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NOTES AND QUERIES

# WHO INVENTED CANTOR'S BACK-AND-FORTH ARGUMENT?

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One reads that Peano's postulates are really Dedekind's and Dedekind's chains are really Frege's and Newton's method was known to Archimedes. As Paul J. Campbell has noted [*1978*, 85], "the history of mathematics is rife with a variety of misattributions, whose continued propagation by oral and written tradition is variously due to widespread ignorance of the historical facts, accepted convention, or just plain complication of the situation." This paper raises the question: Who Invented Cantor's Back-And-Forth Argument?

Cantor's Back-And-Forth Argument (BAF) has emerged as a vital technique in the theory of models. Not to be confused with his Diagonal Argument, which is used most frequently to prove that there are more real numbers than rationals, Cantor's BAF argument is best known for its use in proving the isomorphism of any two countable, dense linear orders (without endpoints). It is a well-entrenched belief among set-theorists and model-theorists that BAF is due to Cantor, who first introduced it to prove that result. Indeed, Cantor is sometimes called "the father of the back and forth argument" (see, e.g., [Barwise *1973*, 5–6]), and is frequently cited for proving the isomorphism theorem using BAF (see, e.g., [Dickmann *1985*, 348]; [Roitman *1990*, 123]; [Shapiro *1991*, 160]; and van Dalen [*1983*, 132]). Such citations, as in the case of most folkloric attributions, however, fail to provide precise bibliographic references.

Many years ago, while in graduate school, I discovered a proof of the isomorphism theorem that did not require the BAF technique. Since this simplification was surprising to several logicians, I pursued the matter a bit at that time. That led me to look up Cantor's famous proof. I was shocked to discover that Cantor did the proof "my way," not using the BAF technique which has become associated with him. So, at that time, two facts were surprising: (1) the isomorphism result did *not require* BAF; and (2) Cantor did not use BAF himself, at least not where he supposedly introduced it (namely, in the two-part

[*1895-97*] *Beiträge zur Begründung der transfiniten Mengenlehre*; see the Philip E.B. Jourdain translation, [Cantor *1915*, *1955* reprint, 123–127]).

More recently, I have investigated the matter further and have a few more details to report. The first occurrence of the use of the BAF argument I have been able to find is in Hausdorff's *Grundzüge der Mengenlehre* written in [*1914*],[1] where (pp. 99–100) he uses it to prove Cantor's result. There is some (perhaps slight) evidence in the book that Hausdorff wished to offer a more convincing proof than Cantor had. The difficulty with the "One-Way Argument" (and why it is "surprising" that it works) is that the "onto" part of the proof seems to break down. At first glance, it appears that a One-Way Argument establishes only that one of the two ordered sets can be made isomorphic to a *subset* of the other set. In his book, Hausdorff first establishes *that theorem* using the One-Way construction. Then, *after* proving that one set can be made isomorphic to a subset of the other one, Hausdorff introduces the BAF argument to prove full isomorphism from one set to the other. This suggests to me that he found the BAF method more convincing than Cantor's One-Way Argument.

Hausdorff was widely read, and it was well-known that the original proof of the theorem we have been discussing was due to Cantor. Thus, it is possible that readers of that book (and of Hausdorff's later *Mengenlehre* [*1927*; *1935*, 50–51]) may have assumed that BAF originated with Cantor.

These speculations are preliminary. The main purpose of this note is to interest others in these topics so that more conclusive results can be obtained. Here are some questions that need to be answered: (1) Who really introduced "Cantor's" Back-And-Forth Argument?; (2) Did Cantor use it somewhere else?; (3) How did Cantor's name come to be associated with it (supposing someone else pioneered it)?; and (4) exactly when is BAF dispensable in favor of a One-Way Argument?[2]

The appendix that follows presents the ideas behind BAF and the One-Way Argument, together with proofs of the isomorphism theorem.

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[1]The dedication page of Hausdorff's book says: "Dem Schöpfer der Mengenlehre Herrn Georg Cantor in dankbarer Verehrung gewidmet [Dedicatad With Grateful Reverence To The Creator of Set Theory, Mr. Georg Cantor]." I wish to thank Michael Losonsky for his translation of the crucial passages of Hausdorff's book from the German.

[2]This question cannot be given a definitive answer until it is more precisely formulated. For simple cases, where the two sets involved are taken to be well-ordered, it seems that there will always be a least element having the property in question. This suggests that it may be the case that many other proofs using BAF can be simplified in favor of One-Way arguments, but the question needs to be better defined before it admits of a precise answer.

## APPENDIX

The purpose of BAF is to create a "shoelace" effect between two sets in order to "tie them together." More formally, the idea is to define two functions, say $f$ and $f'$, such that $f$ goes "forth" from $X$ to $Y$ (i.e., $f: X \to Y$) and $f'$ goes "back" from $Y$ to $X$ (i.e., $f': Y \to X$). Piecing them together creates a one-to-one correspondence, $h$, between the sets $X$ and $Y$. If, in addition, the resulting function, $h$, preserves the order relation(s) on each of the two sets, then the function is an isomorphism.

