ℵ Modern Logic ω

Richard L. Epstein and Walter A. Carnielli, *Computability: Computable Functions, Logic, and the Foundations of Mathematics*, Belmont, California, Wadsworth, Inc., 1989. xvii + 297 pp.

Reviewed by

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What makes recursion theory tick? Why did the field develop — what are the themes that underlie the subject? Three major threads running through recursion theory come to mind quickly:

(A) *Considerations of machine computability*, in evidence from Turing's 1936 paper that introduced Turing machines, right up to current work on computational complexity.

(B) *Logical/foundational matters*, well exemplified by Gödel's use of the class of primitive recursive functions in his 1931 paper on undecidability.

(C) *"Mainstream" pure mathematics*. For example, the opening sentence of the first book ever written on the subject, Péter's *Recursive Functions*, reads: "The theory of recursive functions properly belongs to number theory; indeed, the theory of recursive functions is, so to speak, the function theory of number theory."

Corresponding to these three threads are the groups of people most interested in recursion theory, respectively, computer scientists, logicians (both mathematical and philosophical), and mathematicians. Of course, the threads are often closely interwoven. The very title of Turing's abovementioned article, "On Computable Numbers, with an Application to the Entscheidungsproblem", suggests the interplay among (A), (B), and (C) present in that paper.

The textbook by Epstein and Carnielli has a much shorter title, *Computability*, which hints at an emphasis on thread (A). But the subtitle tells the true, thread (B) story. This book is indeed about *Computable Functions, Logic, and the Foundations of Mathematics* and their mutual connections. In pursuing that theme, the authors develop the material in a strikingly original manner. A recursion theory textbook which waits until page 63 (out of

270) to begin just the *informal* discussion of the concept of algorithm? And with a formal definition of partial recursive function not occurring until page 124? *Computability* is not, nor is it intended to be, a standard text for a standard course in recursion theory or theory of computation. Rather, it is a quite successful introduction to, and exploration of, computable functions within the context of foundational questions.

The first of the book's four main sections raises these questions, beginning with paradoxes of self-reference and Zeno's Paradoxes, discussing (informal) mathematical proof, and culminating in a 14-page excerpt, in translation, from Hilbert's 1925 "On the Infinite", which sets forth his Program.

Part II contains the hard core of *Computability*'s treatment of recursion theory. Following a general discussion of computability and algorithms, Turing machines make a brief appearance. But the main line of development is a classical recursion-based route, which starts with the primitive recursive functions, takes excursions through the Grzegorczyk Hierarchy and nested recursion, picks up the search operator, and arrives at the class of partial recursive functions. An optional chapter addresses the proof that this class is the class of Turing machine computable functions. The approach is geared towards the applications to logic of Part III, with such topics as Gödel numbers and diagonalization. Conversely, other staples of basic recursion theory are de-emphasized; the *s-m-n* Theorem and Recursion Theorem receive compressed treatment, and there is next to nothing on relative recursivity and reducibilities.

In the third section, the authors apply the recursion-theoretic machinery of Part II to proof theory. Here are the decidability and completeness of propositional logic, the representability of the recursive functions, the undecidability of arithmetic, and the unprovability of consistency.

Finally, Part IV reverts to the more philosophical orientation of the opening section. Church's Thesis is scrutinized — what does it really say? what are its pros and cons? The book ends with a discussion of constructivism in various guises, such as the approaches of Brouwer, Bishop, and finitists.

Computability thus provides a rich mix of mathematics and philosophy, but there is another ingredient, too—history. One of the declared purposes of the authors is to present the historical context of the mathematics they deal with. In particular, the book includes a generous supply of writings from key figures in the story. Besides the abovementioned Hilbert address, there are multipage excerpts from (among others) Turing, Post, and Brouwer, as well as shorter passages by (among others) Gödel, Kleene, and Kalmár. These selections combine with the historical remarks of Epstein and Carnielli to give the student a good sense of the roots and chronological development of the subject matter.

Not that the history presented is always totally correct. For example, regarding the standard non-primitive-recursive function f, where

$$f(0,n) = n+1$$

$$f(m+1,0) = f(m,1)$$
$$f(m+1,n+1) = f(m,f(m+1,n)),$$

the authors claim that "W. Ackermann first gave this definition in 1928." However, Ackermann defined a ternary function. The simplification of his function to the binary function f occurs in a 1935 paper of Rózsa Péter (whose given name is misspelled as Rósza throughout *Computability.*) There are also several minor errors in the mathematics that, while easily corrected by instructors, are potentially confusing to students tackling the book on their own. One other drawback is the relatively high price that the publisher has set.

But all in all, *Computability* achieves its aims quite well. Both the text and the exercises are interesting and thought-provoking. And the book is well-written in a lucid style with some nice lighter touches. In the introduction to one of his other books, Epstein quoted Bruno Bettelheim: "It is my conviction that to withstand and counteract the deadening impact of mass society, a man's work must be permeated by his personality" (p. xi, *Degrees of Unsolvability: Structure and Theory*, Lecture Notes in Mathematics vol. **759**, Berlin, Springer-Verlag, 1979). *Computability* is a highly individual book, occupying its own niche. Again, it is not designed to cover the whole spectrum of basic recursion theory. But for investigating the notion of computability and how computable functions relate to Gödel's work, for exposing budding logicians to the interrelationships among various areas of logic, or for awakening mathematics students to foundational questions often neglected in their studies, this book is well worth considering.