known that one of the common characteristics of the usual mathematical theory of nonstandard analysis involves the use of higher-order structures and their enlargements ([Robinson *1966*], and its subsequent simplifications). The usual nonstandard analysis is done in ultrapower enlargement of a type-structure or cumulative superstructure over $\mathbb{R}$ or $\mathbf{N}$ in particular. An apparent disadvantage of the usual approach is the use of higher-order nonstandard models containing all entities pertaining to the particular problem under investigation, i.e. different enlargements are required for different problems. The second advantage in working with higher-order structures is due to the use of cumbersome apparatus of the type-theoretic language. In general, any such approach presents a difficulty reflected either in deriving any effective form of transfer principle or in inescapably involving the appeal to some transfinite principle. The systems of axiomatic nonstandard analysis have been constructed to remedy these disadvantages. The issue has been treated in two ways: first for analysis proper; and the second, for a more general framework of set theory, called *axiomatic nonstandard set theory*.

1. Axiomatic Nonstandard Analysis. It was G. Kreisel [Kreisel *1966*] who initiated the issue of codifying nonstandard analysis. Kreisel asked:

(i) Is there a simple formal system (with a recursive, preferably finite list of rules and axiom schemata) in which the existing practice of nonstandard analysis can be codified? And if the answer is positive:

(ii) Is this formal system a conservative extension of the current system of analysis (in which the existing practice of standard analysis has been codified)?

In general, and as Kreisel also pointed out, the answer depends on what formal system of standard analysis is used, for example, whether or not the axiom of choice is included. In this regard, it is well known that the usual proof of the existence of nonstandard models uses the so-called extended completeness theorem, which in turn implies the axiom of choice, and hence a nonstandard analysis without the axiom of choice or well-ordering will only be a nonconservative extension of a standard analysis. Here either a conservative or a nonconservative extension of the existing standard anaysis (of either $\mathbb{N}$ or $\mathbb{R}$ or ...) is to be formulated axiomatically. We present here a systematised version of the axiomatic formulations of nonstandard analysis presented in [Keisler *1976*], [Kreisel *1966*], [Riečan *1986-1987*] and others.

Let X be any infinite set and let $V(X)$ be the superstructure over X. Now, a nonstandard model of $V(X)$ consists of a superstructure $V(X^*)$ and an embedding $*: V(X) \rightarrow V(X^*)$ satisfying the following two axioms:

Extension Axiom (EA): *X is a proper subset of X^*, and $S = S^*$ for all $S \in X$.*

(For example, $\mathbb{R}^*$ is proper ordered field extension of the complete ordered field $\mathbb{R}$.)

This axiom was first introduced by [Robinson and Zakon *1969*].

Transfer Axiom (TA): *A sentence ϕ in $L(V(X))$ is true in $V(X)$ if and only if its *-transform ϕ^* is true in $V(S^*)$.*

(For example, for $\phi_1, ..., \phi_n \in V(\mathbb{R})$, any elementary statement which is true of $\phi_1, ..., \phi_n$ is true of their *- transforms $\phi S(*,1), ..., \phi S(*,n)$.)
As we have seen, this axiom (for a suitable language) has been the essential aspect of nonstandard analysis.

As a matter of fact, a nonstandard treatment of parts of classical mathematics can be conducted on the strength of (EA) and (TA) alone. For example, one of the hurdles may be introducing internal entities (sets). For this, one starts with the definition that a set in $V(X^*)$ is *standard* if it is of the form A^* for some $A \in V(X^*)$ and then defines a set to be *internal* if and only if it is an element of some standard set. The definition also follows that set A is *hyperfinite* (or **–finite*) if and only there is an $\omega \in \mathbb{N}^*$ and an internal bijection: $A \rightarrow \{1, 2, ..., \omega\}$.

However, to encompass more sophisticated abstractions, particularly for treating topological and functional analysis, we need to include saturation:

SATURATION AXIOM (SA): *Let K be an infinite cardinal. The nonstandard model V(X*) is K-saturated if whenever* $|\Gamma| < K$ *and* $\{A_y\}\ y \in \Gamma$ *is a set of internal sets with the finite intersection property, then*

$$\bigcap_{y\in\Gamma} A_y \neq \varnothing .$$

Equivalently, if $A_1 \supseteq A_2 \supseteq \ldots$ is a countable decreasing chain of nonempty internal sets, then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing .$$

The latter form of the saturation principle is sometimes called ω_1*–comprehenrsion*. The inclusion of saturation was first emphasised in [Luxemburg *1973*]. The three axioms presented above are the outcome of a long and tortuous evolution.

That a large part of classical mathematics can be treated within this axiomatic system is seen reflected in the following THEOREM:

THEOREM. *There exists a set* $\mathbb{R}^*$ *and mapping* *: $V(\mathbb{R}) \rightarrow V(\mathbb{R}^*)$ *satisfying the Extension, Transfer and Saturation axioms.*

Of course, the proof of this theorem can be demonstrated by means of an ultrapower construction. Here, it follows from (EA) and (TA) that $(\mathbb{R}^*, +^*, \cdot^*, <^*)$ is an ordered field which is a proper extension of the real number field and consequently has non-zero infinitesimals. Similarly we have $\mathbb{N}^*$ from $\mathbb{N}$. This formulation, which helps in clarifying the concept of internal entities, has played a vital role in clarifying the foundation of nonstandard analysis. For example, as a result of (TA), we see that either every nonempty internal subset of A of $\mathbb{N}^*$ contains a least element or that every nonempty internal subset A of $\mathbb{R}^*$ has an upper bound. The internal sets are well behaved. As a matter of fact, the degree of ease in introducing internal sets has turned out to be a testing ground for an approach in nonstandard analysis called *finer*. The basic underlying principle is the following:

INTERNALITY. *Let* $A_1\cdot\cdot, \ldots, A_n$ *be internal and let* $\phi(A_1, \ldots, A_n, b)$ *be an elementary statement. Then the set* $\{b \in A_1 \mid \phi(A_1, \ldots, A_n, b)\}$ *is internal.*

Some basic concepts include the following:

(i) The family of internal subsets of A_1 contains all the finite subsets of A_1 and is closed under finite unions and complements.

