Is Cantorian set theory an iterative conception of set?*, **

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Authors such as Boolos (1971), Parsons (1975) and Wang (1974) maintain the view that the notion of set as addressed by Cantor fits into "the iterative conception of set."

Although these and other authors in this field refer to "Cantor's set theory," it would appear that Cantor did not adhere to only one set theory throughout his works. It would also seem, therefore, that there is no such thing as _the_ Cantorian set theory. As I see it, Cantor presents two very distinct theories. One of these is to be found primarily in _Grundlagen einer allgemeinen Mannigfaltigkeitslehre_ (1883) and in _Beiträge zur Begründung der transfiniten Mengenlehre_ (1895-97), and the other appears in his correspondence with Dedekind [Cantor and Dedekind(1899)] and with Jourdain [Grattan-Guinness (1971b)]. There are disagreements as to which concept of set underlies the former but there is no doubt that the latter is a theory of limitation of size. Cantorian set theory is usually considered to be that found in _Beiträge_, and for this reason the immediate question is whether or not the theory in _Beiträge_ belongs to the iterative conception. In my view, which I propose to justify in this work, it does not: in _Grundlagen_ and in _Beiträge_ Cantor propounds the same theory, which I shall henceforth call the "first theory", and it is a paradigm of the naïve conception of set.

The fundamental characteristic of the iterative conception (IC) is the intuitive idea that sets are formed by previously available elements. "Previously available" is usually defined more precisely by structuring the sets in levels in such a way that the elements of a set never form on a higher level than that which comes immediately before the level of formation of the set. Zermelo's set theory (1908) is a good formal definition of the intuitive ideas behind the iterative conception. The axioms which he isolated as being the basic principles of his theory are, first, extensionality, followed by some axioms of existence: null set, pairing, union, power set, infinity, subset formation and choice; each one of these guaranteeing the existence of a certain type of sets. Strictly speaking, all these axioms are related to the IC, with the exception of choice: if this last is added, the resulting theory becomes an extension of the basic concept, but does not belong to the idea which originally inspired it. In other words, according to the iterative concept, we obtain new sets by applying the constructive axioms to previously available sets beginning with the empty one.

According to Boolos, the IC also includes the thesis that at each level of set formation, there exist as many sets as can be constructed from the elements making up the previous level. This formation of all possible sets at a given level is the key to the deduction of Zermelo's axioms from the IC. For example, the empty set exists at the first level; if two sets, A and B, form at some level, not necessarily the same one, then on a subsequent level a set must form which consists exactly of the members of A and B. The idea behind the power set axiom in the IC is that if at a certain level, C and all its subsets form, then the set of all the subsets of C _must_ form at a subsequent level, and so on.

If Cantor's first theory embraces the iterative concept, this would mean, at least, the following:

i) the ideas expounded in Zermelo's theory are those that govern the theory in _Beiträge_, and

ii) since it is assumed at present that the IC generates consistent set theories, it follows that the first theory is not contradictory or at least that any contradictions that may arise are brought about by misunderstandings which could be dispelled by more detailed explanation of the theory. One interpretation of the contradictions as a misunderstanding of Cantorian hypotheses has been put forward by Hao Wang[1].

1. Cantor's ideas on sets and Zermelo's axioms

First, let us consider how Cantor approaches the notion of set; clear characterizations are to be found in only two places in his works. The first is a note at the end of Grundlagen:

> "By the words 'multiplicity' or 'set' I generally understand all those [things] that can be thought of as one, that is, the whole group of definite elements that can all be related to one another in one whole by means of a law."[2]

The second comes at the beginning of Beiträge;

> "By an 'aggregate' we understand any collection M of definite and well-distinguished objects m (called the "elements" of M) of our intuition or our thinking into a whole."[3]

We must decide first of all whether both "definitions" are equivalent, as they are obviously not identical. Both passages suggest in different ways that sets are collections of individuals which may be considered simple objects from some point of view. In the description from Grundlagen the elements of each set are interconnected "by means of a law" which, I think, amounts to saying that the elements of a set fall into one concept, related to Frege's idea. In the first "definition," therefore, the relationship between the elements of the same set is established by a concept. Cantor never speaks of sets as "extensions of concepts," although he calls transfinite numbers -- cardinals and ordinals -- "general concepts" (Allgemeinbegriffen). For this reason, I do not endorse the view that Cantor and Frege were of exactly the same opinion, although it does seems to me that in Grundlagen every property can bring together the elements which fall under it in a set. In this way the definition to which I refer closely approaches Frege's view.

Gödel distinguished two different notions of set: in one, a set may be obtained "from the integers (or some other well-defined objects) by iterated application of the operation 'set of'."[4] This conception does not give rise to contradictions. In the other, a set may be considered as "something obtained by dividing the totality of all existing things into two categories."[5] The former is the iterative concept of set and the latter is Frege's concept. In an attempt to explain the divergent reactions to the contradictions arising in Cantor and Frege which Wang found so surprising, he offers two reasons: (1) the different concepts of set they advocated (Cantor proposing a mathematical notion and Frege a logical one) and (2) the different positions these authors adopted in relation to set theory (Cantor spoke from within, Frege from without)[6].

