ℵ Modern Logic ω

Reflections on the Interplay between Mathematics and Logic

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1. Introduction.

Van Heijenoort has rightly stressed that, in the work of Frege and Russell, logic is universal; neither logician made use of partial universes of discourse, as did Boole and De Morgan, but only *the* universe of everything. One central consequence of this universality, he has stressed as well, is that neither Frege nor Russell raised any metalogical or metamathematical question – nothing about the consistency or independence of the axioms of logic, or about their completeness (van Heijenoort *1967a*, 326).

Let us give an example, one not mentioned by van Heijenoort, of Russell and Whitehead's assertion that it is not possible to stand outside logic. The authors of *Principia Mathematica* insisted that the Principle of Mathematical Induction cannot be used to prove theorems *about* their system of logic (Whitehead and Russell *1910*, 135). This is in utter contrast with work in logic for the last fifty years. A great variety of different kinds of induction (e.g. on the length of formulas, on the kinds of formulas, etc.) are now standard textbook devices for proving theorems about first-order logic, particularly since terms and formulas are given by recursive definitions.

Van Heijenoort contrasts the approach of Frege and Russell with that of Löwenheim (*1915*), who belongs to the Peirce-Schröder tradition, and notes that the latter tradition used universes of discourse. It is precisely the metalogical questions that Löwenheim considers (van Heijenoort *1967a*, 327; *1977*, 183). But let us be clear that the metalogical notions which Löwenheim uses are those of validity and satisfiability, not those of consistency or completeness. Löwenheim does consider independence, but only from a semantic rather than a syntactic perspective.

What van Heijenoort mentions in passing, but does not stress, is that Hilbert's work in logic was intermediate to that of Frege and Russell on the one hand and that of Peirce, Schröder, and Löwenheim on the other (van Heijenoort *1977*,185). That is, Hilbert made use of a formal system, following the lead of Frege and Russell. (Indeed, beginning in his lecture course of 1917, Hilbert used the theory of types as his basic logical system, though this is not well known.) And, like the mathematician that he was, Hilbert did not let his quantifiers vary over every object, but only over some restricted universe of discourse given in advance. Here we must stress that it was Hilbert who brought questions of consistency and completeness to the fore.

Another eminent historian who, like van Heijenoort, has emphasized the differences between the Peirce-Schröder tradition and the Frege-Russell tradition, is Grattan-Guinness. In an article of 1988, Grattan-Guinness surveyed the interactions between logic and mathematics during the period from the French Revolution to the First World War. He followed Schröder in referring to the logic of Boole, Peirce, and Schröder as the algebra of logic, and identified the logic of Peano and Russell with mathematical logic. His article concluded by lamenting the lack of contact between mathematics and logic, both during that period and today:

> The eclipse of algebraic logic by mathematical logic has left rather forgotten the links that algebraic logic forged with certain algebras and with probability theory .... So logic [in 1914] still lived rather apart from mathematics, even though one branch of it was now "mathematical" .... The situation has continued until today, and to a significant degree because of the (lack of) reception of *Principia Mathematica*. Logicians ... are more often found today in departments of philosophy or computing than of mathematics .... The two communities of logicians and mathematicians largely live apart .... (Grattan-Guinness *1988*, 79).

To those familiar with mathematical logic as it exists in 1992, Grattan-Guinness's claims will sound rather odd. Whatever the interactions between mathematics and logic in the work of Boole and Whitehead, and they were considerable, today the interactions between mathematics and logic are very rich indeed. It is the interactions between algebra and logic which merit most attention. In addition, thanks to category theory, there are today substantial interactions even between logic and geometry. Finally, in recent years nonstandard analysis has proved to be a very fruitful area of interaction between logic, analysis, probability, and applied mathematics.

In sum, since the the 1940's mathematics and logic have been interacting more than they ever did before. The present paper discusses some of the most significant ways in which logic and mathematics have affected each other, and examines how closely those interactions are related to the metalogical and metamathematical concerns discussed by van Heijenoort. The paper begins in the 1920's, but it concentrates on the period shortly after the Second World War when model theory crystallized as a discipline and when substantial interactions developed between logic and algebra.

2. Applications to Algebra, 1925-1940: Langford, Tarski, and Maltsev.

Any kind of logic, mathematical or otherwise, can be developed in one of three ways. First, its philosophical or mathematical foundations can be investigated. Second, it can be developed as a system in its own right, possibly undergoing extensions or restrictions in the process, and possibly evolving into a family of systems rather than a single system. Third, it can be applied to other areas of knowledge. Interactions between mathematics and logic will be used here to refer only to the second and third categories, not to foundational considerations.[1]

[1] If, in this context, we understood interactions between logic and mathematics to include the first category, then we would need to add an extensive discussion of intuitionism. In particular, we would need to examine the formalization of intuitionistic logic by Heyting, and the later rise of topos theory (within category theory) as an important part of mathematics that made use of intuitionistic logic. We would also need to examine the interactions between intuitionism and recursive function theory, as embodied in the work

Eventually the most seminal figure in promoting interactions between logic and mathematics was to be Alfred Tarski. At the beginning, however, he developed mathematical logic in its own right, rather than applying it to some area of mathematics or borrowing concepts from mathematics. His doctoral dissertation at Warsaw (*1923*) was formulated in the theory of types, and showed that all the propositional connectives ("and", etc.) can be defined from the sole connective "if and only if", together with the quantifiers. Yet he already cited the work of Schröder as well. Thus, even at the start of his career, Tarski showed the eclectic approach to problems in logic that was to prove so fruitful in his later work.