(ii) Using (SA), it follows that every internal set is either finite or uncountable.

(iii) Let A be an internal set. The family of all internal subsets of A is an algebra of subsets which contains no countable subsets of A. The σ-algebra generated by the internal subsets of A is called the family of *Borel subsets* of A. This σ–algebra is external.

(iv) An important type of internal set is a hyperfinite set. For each hyperfinite integer $H \in \mathbb{N}^*$, the initial segment $\{K \in \mathbb{N}^* \mid K \leq H\}$ is internal.

An internal set $\rho \in V(\mathbb{R}^*)$ is called hyperfinite if there is an internal bijective mapping $\{K \in \mathbb{N} \mid K \leq H\}$ onto ρ, and the internal cardinality of ρ is denoted by H. The hyperfinite sets are known to be a powerful tool. First, all elementary statements which hold for finite sets also hold for hyperfinite sets; and second, that hyperfinite structures could approximate infinite structures in a strong way.

Here we would like to point out that various other fields of mathematics have yet to be brought under the umbrella of nonstandard analysis; for that, additional axioms, i.e., different properties of the nonstandard model, e.g., models satisfying an isomorphism poperty, etc., would have to be developed.

§2. Nonstandard Set Theory (NST). Robinson [Robinson *1966,* 47] not only suggested that one might use axiomatic set theory rather than type theory for the development of higher order nonstandard analysis, but also formulated, jointly with E. Zakon [Robinson & Zakon *1969*], a set-theoretical characterisation of enlargements. A variety of constructions of nonstandard analysis within a set-theoretic environment have appeared ([Grieser *1970*], [Luxemburg *1973*], and [Philips *1973*], to name a few). Nonstandard axiomatic set theory is an attempt to generalise nonstandard analysis to encompass the whole of classical mathematics. In general, an extension of Zermelo-Frankel set theory with the Axiom of Choice (**ZFC**) is known as **NST**. In the following pages, we propose to present a systematic study of this approach.

2.1. Nelson's Internal Set Theory (IST). Nelson [Nelson *1977*] has stated that the axioms of **IST** are the basic properties of internal sets in the usual approach to nonstandard analysis (provided sufficient saturation has been assumed). Accordingly, in **IST**, internal sets are simply treated as sets. In addition to the usual primitive binary predicate ∈ of **ZFC**, a new unary primitive predicate *standard* is added. Thus, Nelson's universe consists of standard sets and nonstandard but internal sets. The axioms of **IST** are those of **ZFC** together with the three listed above. As a consequence, certain sets will be called *standard,* although, as we shall see later, all the theorems of conventional mathematics (**ZFC**) apply to all sets, nonstandard as well as standard.

A formula of **IST** is called *internal* in case it does not involve the new predicate **standard** (implying that all formulæ of **ZFC** are internal), *external* otherwise. Hence, *x standard,* a new undefined notion, is an external formula and has no meaning in **ZFC**, for it is not defined in terms of conventional predicate $\in$. The axioms of **ZFC** say nothing about external predicates. For example, no axiom of **ZFC** permits the existence of a subset S of $\mathbb{N}$ such that

$$\forall n \in S \leftrightarrow n \in \mathbb{N} \text{ and } n \text{ is standard.}$$

A good reason for the success of nonstandard analysis is that a complicated internal notion in classical analysis (for example, the epsilon-delta definition of continuity) is frequently equivalent to a simple external notion for standard sets.

NOTATION:	$\forall^{\text{st}}x$ for $\forall x$ (x standard) $\rightarrow$	$\exists^{\text{st}}x$ for $\exists x$ (x standard) $\wedge$
	$\forall^{\text{fin}}x$ for $\forall x$ (x finite) $\rightarrow$	$\exists^{\text{fin}}x$ for $\exists x$ (x finite) $\wedge$
	$\forall^{\text{st fin}}x$ for $\forall^{\text{st}}x$ (x finite) $\rightarrow$	$\exists^{\text{st fin}}x$ for $\exists^{\text{st}}x$ (x finite) $\wedge$
	$\forall x \in y$ for $\forall x\,(x \in y) \rightarrow$	$\exists x \in y$ for $\exists x\,(x \in y) \wedge$

The three additional axioms are the following:

TRANSFER AXIOM SCHEMA (T): $\forall^{\text{st}}t_1, \ldots, \forall^{\text{st}}t_k\ (\forall^{\text{st}}x\ A(x, t_1, \ldots, t_k) \rightarrow \forall x\, A(x, t_1, \ldots, t_k))$,

where $A(x, t_1, \ldots, t_k)$ be an internal formula with $x, t_1, \ldots, t_k$ as the only free variables.