The problem with defining only one function $f$ from a countable dense linear order $X$ (without endpoints) to another one, $Y$, is that for the proof to succeed $f$ must be an onto function, a surjection. Whereas, if $f$ goes "back" from $Y$ to $X$, it is clear that both sets must be "used up" at the same time (formally, by recursion).

Another issue is whether the so-called simplification of BAF to a One-Way argument is really simpler. Some persons may argue that even though BAF *construction* is more complicated than simply defining a function with the requisite properties one way, the resultant BAF proof is clearer, easier to understand, and therefore simpler. If this is so, it would not always be a "reduction" to show that a BAF argument could be turned into a One-Way argument. If the above speculation about Hausdorff's motivation for introducing the BAF construction has any merit, then perhaps this was his position.

### BAF ARGUMENT

Following is an informal exposition of the BAF to prove that any two countable dense linear orders are isomorphic.

Consider $X$ and $Y$ to be any two countable dense linear orders without endpoints. The elements of the sets can be arranged in two lists, where the order of elements in the lists is indicated by the number of prime marks appended to the letter '$x$' or '$y$', such that the first element in the list has one prime mark, the next one has two prime marks, etc.:

$$X = \{x', x'', x''', x'''', \ldots\}$$
$$Y = \{y', y'', y''', y'''', \ldots\}$$

Start out by letting $x_0 = x'$ and $y_0 = y'$. Then go "forth" to $Y$ and take the next $y$ in the list, which is $y''$. $y''$ now becomes $y_1$, which is greater than or less than $y_0$. Suppose $y_1 < y_0$. Then choose an $x$ less than $x_0$, and make this $x_1$. (If $y_1$ is greater than $y_0$, then choose an $x$ greater than $x_0$ to be $x_1$.)

Now, go "back" to $X$ and take the first element left in

$$\{x', x'', x''', x'''', ...\} - \{x_0, x_1\}.$$

We know it cannot be $x'$ because $x'$ is $x_0$, but it could be $x''$. Let the first element left in this set be $x_2$. $x_2$ is either greater than each of $x_0$ and $x_1$, less than both of them, or in between them. Choose a $y$ from

$$\{y', y'', y''', y'''', ...\} - \{y_0, y_1\}$$

such that this $y$ bears the same relation to $y_0$ and $y_1$ as $x_2$ does to $x_0$ and $x_1$. This $y$ then becomes $y_2$.

Go "forth" again to $Y$, get the next $y$ not yet taken, call this $y_3$, and then choose an $x$ such that this $x$ bears the same relation to the three $x$'s already taken as $y_3$ does to the three prior $y$'s.

The above method constitutes a recursive definition of the two sequences $\langle x_0, x_1, x_2, x_3, ...\rangle$ and $\langle y_0, y_1, y_2, y_3, ...\rangle$ such that the function $h$ defined from $X$ to $Y$, where $h(x_n) = y_n$, is the isomorphism we wanted.

## ONE-WAY ARGUMENT

The One-Way Argument creating an isomorphism is similar to the above construction in one direction, except that we explicitly take "the first" value satisfying the given property.

We define $g: X \rightarrow Y$ as follows, and then prove the onto part: Let $g(x') = y'$, just to fix the initial points. Then, if $x'' < x'$, let $g(x'') =$ the first $y$ such that $y < y'$ (or if $x'' > x'$, let $g(x'') =$ the first $y$ such that $y > y'$). For any $x^n$ (i.e. x followed by $n$ primes), $g(x^n) =$ the first $y$ among

$$\{y', y'', y''', y'''', ...\} - \{ g(x'), g(x''), ..., g(x^{n-1})\},$$

such that $y$ bears the same relation to all of $g(x')$, $g(x'')$, ..., $g(x^{n-1})$, as $x^n$ bears to all of $x'$, $x''$, ..., $x^{n-1}$.

The claim is that $g$ is an isomorphism, as $h$ is. It was easier to see that $h$ is an isomorphism because the back-and-forth character of the construction ensured that the elements of $X$ and $Y$ would be "used up together," i.e., that $h$ is a surjection.

Suppose at least that one element of $Y$ is missed. Take the first such element, $y^m$, such that $y^m$ is *not* the image of any $x$ in $X$ under $g$. This means that all of $y'$, $y''$,..., $y^{m-1}$ are images of $x$'s under $g$. Suppose that it takes $k$–many $x$'s to cover these $m-1$ elements of $Y$, where $k$ is at least as large as $m-1$. That creates $k+1$ "slots", one of which $y^m$ fits into. For example, one slot exists between the next-to largest and the largest element among $g(x')$, $g(x'')$, ..., $g(x^k)$, and another exists past the largest element (since there are no endpoints). Not only does $y^m$ fit into one of these slots, but it is the first such element available that fits. For example, suppose $y^m$ is between $g(x')$ and $g(x'')$. Then, as soon as an $x$ is reached that is between $x'$ and $x''$, $g$ of that $x$ will equal $y^m$. At some point an $x$ will be encountered that fits into that slot (by denseness), and it will be mapped to $y^m$. Therefore $g$ *is* surjective, which is what we wanted to show.

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