The third category – the use of logic to gain information about particular mathematical theories – first became prominent in the work of the philosopher C.H. Langford (*1927*). Using propositional functions that were first-order, in the sense of *Principia Mathematica*, Langford treated "a form of the problem of categoricalness ... for types of dense series" (*1927*, 16). In effect, he gave a decision method for dense linear orders by using the method of elimination of quantifiers, a method whose essentials had already been discovered by Löwenheim (*1915*, §3) and Skolem (*1919*, §4). Langford, who worked in the tradition of *Principia*, gave no indication in his paper of being aware of their work.

Around 1927-1928 Tarski used elimination of quantifiers to extend Langford's work on dense linear orders, but these results were not published until 1936.[2] In the interim, there appeared Tarski's article on definable subsets of $\mathbb{R}$, the set of real numbers (*1931*). This article was formulated in the simple theory of types, and consequently was concerned not only with subsets that were first-order definable, but also those that were second-order definable, third-order definable, and so on. His principal concern was to combine mathematics with metamathematics and, in particular, to express metamathematical notions (such as definability) in mathematical terms:

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of Kleene. For reasons of space, as well as for more substantive reasons, we leave such questions to one side in the present article.

[2] Tarski *1936*, 374–383. Here, as with all his translated articles, we use the page numbers in the 1983 translation.

> The distrust of mathematicians towards the notion [of definability] is reinforced by the current opinion that this notion is outside the proper limits of mathematics altogether.... In this article I shall try to convince the reader that the opinion just mentioned is not altogether correct .... I believe that I have found a general method which allows us to construct a rigorous metamathematical definition of this notion. Moreover, by analyzing the definition thus obtained it proves to be possible ... to replace it by a definition formulated exclusively in mathematical terms. Under this new definition the notion of definability ... can be discussed entirely within the domain of normal mathematical reasoning. (*1931*, 110-111)

Tarski proceeded to mathematize the notion of definability over a structure. In particular, he discovered that a set M is first-order definable in the structure $(\mathbb{R},+,\times)$ if and only if M is a finite union of intervals with algebraic endpoints (or $\pm\infty$). First-order definability in a structure was reduced, in mathematical terms, to the closure of a class of given relations under the Boolean operations and the "geometric" operation of projection.

When Tarski published his results on dense and isolated linear orders in 1936, he also formulated some of his results, obtained by 1930,[3] in a way that presaged his work on model theory twenty years later. In 1936 he introduced the notion of $T(\alpha)$, the set of all first-order sentences true of an order type α, where the only non-logical symbol was for a binary relation. He next introduced the notion that two order types, α and β, are elementarily equivalent, i.e. $T(\alpha) = T(\beta)$.[4] He noted that, thanks to the Löwenheim-Skolem Theorem, there can be only $2^{\aleph_0}$ types that are not elementarily equivalent. As for dense linear orders, there can only be four types that are not elementarily equivalent, depending merely on endpoints. In particular, the order type of the rational numbers and that of the real numbers are elementarily equivalent. Hence properties of order types such

[3] Tarski *1936*, 383.

[4] At this time he introduced the notion of being elementarily equivalent ("elementar äquivalent"). See footnotes 12 and 16 on later changes of terminology.

as being denumerable or being continuous are not expressible in the first-order theory of linear order. Further, he showed that the notion of well-ordering is not expressible in that first-order theory, since $T(\omega) = T(\omega+{}^*\omega+\omega)$. In conclusion, he looked to the future:

> It seems that a new, wide realm of investigation is here opened up. It is perhaps of interest that these investigations can be carried out within the framework of mathematics itself (e.g. set theory) and that the concepts and methods of metamathematics are essentially superfluous: all concepts which occur in these investigations (e.g. the concept of an elementary property of an order type ...) can be defined in purely mathematical terms. (*1936*, 382)

This desire to replace a metamathematical notion by one that was equivalent, but formulated in purely mathematical terms, remained a central feature of his approach to logic in the decades that followed.

Tarski's approach can be clarified by comparing it with Gödel's. Tarski was concerned both with general logical notions ant with their application to particular mathematical structures. These dual concerns recur many times in his career. By contrast, Gödel was concerned almost solely with general logical notions, and almost never with their application to specific mathematical structures.[5] Perhaps that is why Gödel did not apply his Compactness Theorem (*1930*, Theorem X) to obtain any results about algebra, although, to later logicians, it cried out for such applications.

The first work applying the Compactness Theorem to get results about specific mathematical structures was due to the Russian logician Anatolii Maltsev (*1941*). Already in 1936 he had extended the Compactness Theorem, which Gödel (*1930*) used only for a countable first-order language, to the corresponding theorem for a language whose set of

[5] This was true despite the fact that Gödel worked on geometry around 1933 (see Gödel *1986*, 272–281).

primitive symbols had any infinite cardinality whatever.[6] Maltsev's 1936 paper was the first to introduce such an uncountable language for first-order logic,[7] but the use of uncountable first-order languages was not pursued further until the work of Henkin and Robinson late in the 1940's (see §3 below).