The principle (T) plays the central role in working with nonstandard methods. By successive applications of (T), we obtain $A^{\text{st}} = A$, where A is any internal statement (closed formula) and A^{st}, obtained by replacing each occurrence of $\exists x$ and $\forall x$ by $\exists^{\text{st}}x$ and $\forall^{\text{st}}x$ respectively, is called the *relativization* of A to the standazrd sets. This shows that all theorems of conventional mathematics also hold when relativized to standard sets. In practice, to prove an internal theorem, it is sufficient to prove its relativization to the standard sets, which is more convenient than the general case.

In order for the application of (T) to be legitimate, the following two points are to be observed:

(i) The statement to which (T) is applied must be internal.

(ii) All the parameters (including constants like $\varnothing$ for empty set or $\mathbb{R}$ for the set of real numbers, etc.) appearing in the statement must be standard.

Every specific (uniquely described) object, like $\mathbb{N}$, $\mathbb{R}$, the real number π, $\mathbb{R}^{\mathbb{R}}$, the Hilbert space $L^2(\mathbb{R})$, etc., of conventional mathematics is a standard set. It remains unchanged in the new theory **IST**. Hence **IST** is an addition and not a change.

IDEALIZATION (SATURATION) AXIOM SCHEMA (I): $\forall^{st\, fin} z\ \exists x\ \forall y \in z\ B(x, y) \leftrightarrow \exists x\ \forall^{st} y\ B(x, y)$, *where $B(x, y)$ be an internal formula with x and y as free variables together with possibly other free variables.*

The principle (I) asserts that the relationship described by the formula B is simultaneously satisfiable for all standard y if and only if it is simultaneousiy satisfiable for every standard finite set.

Now one can prove the following results:

(1) Every element of a set X is standard if and only if X is a standard finite set.

(2) Every infinite set has a nonstandard element. It implies that there exist nonstandard natural numbers. (T) justifies the claims that 0 is a standard natural number and that finite induction holds (i.e. if n is a standard natural number, then $n + 1$ is standard, etc.), but there does not exist a subset S of $\mathbb{N}$ such that a natural number is in S if and only if it is standard.

(3) There exists a finite set F containing all standard sets x.

Clearly F cannot be a standard set, as otherwise F will be a set leading to Russell's paradox.

(4) If S is any subset of $\mathbb{N}$ which contains all unlimited natural numbers, then the complement of S is a standard finite set.

STANDARDIZATION AXIOM SCHEMA (S): $\forall^{st} x\ \ \exists^{st} y\ \forall^{st} z\ (z \in y \leftrightarrow z \in x\ \wedge C(z))$,

where $C(z)$ is a formula, internal or external, with free variable z and possibly other free variables The uniqueness of the set y given by (S) is guaranteed by (T), which asserts that two standard sets are equal if they have the same standard elements.

The Principle (S) provides a substitute for situations where we must use external predicates to which we are not formally entitled. For example, we have

$$y = {}^{S}\{z \in x \mid C(z)\},$$

which is correctly read as "the standard subset x whose standard elements are those which satisfy the predicate C," and not read as "the set of all standard elements in x which satisfy

C." The principle (S) does not provide a direct criterion for deciding whether or not a nonstandard element of x is in y.

For example, ${}^{S}\{z \in \mathbb{N} \mid z \in n\} = \mathbb{N}$, where n is a nonstandard natural number, holds by (T) because the standard elements in the sets on both the sides of = are the same, though there are other elements of $\mathbb{N}$ which are not $< n$. Similarly, ${}^{S}\{z \in \mathbb{N} \mid z \geq n\} = \varnothing$ though there are natural numbers z with $z \geq n$. It is also significant to note that $C(z)$ with $z < n$ is an internal formula and hence we can legitimately form the nonstandard set ${}^{S}\{z \in \mathbb{N} \mid z < n\}$, which is a proper subset of $\mathbb{N}$. It may appear objectionable at first sight (considering "nonstandard" as an external notion and n being nonstandard). In fact, since $z < n$ is internal, for every natural number n (standard or nonstandard), we can form $\{z \in \mathbb{N} \mid z < n\}$.

The principle (S) legitimizes the existence of functions, and hence greatly facilitates the introduction of propositions.

For any standard sets X and Y, if for all standard x in X there is a standard y in Y such that $A(x, y)$, where $A(x, y)$ is any formula (internal or external) with free variables x and y together with possibly other free variables, then there is a standard function $\tilde{y} : X \to Y$ together, by definition, with $A(x, \tilde{y}(x))$ for all standard x in X.

This completes the description of **IST**. In [Nelson *1977*, Powell's appendix] **IST** has been formulated as a conservative extension of **ZFC**, i.e., every internal statement which can be proved in **IST** can also be proved in **ZFC**, and uses unlimited idealization. That is, it is shown that the whole of the usual nonstandard analysis can be done with greater ease within **IST**. In particular, a real number x is called *infinitesimal* only in case $|x| \leq \varepsilon$ for all standard $\varepsilon > 0$, limited in case $|x| \leq r$ for some standard r, and unlimited in case it is not limited. By (T), the only standard infinitesimal is 0. For any $x \neq 0$, there is an integer n such that $nx \geq 1$, where n is unlimited if x is an infinitesimal. Two real numbers x and y are infinitely close in case $x - y$ is infinitesimal, denoted by $x \approx y$. It follows that every limited real number x is infinitely close to a unique standard number, denoted by st$x$. If x is not an infinitesimal, then x^{-1} is limited. By principle (S), one can define any function f at a point x (in particular a standard function f and a standard point x) to be continuous if $\langle f, x \rangle$ belongs to ${}^{S}\{\langle f, x \rangle \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \mid \forall y\, (y \approx x \rightarrow f(y) \approx f(x)\}$.

