Jean van Heijenoort, *El desarrollo de la teoría de la cuantificación.* No. 23, Instituto de investigaciones filosóficas. Universidad Nacional Autónoma de México, México, D.F. 1976 (copyright 1975). 57pp.

Reviewed by
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Introduction. In Spring 1973, Jean van Heijenoort delivered a series of lectures on the development of quantification theory at the Philosophy Institute of the National Autonomous University of Mexico in Mexico City. The booklet under review is the published record of these lectures and served in turn as an outline of his Fall semester of 1976 seminar at Brandeis University on "Theories of Quantification."

What follows is the unaltered reproduction of my manuscript review of van Heijenoort's *El desarrollo de la teoría de la cuantificación* first written on 14 September 1978 and privately printed in my *Introduction to Proof Theory: Papers in Metamathematics* (Itta Bena, Mississippi Valley State University, 1980), pp. 8-42. I do not now necessarily agree with all the statements or interpretive details contained in this old review, as readers of my paper "The Löwenheim-Skolem Theorem, Theories of Quantification, and Proof Theory," in T.L. Drucker (editor), *Perspectives on the history of mathematical logic* (Boston/Basel/Berlin, Birkhäuser, 1991), pp. 71-82, will recognize.

Review. There are several theories of quantification competing for the logician's attention. By far the most familiar is the axiomatic method, which had its auspicious beginning in Frege's *Begriffsschrift* and reached its peak in the *Principia Mathematica.* Hilbert-type systems and Frege-type systems thrived until 1931 with the publication of Gödel's results on incompleteness. The Gödel results, together with the Russell paradox and the complications introduced into set theory to resolve the paradox, dealt a heavy blow to the axiomatic method. The philosophical repercussions to the Gödel results were the rivals of logicism, formalism, and intuitionism. The technical repercussions included development of alternative theories of quantification. It is not coincidental that the majority of quantification theories to rival the axiomatic method arose within a few years of the publication of Gödel's results. Of the major alternatives, only the Herbrand system preceded the incompleteness results.

It is clear from Herbrand's own comments that his investigations were undertaken to clarify the concept of *being a proof* for a Hilbert-type quantification system. This suggests that the Gödel results gave impetus to investigations in proof theory and into the concept of *validity*, but did not serve as the point for initiation of such studies We are further justified in supposing that the Gödel results did not serve to initiate these investigations by evidence that much of Herbrand's concern centered on the Löwenheim-Skolem theorem, and particularly on the concept of *satisfiability* as applied to $\aleph_0$-satisfiability obtained from the k-satisfiability for every finite k of a Löwenheim infinite conjunction.

The alternatives to the axiomatic method came fast and furious after 1931; but the example of Herbrand should show that the Gödel results did not precede – or cause – the development of alternative quantification theories.

The axiomatic method provides results from the concept of *formal system*; we are provided with a set of axioms, a list of formulas, from which we derive other formulas, the theorems of the system, and the concept of *proof* is not considered. A proof is simply a string of formulas, or axioms, the last line of which is a theorem proven in the system. (The Hilbert-type system, then, is closely defined by the Hilbert program, with its underlying formalist philosophy.) Herbrand, as noted, following the Löwenheim-Skolem results on satisfiability, was led to seek a clarification of satisfiability or validity. The Herbrand expansion is a special case of the infinite conjunction used by Löwenheim in his results on $\aleph_0$-satisfiability. It is a special case because it is the fundamental theorem for the Herbrand quantification system Q_H. Herbrands's theorem states that we can effectively generate an infinite sequence of quantifier-free formulas from a given formula of classical quantification theory; and if F_Q is a formula of a classical quantification theory without identity, then for the k^{th} Herbrand expansion formula F_{QH_k} for any k, $\models F_Q \Rightarrow \models F_{QH_k}$ and F_{QH_k} is sententially valid.

Perhaps the single most representative feature of Herbrand's system is its elimination of *modus ponens*. This rests on set of connectives of Herbrand expansion (conjunction) and disjunction, and negation being a base. Gentzen's Sequenz calculus (1934) rests on the results of Herbrand, but goes further by giving an analysis of the sentential parts of the proof of validity. However, unlike Q_H, the Gentzen theory contains a *Hauptschnitt* which is first of all extendable to the intuitionistic calculus and to modal logic, but otherwise is not as general as the Q_H elimination of *modus ponens*. The method of natural deduction, however, places extreme emphasis on *modus ponens* as a method of inference. Its central feature is the rule of conditionalization, supported by its rules for introduction and elimination of the connectives and quantifiers of classical quantification theory. In reliance upon conditionalization, natural deduction has a close affinity to the Gentzen Sequenz-calculus.

It was Jaśkowski in 1934 who first made use of the idea of Łukasiewicz to found proofs on the rules of intuitive logic. Jaśkowski's system made use of **UI** and **UG**, and was therefore insufficiently intuitive. The N-sequenzen (natural sequences) were found to be more intuitively acceptable, allowing existential as well as universal quantification. Nevertheless, despite all the work of Quine, Copi, Prawitz and others, **UG** and **EI** persisted as sore spots for natural deduction. We in fact are brought back to the question of satisfiability for quantified formulas involving **UG** and **EI**. We can show, for example that some formula F' of quantification theory with identity $Q_=$ in prenex normal form is k-valid but not $(k\text{-}1)$-valid, or that some formula $F''$of $Q_=$ in prenex normal form is k-satisfiable for and finite k but not $\aleph_0$-satisfiable. So we seem to have gone full circle, back to Herbrand's question on the Löwenheim-Skolem concept of *satisfiability* or *validity*. (Note that the Law of Lesser Universes applies only to quantification theory in which formulas occur only in disjunctive normal form, and in which there is neither identity nor a universal quantifier, although even for $Q_=$, satisfiability/validity depends to a great extent on the cardinality of the universe.) We find, however, that Prawitz has provided a measure to rescue us from facing Herbrand's question, by his introduction into the natural deduction method of an analogue of Gentzen's *Hauptsatz* on the *Schnitt*-elimination. We have, then, a bit of solid ground on which to rest.

Thus, the competing multiplicity of quantification theories, as viewed by van Heijenoort, rather than providing an opportunity for possible conflict, represents an evolution or development (*desarrollo*) of quantification theory, starting from the definition of the Hilbert program and the Herbrand system Q_H, through the Gödel incompleteness results, to the Gentzen-sequenzen and natural deduction, in an attempt to clarify the concepts *proof*, *satisfiability*, and *validity*. We may even presume to have made progress in this. Van Heijenoort's task is to trace the history of this development and provide a brief technical sketch of each of these systems. The brief introduction sets the stage for us, calling the theory of quantification "a family of formal systems" and giving an outline of classical logic and the creation of classical quantification theory in Frege's *Begriffsschrift* (1879). In the first chapter, we find an abbreviated technical discussion of classical quantification as it appears in the axiomatic method, with a stress on Hilbert-type systems, but avoiding consideration of neither the *Principia* nor the Frege-type system. It is the briefest chapter, followed respectively by chapters on Herbrand, Gentzen, and Natural Deduction, each of which are of equal length, all three of which present a sketch of the technical aspect, as the systems were developed to contend with and elucidate, the nature of *proofs*. In a final chapter, all four members of the "family of formal systems" treated are brought into juxtaposition, their respective characteristics highlighted, so as to stress their unity and the distinct contributions which they make towards the development of quantification theory and the clarification of *proof*, *satisfiability*, and *validity*.