In his 1941 paper, Maltsev applied the Compactness Theorem to group theory. He noted that quite a few theorems in algebra were of the following form: If a certain property *P* holds for all the finitely generated subalgebras of a given algebra (group, ring, etc.), then *P* holds for the algebra itself. His aim was to give new, and simpler, proofs for many of these theorems by showing that they were immediate consequences of the Compactness Theorem for an uncountable language. His first example was a theorem originally proved by Chernikov: If every finite subgroup of a locally finite group has a Sylow sequence, then so does the group itself. The second was a theorem due to Fuks-Rabinovich, whose original proof was quite complicated: Every locally free group is not simple.

3. Applications to Algebra, 1945-1955: Robinson, Henkin, and Tarski.

One of the main ways in which logic and mathematics have interacted in the recent decades can be conveyed by a metaphor: logic is to mathematics as mathematics is to applied mathematics. That is, logic can be applied to give specifically mathematical results about mathematical systems. As events unfolded, one of the essential tools of these applications was the use of a symbolic language with uncountably many symbols and the use of the Compactness Theorem, or the Strong Completeness theorem, for such languages. In other words, this tool was an extension of Gödel's Completeness Theorem for first-order logic to an uncountable language.

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[6] Henkin and Mostowski later questioned the validity of Maltsev's proof. See Henkin and Mostowski *1959*, 56–57.

[7] Earlier, Gödel had briefly considered a language for propositional logic that could be countable or uncountable (*1932*). Zermelo (*1931*, *1935*) had considered infinitary languages with uncountably many primitive relations. Maltsev did not appear to be aware of these three articles in his *1936* or *1941*.

Besides Tarski in Poland and Maltsev in the Soviet Union, one of the principal figures who turned this metaphor into a reality, by initiating the application of mathematical logic to algebra, was Abraham Robinson. While he was an undergraduate at the Hebrew University of Jerusalem, and a student of Fraenkel's, Robinson published his first paper in the *Journal of Symbolic Logic* (*1939*). He spent the war years in England, where his research was on aerodynamics. After the war, his interests returned to logic, and he did his Ph.D. at the University of London with the dissertation "The Metamathematics of Algebraic Systems" (*1949*). A report that he gave on his dissertation in February 1948 illustrates his conception of the role of logic in mathematics: "Time and again in the development of Mathematical Logic there arose a desire to use the new science not only to clarify and crystallise the conquests of Mathematics proper, but as a tool for the discovery and demonstration of actual mathematical theorems." As two examples, he cited work by Leibniz and, early in this century, Poincaré's disdain for Peano's logic. Robinson concluded: "Conditions have changed in the last thirty years, and Symbolic Logic now receives a good deal of attention even in purely mathematical circles."[8]

Robinson's dissertation was published in 1951 as a book, *On the Metamathematics of Algebra*. It opened by explicitly advocating the application of logic to modern abstract algebra:

> The principal object of the present work is the analysis and development of Algebra by the methods of Symbolic Logic. In fact, in view of their transparent logical character, the algebraic theories of fields, rings, and of similar structures appear to be eminently suited to such treatment, more perhaps than any other branch of Mathematics ....
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> Instead of formulating and proving individual theorems as in orthodox Mathematics, we may consider statements about theorems in general. In particular, we may be able to show that any theorem (of a certain class) which is true for one type of mathematical structure is also true for another type of mathematical structure. An instance of such a metamathe-

[8] This passage is from a page of manuscript printed in his *Selected Papers* (*1979*, xii).

> matical statement is provided by the classical principle of duality in Projective Geometry. (*1951*,1)

Robinson then offered three examples of such statements about theorems in general, including the following:

> 1. Any theorem of the first-order theory of fields that is true for all non-Archimedean ordered fields is true for all ordered fields.
>
> 2. Any theorem of the first-order theory of fields that is true for the field of all algebraic numbers is true for any other algebraically closed field of characteristic 0. (*1951*, 3)

He concluded his discussion by noting that

> the above examples will suffice to show that Symbolic Logic can be an effective tool for the discovery and proof of mathematical theorems: while on the other hand the analysis of the procedures used in a highly developed mathematical discipline throws light on points of Symbolic Logic which might be overlooked in purely abstract investigations. (*1951*, 6)

The main tool in Robinson's dissertation was the Strong Completeness Theorem, whereby every consistent set of first-order sentences has a model. He employed this theorem in a way that amounted to applying the Compactness Theorem for an uncountable language. At this time he was apparently not aware of the work of Maltsev, where the latter theorem was used directly. Later, Robinson very much admired Maltsev's work.[9]

In order to define the satisfaction of a sentence by a structure, Robinson extended the symbols in the language by introducing a new constant symbol for each element in the structure. Thus, if the structure

[9] See Robinson *1962*.

was uncountable, he did not see how such one could avoid using uncountably many symbols (*1951*, 21). In this he was mistaken.[10] Where the use of uncountably many symbols was unavoidable, it seems, was in proving the Strong Completeness Theorem for uncountable sets of sentences and in proving its corollary, the Compactness Theorem.