Now it is clear that a variety of nonstandard constructions can be performed in **IST**. However, the usual presentations of nonstandard analysis make use of external predicates to define subsets, which **IST** lacks because it deals with only standard and internal sets. External sets, for example, monads, galaxies, the set ${}^{O}A$, etc., which are common in nonstandard arguments, do not get explicit treatment in **IST**. As Nelson himself has suggested, external sets can be introduced for the price of working with a model of set theory, whereas **IST** claims to be concerned with aspects of nonstandard analysis which do not involve model theory. This is unsatisfactory. Further, Nelson claims that **IST**, by virtue of using unrestricted idealization, achieves the reduction algorithm in a more natural way than that found in [Robinson *1966*], which uses the restricted idealization principle. However, we see in [Fletcher *1989*] that the unrestricted use of idealization conflicts with the wish to have a full theory of external sets.

2.2. Hrbáček's Axiomatics for Nonstandard Set Theory. Independently of Nelson's approach, Hrbáček [*1978*, *1979*] presented an axiomatic system (in fact, three such systems, differing in describing the properties of external sets—some of them are conservative extensions of **ZFC**, and others are essentially stronger) for nonstandard set theory. We shall present here a comprehensive (sufficient for the formalization of nonstandard mathematics) version of Hrbáček's axiomatics and **NZFC** (nonstandard Zermelo-Fraenkel set theory). Alternatively, if **ZFC** is represented by Σ, then the nonstandard extension of Σ may be denoted by **NS**(Σ).

NZFC deals with three sorts of objects: standard sets, internal sets, and external (noninternal) sets. Accordingly,variables of **NZFC** range over external sets, the most encompassing type. The usual sets (of **ZFC**) are called *standard*. However, standard sets may have nonstandard elements and the extended universe may be further enriched by admitting noninternal objects. For formalization, in addition to the elementhood predicate $\in$, two other primitive concepts, namely, the relations (unary) of being standard S(.) and internal I(.), are introduced. Unless explicitly mentioned, Roman letters are used to denote standard objects, Greek letters denote internal objects, and boldface Roman letters denote arbitrary (external) objects. However, because of typographical problems, the distinction between standard and external object symbols will be understood contextually. That is, S(A) expresses the fact that A is a standard object and $\mathcal{I}(A)$ says that A is an internal object. Analogousiy, a statement is called *standard* (*internal*) if all quantified variables in it range over standard (internal) objects, and the concepts (relations, operations, constants) are standard (internal) if defined by standard (internal) statements. Thus, the set of wffs of **ZFC** is extended to include two additional wffs, namely S(x) and $\mathcal{I}(x)$ where x is a variable.

Intuitively, internal objects may be standard or nonstandard (that which is not standard) while all standard objects are internal. Besides, one may like to extend the universe of discourse to include collections of internal sets which are not themselves internal (for example, the collection of all nonstandard natural numbers is a noninternal set), called noninternal sets. Thus, external objects are either internal or noninternal. For notation, we use ${}^{0}A$ for $\{x \in A \mid \delta(x)\}$. Now, ${}^{0}A$ is infinite if A is infinite, ${}^{0}A$ is an internal finite set, and nonstandard. The set $A - A$ is external. [Robinson *1966*] uses "external" in the sense of "noninternal" used here. This makes the usual standard concepts correspond to external concepts, for example, ${}^{\mathcal{S}}\mathbb{N} = {}^{0}\mathbb{N}$, etc. In fact, the external objects may be visualized as subsets of the universe of internal objects, sets of such subsets, sets of sets of such subsets, etc. It is to be noted that noninternal objects cannot belong to internal objects, of course. Some external sets are already internal. The universe of external sets satisfies **ZFC**, except for the full strength of Fraenkel's Axiom Schema of Replacement where care has to be taken that not every external well ordering is isomorphic to an external ordinal. As a matter of fact, external infinite ordinals and cardinals are not employed in the development of nonstandard analysis In order to work in the most encompassing universe of external objects (sets), note that unlike the internal *vs*. standard distinction, not every internal concept automatically extends to its external counterpart. For example, although every nonempty internal subset of $\mathbb{N}$ has a (standard) least element, ${}^{\mathcal{G}}\mathbb{N} = {}^{0}\mathbb{N}$, the collection of all nonstandard natural numbers, does not have a least element (if ${}^{\mathcal{G}}\mathbb{N} = {}^{0}\mathbb{N}$ then $(n-1) \in {}^{\mathcal{G}}\mathbb{N} = {}^{0}\mathbb{N}$), which in turn, proves that ${}^{0}\mathbb{N}$ (nonstandard) is noninternal. However, all elementary set-theoretic relations and operations do agree. Sometimes a superscript $\mathfrak{S}$ will be used for denoting external concepts.

Axiomatization of NZFC.

Notation: If $\phi(v_1,...,v_n)$ is a formula of Σ (= **ZFC**), then Φ is a formula obtained by appropriately replacing all variables of ϕ by variables of **NS**(Σ), ϕ^s ($\phi^{\mathcal{G}}$) is obtained from Φ by replacing all its bound variables by variables ranging over standard (internal) sets.

Axioms:

(A) ϕ^s *is an axiom of* **NZFC** *whenever the sentence* ϕ *is an axiom of* Σ.