Shortly before Robinson wrote his dissertation in London, Leon Henkin completed his own at Princeton, where it was accepted in October 1947. Directed by Alonzo Church, Henkin's dissertation shared a number of concerns with Robinson's. One of those concerns was the use in logic of uncountably many symbols (*1947*, 21) and another was the importance of applying logic to algebra. In his introduction, Henkin described the general situation in logic as he saw it:

> From one point of view formal logic is the application of mathematics to the analysis of the way in which symbols are used in scientific discourse. When the language of mathematics itself becomes the object of investigation, we can expect three general types of result:
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> 1. Insight into the foundations and philosophy of mathematics;
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> 2. Discovery of limitations imposed by the nature of mathematical tools on the kinds of problem which can be stated or proved; and
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> 3. The development of new and improved methods to strengthen the attack on problems in the various branches of mathematics.
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> Historically, problems of the first category were principally responsible for the development of modern logic ....
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> As for the third class of results, it must be admitted that these are most meager. Whereas the methods and results of

[10] It was possible to avoid the use of uncountably many constant symbols, in the definition of satisfaction, by using valuations. These were functions whose domain was the countable sequence of free variables and which assigned an element of the structure to each free variable. See, e.g., Mendelson *1964*, 50. Nevertheless, Robinson's definition of satisfaction was entirely correct. It was used, for example, in Shoenfield's textbook for logic (*1967*, 18) and in Sacks's textbook for model theory (*1972*, 20).

> algebra, geometry and analysis are richly intertwined, mathematical logic has held a rather isolated position within the family of mathematical disciplines .... Boole's original development of the algebraic structures now named after him is perhaps as prominent an example as any. (*1947*,1-2)

Despite the limited success which Henkin saw in past attempts to apply logic to mathematics, he was more optimistic for the future: "Our conjecture is that the potential aid ... which mathematical logic can render to older branches of mathematics is abundant and accessible, and that a rich harvest lies just below the surface" (*1947*, 4). As a contribution to that harvest, he applied his logical results to obtain a new and shorter proof of the Representation Theorem for Boolean algebras (due to Marshall Stone) and for distributive lattices (due to Garrett Birkhoff). Henkin relied on both the Strong Completeness Theorem and the Compactness Theorem for his algebraic results (*1947*, 26-31). To do so, he needed those theorems as expressed for a language with uncountably many constant symbols (*1947*, 21).

Robinson's dissertation gave no indication of the importance of Henkin's work or that of Tarski. But when Robinson spoke to the International Congress of Mathematicians in 1950, he stated that "we may mention the names of K. Gödel, L. Henkin, and A. Tarski as representative of those who either directly or indirectly contributed towards the establishment of symbolic logic as an effective tool in mathematical research" (*1952*, 686). Robinson's lecture was primarily based on his dissertation, and used the Strong Completeness Theorem to obtain various algebraic results.

One of the striking features of this period from 1947 to 1950 is the independent discovery, by Henkin, Robinson, and Tarski, that the Strong Completeness Theorem can be used to prove theorems about classes of algebraic structures.[11] This situation led to some duplication of results. In particular, Robinson's theorem 2 above, about algebraically closed fields,

[11] A detailed discussion of these independent discoveries can be found in Henkin *1953*, 427.

had been found independently by Tarski, as Robinson acknowledged (*1952*, 686).

Tarski's first thorough discussion of such uses came in the paper which he gave to the same session of the 1950 International Congress that Robinson had addressed. In the paper, "Some Notions and Methods on the Borderline of Algebra and Metamathematics", Tarski introduced the notion of an arithmetical class of structures. In general, he allowed a structure $\mathfrak{A}$ to consist of a non-empty set A together with a finite number of operations on A, relations on A, and distinguished elements of A. However, in order to simplify his definitions and theorems, he restricted himself in the paper to structures of the form $\mathfrak{A} = (A,+)$, where A was a set and + was a binary operation on A. Two such structures $\mathfrak{A}$ and $\mathfrak{B}$ were said to arithmetically equivalent, i.e. $\mathfrak{A} \equiv \mathfrak{B}$, if precisely the same first-order sentences were true in $\mathfrak{A}$ and in $\mathfrak{B}$.[12] Then a class $\mathcal{A}$ of systems was defined to be an arithmetical class if, for some $\mathfrak{A}$, $\mathcal{A}$ was the class of all models of some finite set of sentences in the language of $\mathfrak{A}$. In a statement reminiscent of that quoted above from his 1931 paper, Tarski observed:

> The notion of an arithmetical class is of a metamathematical origin; whether or not a set of algebraic systems is an arithmetical class depends upon the form in which its definition can be expressed. However, it has proved to be possible to characterize this notion in purely mathematical terms and to discuss it by means of normal mathematical methods. The theory of arithmetical classes has thus become a mathematical theory in the usual sense of this term, and in fact it can be regarded as a character of universal algebra.[13]

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[12] In 1936, as we saw above, he used "elementarily equivalent" instead of "arithmetically equivalent". By the 1960's, "elementarily equivalent" became the common term. Cf. footnote 16.

[13] Tarski *1952*, 705. Although the name "universal algebra" had originated with Whitehead (*1898*), the subject only began its continuous development in the 1930's with the work of Garrett Birkhoff.

Tarski obtained this mathematical definition of an arithmetical class by using the same general technique that he had employed in 1931 to mathematize the notion of definability in a structure $\mathfrak{A}$. Namely, he replaced the operations on A by the corresponding relations, and treated the relations as sets of sequences of elements of A. He then closed under the Boolean operations of union, intersection, and complement, as well as projection.[14] The result was called CL($\mathfrak{A}$). Then a set C of structures was defined to be an arithmetical class if $C = \mathrm{CL}(\mathfrak{A})$ for some $\mathfrak{A}$. The notion of arithmetical class, defined in this way, made use of standard mathematical notions, not metamathematical ones.

In the same vein, Tarski introduced the notion of arithmetical type. A class $\mathcal{A}$ of structures is an arithmetical type if it is the class of all models of some set, finite or infinite, of first-order sentences. Equivalently, since his language was countable, $\mathcal{A}$ is an arithmetical type if it is the intersection of countably many arithmetical classes.[15] In view of the previous paragraph, this notion of arithmetical type was defined in mathematical, rather than metamathematical, terms.

Every arithmetical class is an arithmetical type.[16] The applications to algebra came from the failure of the converse, i.e. the existence of arithmetical types that are not arithmetical classes. In particular, Tarski proved that the class of all fields of characteristic zero is an arithmetical type that is not an arithmetical class. He noted that the same was true for the class of all algebraically closed fields and for the class of all torsion-free groups (i.e. those without elements of finite order greater than 1).

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[14] Strictly speaking, he now spoke not of the geometric operation of projection (as he had in 1931), but of "outer cylindrification". This was an algebraic analogue of the existential quantifier. He also used "inner cylindrification", which was an algebraic analogue of the universal quantifier.

[15] Henkin pointed out in his *1953* that these two definitions of arithmetical type are not equivalent, in general, if the language is uncountable.

[16] Tarski's terminology of "arithmetical class" and "arithmetical type" (the latter in the sense of a class of the models of a set of sentences) was varied considerably by later researchers; cf. Lyndon *1959*, 144. Tarski himself replaced "arithmetical type" by "arithmetical class in the wider sense" (*1954*, 576–577). In place of "arithmetical class" and "arithmetical type", Grätzer used "elementary class" and "axiomatic class" respectively (*1968*, 256), while Chang and Keisler used "basic elementary class" and "elementary class" (*1973*, 173).

Finally, Tarski considered the notion of being "arithmetically closed". A class $\mathcal{A}$ of structures is arithmetically closed if, whenever it contains $\mathfrak{A}$, it contains every $\mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$. Thus if $\mathcal{A}$ is an arithmetical type, then $\mathcal{A}$ is arithmetically closed, but in general not conversely.

Tarski was particularly concerned to determine whether or not various classes of structures were arithmetical. Sometimes the most convenient way to show that a class failed to be arithmetical was to prove that it was not even arithmetically closed. This was the case, he pointed out, for the class of all Archimedean ordered rings. It followed that for every ordered ring there was an arithmetically equivalent ring which was non-Archimedean. By considering the real numbers, this provided an easy proof for the theorem, previously shown by more complicated methods, that there exists a non-Archimedean ordered field. Likewise, it had recently been established that the class of all simple groups was not arithmetically closed.

Something of Tarski's hopes for these methods, which made essential use of the Strong Completeness Theorem (but only for a countable language), was apparent in the conclusion to his paper: "At first sight the mathematical theory of arithmetical classes seems to be merely a translation of the metamathematics of arithmetical formalism; actually this theory paves the way for constructions and derivations which go far beyond purely metamathematical procedures" (*1952*, 719).

In fact, Tarski had in mind a general research program whose roots lay in his 1931 work on definable sets of real number. As in that earlier work, the elimination of quantifiers was an essential technique.[17] For a given class $\mathcal{A}$ of algebras, he wished to give a complete description of all arithmetical classes relative to $\mathcal{A}$. In the cases where he or his co-workers had succeeded in obtaining such a complete description, they had generally done so by eliminating quantifiers from the theory whose models were $\mathcal{A}$. These classes $\mathcal{A}$ were those of all Abelian groups, of all algebraically closed fields, all Boolean algebras, and all well-orderings. The aim was to

[17] On Tarski's use of elimination of quantifiers, see Doner and Hodges *1988*. On Tarski's work of 1931, see van den Dries *1988*.

classify the structures of the class $\mathcal{A}$ of algebras up to arithmetical equivalence (or, as we would say now, up to elementary equivalence).

The other half of Tarski's research program came into effect when the elimination of quantifiers could not be applied to a class $\mathcal{A}$ of algebras. Then he aimed to prove that the corresponding theory was undecidable. By extending the methods of Gödel, it had been possible to prove the undecidability of all groups, all fields, and all lattices (*1952*, 716). This second half of Tarski's program was visible in the book, *Undecidable Theories*, that he published with Andrzej Mostowski and Raphael Robinson (*1953*), where there was some discussion of Julia Robinson's result that the theory of the rational numbers, that of all fields, and that of all ordered fields are undecidable.