(B1) $(\forall x)\ \mathcal{G}(x)$. Equivalently, $\forall^{st}x \rightarrow \mathrm{I}(x)$.

Informally, all standard sets are internal.

(B2) $(\forall x)(\forall \xi)\ (x \in \xi \rightarrow \mathcal{G}(x))$. Equivalently, $(\forall x)(\forall^{I}y) \rightarrow \mathcal{G}(x))$. The universe of internal sets is transitive.

(B3) Axiom of Embedding (Transfer): $(\forall x_1, ..., x_n)(\phi^s(x_1, ..., x_n) \leftrightarrow \phi^{\mathcal{G}}(x_1, ..., x_n)$.
The standard objects have a standard property if and only if they have the corresponding internalized property. In other words, the universe of internal sets is an elementary extension of the universe of standard sets.

For example, the standard relation of set inclusion (for standard sets in their standard elements) is equivalent to its extended counterpart (internalization), namely

for all internal sets A and B, $A \subseteq^{\mathcal{G}} B$ if and only if for all internal x, $x \in A \rightarrow x \in B$,

(B4) Axiom Schema for Saturation (Idealization of Enlargement):

$$(\forall x_1, ..., x_n)(\forall A)\ [(\forall a)\ (a \subseteq A \wedge a \textit{ finite} \rightarrow (\exists b)(\forall x \in a)\phi^s(x, b, A, x_1, ..., x_n)) \rightarrow (\exists \beta)(\forall x \in A)\ \phi^{\mathcal{G}}(x, \beta, A, x_1, ..., x_n)].$$

In other words, for any standard property ϕ^s, if A is a standard set, and if for standard finite $a \subseteq A$ there is a standard y such that $\phi(x, y)$ holds simultaneously for all standard $x \in a$, then there is an internal y such that $\phi^{\mathcal{G}}(x, y)$ holds simultaneously for all standard $x \in a$.

Informally, this postulate is intended to have as many ideal elements available as possible, provided that mutual consistency is available. The saturation principle of [Nelson *1977*] is (B4). One can prove the following propositions as direct consequences of the saturation principle.

PROPOSITION 1: *Every standard infinite set has nonstandard elements.*

PROPOSITION 2: *For every standard set A there is an internal finite set* $\alpha \in A$ *containing all standard elements of A.*

(B4) is weaker than Nelson's idealization axiom. However, there is a stronger version of (B4) as follows.

NOTATION: An external set A has standard size, Ss(A), if there is a standard set A and a function f such that $x \in A$ if and only if $x = f(x)$ for some $x \in A$.

$$\text{(B4+)}\quad (\forall \eta_1, \ldots, \eta_n)(\forall A \mathrm{Ss}(A))\,[(\forall a)\,(a \subseteq A \wedge a \text{ finite} \rightarrow (\exists \beta)(\forall \xi \in a)\phi^{g}(\xi, \beta, A, \eta_1, \ldots, \eta_n)) \rightarrow (\exists \beta)(\forall \xi \in A)\,\phi^{g}(\xi, \beta, A, \eta_1, \ldots, \eta_n)].$$

In other words, for any standard property ϕ, if A is a set of standard size and all elements of which are internal, and if for every external finite $a \subseteq A$ there is an internal y such that $\phi^g(x, y)$ holds simultaneously for all $x \in a$, then there is an internal y such that $\phi^g(x, y)$ holds simultaneously for all $x \in A$.

As compared to saturation principle of [Nelson *1977*], B4 is stronger because here ϕ may have internal parameters as well.

(CO) AXIOM OF TRANSFER (STANDARDIZATION): $(\forall A)(\exists A^*)(\forall x)(x \in A^* \leftrightarrow x \in A)$.

Informally, for every external set A there is a standard set $A^* = \{x \in A \mid S(x)\}$, the standard kernel of A, having the same standard elements as A. This axiom is often referred to as the *principle of standardization* in literature of nonstandard analysis. This is a strengthened form of Nelson's standardization axiom. It permits unlimited use of external sets in constructing standard sets.

Since A is internal, the noninternal elements of A belong to A^*; however, as the axiom of standardization does not say anything about the nonstandard elements of A, it may happen that A may have nonstandard elements which may or may not belong to A. Clearly $A^* = A$ if A is standard.

The Axiom of Transfer extends the standard induction principle to the following:

If A is a such that

(i) $0 \in A$, and
(ii) for all standard natural numbers n, if $n \in A$, then $(n + 1) \in A$,

then A contains all standard natural numbers (i.e., ${}^0\mathbb{N} \subseteq A$).

But, one cannot conclude that $\mathbb{N} \subseteq A$) unless A is standard.

The following propositions are among the characteristic consequences of the Axiom of Transfer.

An external set is standard finite if and only if it is external finite and all its elements are standard. The standard natural numbers coincide with the external natural numbers. Standard algebraic operations on standard natural numbers conicide with the corresponding external operations. In particular, the Axiom of Transfer implies that all elements of a standard finite set are standard.

Let us assume the system **NS** consists of all the previous axioms plus (C1) the Axiom of Extensionality, (C2) the Axiom of Pairing, (C3) the Axiom of Union, and (C4) the Axiom Schema of Comprehension (involving formulæ which may contain 'st' and 'I') for the external inverse (i.e. without any relativization of quantifiers).