Julia and Raphael Robinson were at Berkeley, and they formed part of the group of logicians that, in the late 1940's, began to expand there under his tutelage. In a reminiscence decades later, John Kelley described the environment in which this occurred:

> It is very easy to describe the Berkeley math department of that period: very strong in analysis, statistics, set theory and the foundations of mathematics, and not strong in other areas.... There was also occasionally a little nervous hostility toward the work in foundations .... This hostility has now pretty well vanished. (Kelley *1989*, 485–486)

4. Preservation Theorems.

One fundamental kind of interaction between logic and algebra that developed during the 1950's can be described as follows: Consider an algebraic procedure, such as taking the direct product of two algebras or applying a homomorphism to an algebra; determine a syntactic form F such that a class of algebras is closed under that algebraic procedure if and only if the class can be axiomatized by sentences of form F. A general problem of this sort would not have arisen prior to the formulation of certain procedures of abstract algebra that occurred in the 1920's and were codified, to a considerable extent, in van der Waerden's *Moderne Algebra 1930*), where the notions of direct product, homomorphism, and isomorphism of groups (and some other structures) were prominent. As van der Waerden was aware, the direct product of two groups is a group.

The earliest instance of this kind of interaction can be found in Birkhoff's paper "On the Structure of Abstract Algebras" (*1935*). He defined an "abstract algebra" to be an ordered pair $A = (\mathfrak{S}, F)$ such that each member of F is an operation on the set $\mathfrak{S}$.[18] In particular, any group, ring, field, lattice, or Boolean algebra was an abstract algebra. The species (or, as we would say now, the arity) of an abstract algebra was an n-tuple of integers, the integer in the i th place being the number of arguments of the function f_i. He called a "law", for an algebra A, any equation between functions f and g of A holding for every substitution of elements of A for the primitive symbols. More generally, a "law" for a set of algebras $\mathfrak{A}$ (of a given species) is a law holding for every algebra in the set. Thus, in this sense, the associative law was a law for groups. As a direct consequence of his definitions, he noted that if P is a law for a set $\mathfrak{A}$ of algebras, then P is a law for any subalgebra (or any homomorphic image) of any member of $\mathfrak{A}$ and for any finite direct product of members of $\mathfrak{A}$. In other words, P is preserved by subalgebras, homomorphisms, and direct products. This simple statement was later regarded as the beginning of equational logic (i.e. no relations other than identity occur). It was also the beginning of the study of such preservation properties.

Birkhoff's notion of abstract algebra was taken up by the logician J. C.C. McKinsey in a paper of 1943 devoted to decision problems for universal sentences in a class $\mathcal{A}$ of algebras, particularly lattices. He was interested in the case when $\mathcal{A}$ was closed under finite direct product. By way of example, he noted that the class of all groups is closed under finite direct product, as is the class of all Boolean algebras. But, he added, the class of all fields is not closed under finite direct product, since the direct product of two fields is not in general a field. He observed that a class $\mathcal{A}$ of algebras is closed under finite direct product if the class can be defined by a set of axioms having a certain form. In particular, this was true if the axioms were all equations or were the negation of a conjunction of equations (*1943*, 65).

[18] He does not appear to have allowed 0-ary functions, i.e. constants.

In the spring of 1947 Tarski posed in his seminar the problem of ascertaining which sentences are preserved by direct products of algebras. It was there that Alfred Horn learned of the problem. Like Birkhoff and McKinsey, Horn made use of equational logic. McKinsey had considered what are now known as Horn sentences, i.e. sentences in prenex normal form whose propositional part is of the form "if *p*, then *q*", where *q* is an equation and *p* is a conjunction of *n* equations, for some *n* (McKinsey *1943*, 65). Horn proved that a universal sentence is preserved by direct products if and only if it is equivalent to some universal Horn sentence. He also showed that Horn sentences are the largest class of prenex sentences preserved by direct products, provided that the form of propositional part of the sentence is considered (*1951*). Questions surrounding the connection between Horn sentences and direct products proved to be quite complicated, and were the subject of research for a number of years.

When reviewing Horn's paper that same year in the *Journal of Symbolic Logic*, R.C. Lyndon noted that one of Horn's theorems could be strengthened as follows. Horn had shown that if a sentence in equational logic is positive[19] and is true in a direct product, then the sentence is true in each component of the product. Lyndon claimed that this theorem may be strengthened to the following: A positive sentence, true in an algebra, is true in any homomorphic image of the algebra (Lyndon *1951*, 217).

By the time that Lyndon published his proof of this theorem in 1959, he had strengthened it to apply not only to algebras but to structures in general. That is, the structure could have relations as well as functions and distinguished elements. He had also strengthened it to an equivalence: A first-order sentence is preserved under homomorphisms if and only if it is equivalent to a positive sentence. Lyndon acknowledged discussions about this theorem with Tarski, Henkin, and Robinson, all of whom improved aspects of the proof.

At that time Lyndon also published a preservation theorem about subdirect products, i.e. those which are substructures of direct products

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[19] A sentence is positive if it is built up from conjunctions, disjunctions, and quantifiers only.

(*1959a*): A first-order sentence holds in a subdirect product of structures if and only if it is equivalent to a special Horn sentence.[20]

Meanwhile, building on the ideas in his 1950 Congress paper discussed above, Tarski had heralded the emergence of "a new branch of metamathematics ... *the theory of models* " *(1954*, 572). One of the questions at the heart of this new theory was the following, which was really a general formulation of the notion of preservation theorem:

> Knowing some structural (formal) properties of a set Σ of sentences, what conclusions can we draw concerning mathematical properties of the correlated class **K** of models? Conversely, knowing some mathematical properties of a class **K** of mathematical systems, what can we say about structural properties of a set Σ which constitutes a postulate system for **K**? (*1954*, 572)

In this spirit, Tarski analyzed the simplest kind of arithmetical types, those that are classes of models for some set Σ of prenex sentences sentences whose quantifiers are universal. Calling such classes universal he proved a preservation theorem about them: If a class **K** of structures is an arithmetical type, then **K** is universal if and only if **K** is closed under substructures (*1954*, 584).

Stimulated by Tarski's paper and by Robinson's *1951a*, Henkin proved an analogous preservation theorem for existential sentences, i.e. those prenex sentences all of whose quantifiers are existential: If **K** is an arithmetical class, then **K** consists of all the models of some existential sentence if and only if is closed under extensions (*1956*, 31).

Similar activity had been under way in Poland, especially by Jerzy Loś at Toruń. In his 1953 work on extending models, Loś was influenced by earlier researches of Tarski, Robinson, and Maltsev. As part of his work, Loś independently discovered Tarski's result, mentioned above, on

[20] A special Horn formula is a formula having the form "if p, then q", where p is positive and q is an atomic formula, or is obtained from a formula of that form by conjunction and universal quantification. A special Horn sentence is a special Horn formula with no free variables.

universal classes. More generally, Loś shared with Robinson a desire to find conditions under which a model of a certain set $\mathcal{A}$ of sentences was also a model of an additional set $\mathfrak{C}$ of sentences.[21] Such a condition, Loś noted, was familiar for the extension of a commutative ring to a field, and had been found by Maltsev for the extension of a semigroup to a group. Loś solved the problem for the extension of a group to an ordered group, and emphasized the open problem of extending a semigroup to a ring.

Loś, together with R Suszko at Warsaw, continued to pursue related questions on extending models. They gave a condition for extending a family of models (*1955a*). That same year they announced a generalization for earlier preservation theorems by considering universal-existential sentences, i.e. those prenex sentences such that all universal quantifiers precede all existential ones: If **K** is an arithmetical type, then **K** consists of all the models of some set of universal-existential sentences if and only if **K** is closed under unions of chains in **K**.[22] At Berkeley, this theorem of Loś and Suszko was extended by Chang (*1956, 1959*).

In 1955 Loś also introduced the important notion of ultraproduct. He based it on his earlier reduced products ("champs logiques"), which were like quotient algebras. His most important result, later known as Loś's Theorem, was that ultraproducts preserve first-order sentences.[23]

Ultraproducts quickly led to a great deal of research, largely stimulated by Tarski. This began when he recognized that a special case of direct product, one used by Chang and Morel (*1956, 1958*) to show that if a class **K** of structures is closed under direct product, then **K** need not be definable by Horn sentences, yielded a proof of the Compactness Theorem for a certain classes of sentences. T.E. Frayne and Dana Scott formulated a better definition of reduced product, which they announced with Tarski (*1958*). Frayne and Scott came close to expressing elementary equivalence

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[21] Loś *1955a*.For this phenomenon Loś borrowed the term "persistent" from Robinson *1951a*, where it had been applied to Abelian groups and fields.

[22] Loś and Suszko *1955*. A detailed proof was given in their *1957*.

[23] Los̨ *1955*, 105. Kochen recognized that Skolem's construction of a nonstandard model of arithmetic (*1934*) used, in effect, a special case of the ultraproduct construction; cf. Frayne, Morel, and Scott *1962*,195.

in algebraic terms by showing that if two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, then some ultrapower of $\mathfrak{A}$ is isomorphic to some elementary extension of $\mathfrak{B}$ *(1958)*. The proofs of these results appeared in a joint paper by Frayne, Morel, and Scott (*1962*).

Soon after the announcements of those results in 1958, the notion of elementary equivalence was expressed in algebraic terms by using ultraproducts. The first such result was discovered by Simon Kochen at Princeton, where his doctoral dissertation (*1959a*) dealt with ultraproducts: Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if they have isomorphic limit ultrapowers (*1959, 1961*). Keisler, who was then doing his doctoral dissertation under Tarski, found many results on ultraproducts, and improved Kochen's result to give a simpler characterization of elementary equivalence: Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if they have isomorphic ultrapowers (Keisler *1961*). At the time Keisler's result assumed the Generalized Continuum Hypothesis. A decade later Saharon Shelah, now widely regarded as the greatest living logician, gave a new proof dispensing with that hypothesis (*1972*).

Using reduced products, Keisler also settled the open problem, discussed above, of determining which products are preserved by Horn sentences. By means of the Generalized Continuum Hypothesis (*196la*), and later merely by the Continuum Hypothesis (*1965*), Keisler showed that a sentence is a Horn sentence if and only if it is preserved by reduced products. Fred Galvin, in his doctoral dissertation, demonstrated the same result without using the Continuum Hypothesis (*1965*).

Ultraproducts quickly became a standard tool in model theory and served as the focus of the first textbook in the subject, Bell and Slomson's *Models and Ultraproducts* (*1969*). Moreover, early in the 1960's ultraproducts were used to give two major results outside of logic.