The axioms of **ZFC** that have been left out are the External Power Set axiom (EP), External Replacement (ER), External Choice (EC), and External Well-foundedness (EWF). In order to capture the notion of external sets in full generality, we would not expect EWF to hold since, for a finite nonstandard ordinal α, $\{\alpha, \alpha-1, ...\}$ is an infinite descending chain. Thus, in an ideal sense, one would define

$$\mathbf{NST}\ (\mathbf{NZFC}) = \mathbf{NS} + \mathrm{EP} + \mathrm{ER} + \mathrm{EC}.$$

Hrbáček has shown:

(a) **NS** + ER + EP is inconsistent.
(b) **NS** + ER + EC is inconsistent.

The reason for these inconsistencies is the unlimited use of the standardization axiom which standardizes very large sets. The standard set of (standard and nonstandard) natural numbers must be larger than ${}^0A$ for standard A because, due to the enlargement axiom, for standard A there is an internal B such that $A \supseteq B \supseteq {}^0A$ and B is formally finite, i.e., there is an internal injection from B to the standard set of natural numbers. In **IST**, this problem does not arise since the comparison holds with only internal sets. In the full theory of external sets, arbitrarily large standard ordinals can be obtained, which in turn gives rise, via strong standardization, to the set of all ordinals, a contradiction.

We must emphasize that although any formula ϕ of **ZFC** is also a formula of **NZFC**, the semantics are different simply because the ranges of quantified variables are different (standard and external entities respectively). Thus, any concept (relation, constant, operation) defined in **ZFC** by a formula ϕ may have three distinct formulations in **NZFC**: the standard analogue, denoted by ϕ^S; the internal one, denoted by $\phi^{\mathcal{I}}$; and external one denoted by $\phi^{\mathcal{E}}$. Quite importantly, for developing a variety of **NZFC**, one must make use of the semantic fact that, for a formula ϕ^S of **NZFC** to have the same meaning as a given formula ϕ of **ZFC**, all the quantifiers in ϕ have to be relativized to S, i.e., we must replace $(\forall x)$ ... by $(\forall x)(S(x) \rightarrow ...)$ and $(\exists x)$... by $(\exists x)(S(x) \wedge ...)$. In neither [Hrbáček *1976*] nor [Hrbáček *1979*] can strong set-theoretic arguments be consistently conducted. For example, for defining the external cumulative hierarchy, one needs EP to generate each new level and EC and ER to justify transfinite recursion. It is our opinion that, were it not for the one fact that Hrbáček's system distinguishes objects of the universe of discourse in a more encompassing way than does Nelson's system, it would not be more powerful than Nelson's system.

2. 3. Kawai's Axiomatics for Nonstandard Set Theory. A variant of **NZFC**, similar to Hrbáček's, has been proposed in [Kawai *1979*; *1980*; *1983*]. Kawai's systems like Hrbáček's, deal with external, standard and internal objects, and hence can be understood as an extension of Nelson's system. In fact, [Kawai *1983*] is the final version which includes the central idea of [Kawai *1979*; *1980*]. Kawai's **WNST** (Weak-**NST**) corresponds to a comprehensive enlargement or a countably saturated enlargement in the model-theoretic approach. A stronger version, symbolized by **NST**, corresponds to a saturated model and has all the axioms of **ZFC** in the universe of external sets except the Axiom of Regularity or Well-foundedness (ZF 3), which is replaced by an axiom in a restricted form:

$$(\forall x)[A \neq 0 \ \wedge \ A \cap I = 0 \ \rightarrow \ \exists x \in A \, (x \cap A \ = \ 0)],$$

where I is constant symbol which stands for an specified external set An external object belonging to I is called internal. Kawai's system is more comprehensive than Hrbáček's in the respect that the former contains both the axioms of power set and replacement compatibly whereas Hrbáček's does not. Kawai's system escapes the inconsistency arguments found in Hrbáček's sustem by weakening the standardization so that very large sets, such as the one in Hrbáček's theory, cannot be standardised. Kawai's system uses stronger idealization. It formalizes saturation with respect to not only the standard universe S, but to any S-sized external set. However, strong set-theoretic principles do not, as in Hrbáček's system, work. For example, Kawai's system does not always allow us to gather only standard elements of a given external set into a standard set containing only those standard elements. Further, in Kawai's system, because the standard and internal universes are themselves external sets, the intuition guiding "the limitation of size" theory of sets gets disrupted; the criterion which would determine a collection as being a set becomes wild. In the end, Kawai describes an axiom system from the usual viewpoints and calls it **UNST**. Every set which **UNST** deals with is called an *external set*. The specific external sets are denoted by constants U, I and *. An external set belonging to $U(I)$ is called a *usual* (*internal*) *set*. Intuitively, U can be identified with the universe of **ZFC**. If ϕ is a formula of **ZFC**, then ϕ^U(resp. ϕ^I, ϕ^E) denotes the formula obtained by replacing all variables of ϕ by variables ranging over usual (resp. internal, external) sets.

The axioms of **UNST** are the following, U1 through U9:

(U1) *If* ϕ *is an axiom of* **ZFC**, *then* ϕ^U *is an axiom of* **UNST**.

(U2) *If* ϕ *is any axiom of* **ZFC** *except the Axiom of Regularity, then* ϕ^E *is an axiom of* **UNST**.

(U3) (Axiom of Regularity in Restricted Form): $(\forall A)\,[A \neq 0 \ \wedge\ A \cap I = 0 \ \rightarrow$ $\exists x \in A\,(x \cap A\ =\ 0\,)]$.

(U4) $* : U \rightarrow I$. (* is a mapping of U into I.)