The first of these results was in set theory. Using the work of his student William Hanf on compactness in infinitary logic, Tarski solved a problem that had been open for thirty years by showing that the first strongly inaccessible cardinal was not a measurable cardinal. In fact, Tarski established that measurable cardinals are very large, in the following sense: If κ is a measurable cardinal then there are κ strongly inaccessible cardinals below κ (*1962*). Aware of Tarski's result, Scott *(1961)* used ultraproducts to show that if there is a measurable cardinal then Gödel's Axiom of Constructibility is false. Scott's result was refined by Frederick Rowbottom (*1964*), in his dissertation done under Keisler: If there is a measurable cardinal, then the Axiom of Constructibility already fails for the real numbers. This line of results has since become one of the most fruitful in contemporary set theory.

The second result using ultraproducts was Robinson's discovery, in the fall of 1960, of non-standard analysis. He was inspired in part by the non-standard models of arithmetic first found by Skolem (Robinson *1966*, vii). Ultraproducts were one way of obtaining such a non-standard model of the real numbers, the Compactness Theorem for an uncountable language being the other. Non-standard analysis, which showed that infinitesimals can be used rigorously, has proved very fruitful in analysis and in applied mathematics. Albeverio *et al.* (*1986*) discuss recent applications of non-standard analysis to probability theory, stochastic processes, and Brownian motion.

5. Some Concluding Comments.

By way of conclusion, two general remarks should be made. The first is that mathematicians whose research is not in logic have occasionally supervised graduate students in logic. An important instance is the algebraist Saunders Mac Lane, whose own doctoral dissertation, supervised at Göttingen by Paul Bernays and Hermann Weyl, was in logic. In the late 1950's, Mac Lane had a graduate student named Michael Morley, who came to Mac Lane with an application of the Compactness Theorem. Morley hoped that this would be his thesis, but Mac Lane told him to find a deeper theorem (Mac Lane *1989*, 520). So Morley went to Berkeley to learn more logic, and, after working with Robert Vaught (a former student of Tarski), found the seminal result known as Morley's Theorem: If a countable first-order theory is categorical in one uncountable power, it is categorical in all uncountable powers. In effect, Mac Lane and Vaught functioned as joint supervisors for Morley's dissertation, to the enormous enrichment of logic. The subject which began with that dissertation, stability theory, has become one of the chief foci for logic today.

The second remark concerns the light that textbooks, and handbooks, can shed on the transmission of interactions between logic and mathematics, in the period under discussion. These interactions have, in good part, taken place between model theory in logic and universal algebra in mathematics. They were already enshrined in the first textbooks of universal algebra, by Cohn (*1965*) and Grätzer (*1968*). They were apparent as well in the most important textbook of model theory, that of Chang and Keisler, who noted: "The line between universal algebra and model theory is sometimes fuzzy; our own usage is explained by the equation universal algebra + logic = model theory" (*1973*, 1). The fundamental role of model theory within mathematical logic is suggested by the part that model theory plays in the *Handbook of Mathematical Logic* (Barwise *1977*). Its editor, Jon Barwise, opens the volume by discussing which algebraic notions can be expressed in first-order logic, and which not. The volume considers model theory before turning to the other parts of logic.

Mathematical logic can be found even in some general textbooks in abstract algebra, such as *Basic Algebra I* by Nathan Jacobson. Its fifth chapter culminated with Sturm's theorem of 1836 (for determining the exact number of real roots of a polynomial) and, generalizing that theorem, Tarski's decision procedure for real closed fields (1948). "It is worth mentioning also", Jacobson concluded, "that Tarski's theorem has had an important application to partial differential equations. This is a striking example of the interconnectedness of mathematics in that a result which originated in mathematical logic has an important consequence in one of the most applied parts of mathematics" (*1985*, 341).

Despite this evidence that logic has contributed to both algebra and analysis, model theory has not always found a comfortable home among logicians. The recursion theorist Gerald Sacks tells us that "Burton Dreben ... once asked with characteristic sweetness: 'Does model theory have anything to do with logic?" (*1972*, 1). One suspects that van Heijenoort, who shared a Quinean bias with his friend Dreben, held similar views. "Model theory", van Heijenoort tells us, "has bloomed into an extensive science. But this new discipline does not seem to care much about what was the original object of semantics, namely meaning. It has become an abstract mathematical science, whose philosophical implications are scant" (*1977*, 185).

Perhaps it is the lack of philosophical concern, on the part of modern mathematical logic, that troubled both van Heijenoort and Grattan-Guinness. Perhaps this is why Grattan-Guinness saw the mathematicians as dominating the logicians in Poland between the two World Wars, especially in the journal *Fundamenta Mathematicae* (*1988*, 80).

Yet one may have a different vision of mathematics and logic in Poland between the wars. During those years, Tarski published many of his papers on mathematical logic in *Fundamenta*, just as he published many of his papers on set theory there, thus encouraging interactions between logic and the rest of mathematics. Tarski was the chief catalyst of such interactions after the Second World War, when their center was no longer in Europe but in the United States. More and more, these fundamental interactions have depended on using mathematical logic to give significant results about parts of mathematics beyond the pale of logic, and of philosophy.

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