(U5) *Taking the symbol* a^* *to be used for* $(a)^*$, $a \in U$,

$$\forall A\ \forall B\,(A \in B\ \wedge\ B \in I\ \rightarrow A \in I\,)$$

(U6) (Transfer Principle): $\forall x_1 ,\dots , x_n \in U\,[\phi^U(x_1 ,\dots , x_n) \equiv \phi^U(x_1^*, \dots , x_n^*)]$

(U7) $\forall A \forall B\,(A \in B \wedge B \in U\ \rightarrow A \in U\,)$.

(U8) $\forall A \forall B\,(A\ \subset B \wedge B \in U\ \rightarrow A \in U\,)$.

Definition: D is *U–size* $\equiv \exists F\,(F\!: U \rightarrow\ D_{(\text{onto})})$.

(U9) (Axiom Schema of Saturation): *Let* $\phi(a, b, x_1 , \dots, x_n)$ *be a formula of* **ZFC**, *all of whose free variables are among* $a, b, x_1 , \dots, x_n$. *Then*

$$\forall D\,(D \text{ is } U\text{–size})\ \forall x_1, \dots, x_n \in I\,[\forall d \in \mathrm{I}\,(d \textit{ is external finite} \wedge\ d\ \subset\ D\ \rightarrow \exists b \in I\ \forall\, a \in d\ \ \phi^I\ (a, b, x_1, \dots, x_n)] \rightarrow \exists B \in I\ \forall\, A \in D \cap I\ \ \phi^I\ (A, B, x_1, \dots, x_n)].$$

Here $A, B, C,\dots, a, b, c,\dots$ denote variables ranging over external sets.

An immediate consequence is the existence of a unique external set S such that

$$\forall y\,[\,y \in \mathrm{S} \equiv \exists x \in U\,(\,y = x^*)]\,.$$

An external set belonging to S is called *standard*. One of the pleasing features of **UNST** is that it is equivalent to **NST** and in particular, if ψ^U is a theorem of **UNST** then ψ is a theorem of **ZFC**, a conservation result.

Recently Kinoshita [Kinoshita *1987*] has proposed a refinement in Kawai's **UNST**, which he calls ***NST**. He understands that the set U of all usual sets need not be a model of **ZFC**, but, similarly to Hrbăček's system, may be a model of **ZFC** without the Axiom of Replacement (which is sometimes known as **ZC** or **ZFC⁻**), and within it all of modern mathematics can be developed. The basic symbols of the language of ***NST** are $\in$ (the usual membership), I ("$x \in I$" means that x is an internal set), and *. It takes "sets" for "external sets" and defines *external set* as noninternal set. We find that it resembles [Robinson *1966*] in specifying objects.

The axioms of ***NST** are the following:

Axiom 1. *The axioms of* **ZFC⁻** *are the formulæ of* ***NST**.
Axiom 2. (Nonstandard analysis):

(a) *The set I is transitive* : $x \in I \rightarrow x \subseteq I$.

(b) *There exists a limit ordinal* $\xi > \omega$ *such that* $*: \mathrm{R}(\xi) \rightarrow I$.

Let $U = \mathrm{R}(\xi)$, which is a transitive model of **ZC**.

(c) (Transfer Principle): $\phi^U(x_1, ..., x_n) \leftrightarrow \phi^I(x_1^*, ..., x_n^*)$

(d) (Saturation Principle): *Let* $\phi(x, X, w_1,..., w_n)$ *be a formula in* **ZFC**, $w_1, ..., w_n$ *be in* I, *and* S *be a set of* U*–size, contained in* I. *Then*

$$\forall F : \text{finite} \subseteq S \ \exists x \in I \ \forall X \in F \ \phi^I(x, X, w_1,..., w_n) \rightarrow \exists x \in I \ \forall X \in S \ \phi^I(x, X, w_1,..., w_n).$$

Axiom 3. *Foundation over* I *is equivalent to the restricted form of the Axiom of Regularity:* $X \neq 0 \wedge X \cap I = X$ *has an* $\in$*–minimal element.*

2.4. Fletcher's Stratified Nonstandard Set Theory (SNST). The idea of [Fletcher *1989*] is not to use unlimited idealization but only sufficient (up to a given cardinality) idealization, and in this sense it could be regarded as a refinement of Hrbácek-Kawai **NST**, or as a formalization of the Robinson-Zakon method of the hierarchy M of models to suit all circumstances.

Definition: An elementary extension I of S (a model for the standard universe) *satisfies* κ-*idealization with reapect to* S if and only if, for every relation $R(x, y)$ of I such that $|R \cap S| \leq \kappa$ and for all finite sets $\{a_1, ..., a_n\} \subseteq \mathrm{dom}(R \cap S)$, $(\exists y \in I)R(a_1, y) \wedge \ldots \wedge R(a_n, y) \in \mathrm{dom}(R \cap S)(R(a, y_0))$.

Define I_κ for each cardinal κ by $I_0 \equiv S$, $I_\kappa \equiv$ elementary extension of the direct limit of $\{I_\lambda \mid \lambda < \kappa\}$ satisfying κ–idealization with respect to I_0. If we take $S = V$ (the universe of sets), each I_κ is a class with an $\in$–relation. Thus we would interpret "standard" to mean "in S" and "internal" to mean in "I_κ". To accommodate external sets we could build up a set-theoretic superstructure E_κ on each I_κ as follows. E_κ is the direct limit of $\{E_\kappa{}^\gamma \mid$ ordinal $\gamma\}$, where $E_\kappa{}^0$ is I_κ. Thus we would interpret "external" to mean "in E_κ". We can choose to work in the structure $M = \{s, I_\kappa's\ E_\kappa's\}$, and ignore the full universe I and E of internal and external sets respectively, as a way to observe the theoretical limitation on size of sets. For relativization, define quantifiers $\forall^\kappa$, $\exists^\kappa$ relativized to I_κ, and $\forall^{\text{ext},\kappa}$ and $\exists^{\text{ext},\kappa}$ relativised to E_κ. Thus, $\forall^{\text{st}}$ and $\exists^{\text{st}}$ are replaced by $\forall^0$ and $\exists^0$ (or without superscript 0), $\forall^I$ and $\exists^I$ are replaced by $\forall^\kappa$ and $\exists^\kappa$, and $\forall^E$ and $\exists^E$ are replaced by $\forall^{\text{ext},\kappa}$ and $\exists^{\text{ext},\kappa}$ respectively. Consequently, the size of a monad depends on cardinal κ.

The axioms of the formal system **SNST** are:

(1) *All of the axioms of* **ZFC** (*implicitly relativized to* S).

(2) ($I_\alpha \subseteq I_\beta$ for $\alpha \le \beta$) $\forall\alpha\forall\beta\ (\alpha \le \beta \to \forall^\alpha x\ \exists^\beta y\ \ x = y)$.

(3) ($E_\alpha \subseteq E_\beta$ for $\alpha \le \beta$) $\forall\alpha\forall\beta\ (\alpha \le \beta \to \forall^{\text{ext},\alpha} x\ \exists^{\text{ext},\beta} y\ \ x = y)$.

(4) ($I_\alpha \subseteq E_\alpha$) $\forall\alpha\forall^\alpha x\ \exists^{\text{ext},\alpha} y\ \ x = y$.

(5) $\forall\alpha\forall^{\text{ext},\alpha} x\ (\exists^\alpha w\ \ x = w \lor \forall\beta\forall^{\text{ext},\beta} y \in x\ \exists^{\text{ext},\alpha} z\ \ y = z)$. (The strictly external part of E_α is transitive).

Hence, any external set belongs either to some $E_\alpha{}^0$ ($= I_\alpha$) or some $E_\alpha{}^y$ with $y > 0$ (in which case it is built up from elements of E_α). Here I_α is not transitive.

(6) $\forall\alpha\forall\beta\forall^\alpha x\forall^{\text{ext},\beta} y \in x\ \exists^{\text{ext},\alpha} y \in x\ \exists^\beta z\ \ y = z$. (The internal universe is transitive.)

(7) (TRANSFER): $\forall\alpha\forall\beta\ (\alpha \le \beta \to \forall^\alpha x_1, ..., \forall^\alpha x_n\ \phi^\alpha(x_1, ..., x_n) \leftrightarrow \phi^\beta(x_1, ..., x_n))$ *for any standard formula* ϕ *whose free variables are* $x_1, ..., x_n$.

(8) (IDEALIZTION): $\forall\alpha\forall^\alpha R$ (*R is a binary relation and* $\exists f f({}^0\alpha) = {}^0R$) $\to [(\forall A$ *finite* $(A) \land A \subseteq \text{dom}(R) \to \exists^\alpha b \forall a \in A\ R(a, b)) \to \exists^\alpha b \forall a \in \text{dom}(R)\ \ R(a, b))]$.

It says that I_α satisfies α–idealization with respect to S.

(9) (STANDARDIZATION): $\forall\alpha\forall^{\text{ext},\alpha} X\ \ \exists Y\, \forall z\ (z \in X \leftrightarrow\ z \in Y)$.

(10) $\forall\alpha\forall^{ext,\alpha}$, *for each axiom* ϕ *of* **ZFC** *in the language of* **SNST**.

That is, instances of the replacement and comprehension schemata using nonstandard formulæ are permltted; however, the axiom of regularity is replaced by its restricted form, as in Kawai's systems.

The justification for **SNST** is the following theorem:

THEOREM: **SNST** is conservative over **ZFC**.

That is, for every standard formula ϕ, **SNST** $\Vdash \phi$ if and only if **ZFC** $\Vdash \phi$.

PROOF. **ZFC** $\Vdash \phi \rightarrow$ **SNST** $\Vdash \phi$ follows from (1).

Proof of the converse needs a bit of manipulation in relativizing the argument to the structure M.

SNST $\Vdash \phi$ implies $\psi_1, \ldots, \psi_n \Vdash \phi$ for some **SNST** axioms $\psi_1, \ldots, \psi_n$; and clearly, due to relativization to M, **ZFC** $\Vdash \psi\mathbf{S}(M,1), \ldots, \psi\mathbf{S}(M,n)$, so we have **ZFC** $\Vdash \phi^M$, i.e. **ZFC** $\Vdash \phi^{st}$ (since all quantifiiers in ϕ are standard). Hence **ZFC** $\Vdash \phi$.

A number of studies would be required for establishing a relation between **NST** and alternative set theory **(AST)**, insofar as the latter is considered to be an axiomatization of nonstandard methods, particularly, by way of allowing the construction of ultrafilters with special properties i.e., constructing various types of automorphisms and endomorphisms. However, because all of these set-theoretic approaches use considerable mathematical sophistication, a possible alternative would be to introduce some restricted form of infinitesimal analysis not depending on explicit use of the transfer principle.

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