Taking Gödel and Wang's lead, we may call <u>the logical conception of sets</u> the view that sets are defined by splitting up the Universe into two parts. To one of these parts belong those objects which have a specified property and to the other, those which do not, the paradigm of which is the work of Frege. Let us call <u>the mathematical conception</u> the view that sets are formed in accordance with some definite laws which in the 20th Century have given rise to the axioms of set theory.

In my opinion, both Cantorian "definitions" of the notion of set, as seen in <u>Grundlagen</u> and in <u>Beiträge</u>, belong to the logical conception. It is evident that the notion of set in <u>Grundlagen</u> is the logical one. In the second "definition," however, the underlying concept of set is more ambiguous, so that the definition in <u>Beiträge</u> cannot be automatically ascribed to the logical conception, even in the loose sense that <u>Grundlagen</u>'s can. Because of this ambiguity, some authors, Wang among them, are convinced that the <u>Beiträge</u>'s definition does not belong to the logical conception at all but rather to the mathematical one. I shall endeavour to demonstrate that the notion of set in both works is the same and that the theory in Beiträge is as naïve as that in Grundlagen. There is nothing in Beiträge that is not to be found in Grundlagen, with the sole exception of the definition of covering. The latter is a philosophical, rather chaotic work, whereas the former is a very systematic one, written for mathematicians, without any of the philosophical considerations of Grundlagen. Beiträge may be seen as the mathematical skeleton of Grundlagen. But before putting forward my reasons for supporting that the theory found in these two works, i.e., the first theory, encloses a naïve conception, let us look briefly into the matter of paradoxes.

2. Cantor's Paradoxes

Let us first discuss whether Cantor's first theory is consistent, and if not, whether Cantor himself was aware of this. Both Wang and Hallett agree that the theory in Grundlagen and Beiträge is consistent, although for different reasons. While Wang mantains that the concept of set underlying the definition in Beiträge is iterative and even calls it "genetic"[7], Hallett endorses the view that Cantor's is a theory of limitation of size from the outset. According to Hallett, the differences in style between Grundlagen and the correspondence with Dedekind and Jourdain merely reveal a superficial shift from a theological account of his theory of transfinite numbers to a mathematical one and not a profound transformation of a naïve concept of set into a consistent theory of limitation of size[8].

Fraenkel[9], Grattan-Guinness[10], Dauben[11] and Meschkowski[12] take the view that it was Cantor himselfwho first discovered contradictions in his system, and so he attempted to clarify them in his correspondence with Dedekind and with Jourdain. Fraenkel and Dauben do not, however, give any direct evidence to support this hypothesis and Grattan-Guinness[13] and Meschkowski[14] rely on

Bernstein's report of the letter written by Cantor to Hilbert in 1896, now lost. On the other hand, Moore and Garciadiego[15] propound the idea that Cantor himself realized these paradoxes is a myth which came about through Bernstein's misinterpretation of Jourdain's paper (1904)[16]. Bernstein understood that Jourdain stated that Cantor discovered Burali-Forti's contradiction. In fact, Jourdain merely claimed that in 1895 Cantor had found a proof of the well-ordering principle. If we are to believe Moore and Garciadiego, there is little support to the hypothesis that Cantor was aware of the so-called Cantor and Burali-Forti paradoxes before 1899. Moore and Garciadiego are convinced that neither Cantor nor Burali-Forti considered these paradoxes as anything more dangerous than a contradiction in an indirect proof. According to Moore and Garciadiego, Cantor directed what we call the paradoxes of the greatest cardinal and the greatest ordinal against the thesis that there are a maximum in the series of cardinal and ordinal numbers and Burali-Forti directed the contradiction against the idea that every set can be well-ordered. They are right in the case of Burali-Forti, but not, I think, as far as Cantor is concerned. In support of their point they quote[17] an unpublished letter that Cantor wrote to G.C.Young in 1907. Fraenkel, who believed that Cantor discovered the Burali-Forti paradox before 1895, knew of this letter from Cantor to Young but did not interpret it as a proof of the consistency of the first theory.

In the letter in question, Cantor refers to his notes (1) and (2)[18] at the end of <u>Grundlagen</u>, where he calls the series of transfinite numbers "absolutely infinite," as evidence that his theory was conceived from the very beginning as incorporating the idea of limitation of size, and therefore as a consistent theory in which Burali-Forti's paradox could not appear. In the letter he claims that sets are just those multiplicities which can be conceived as unities and that absolutely infinite multiplicities, such as the totality of transfinite numbers, cannot[19] and suggests that he had this view on sets at least since <u>Grundlagen</u>.

If Cantor had remembered correctly, then, as Hallett proposes, there would be only one Cantorian set theory incorporating a device of limitation of size which would immunize it against contradictions. But, I think, his memory failed him: in the conclusion of the famous paper published in 1890[20] in which he presents his "diagonal method" Cantor speaks of the series of cardinal numbers as a "well-ordered set." I quote:

> "In Grundlagen einer allgemeinen Mannigfaltigkeitslehre I have already shown by different means that the powers have no maximum; there it has been proved that the multiplicity of all powers forms a 'well-ordered set', when we think of these latter as ordered by their magnitude [...]."[21]

For our purposes, it is irrelevant what he did, but the fact is that in 1890 he still believed that he had two different proofs that the totality of transfinite cardinal numbers could be arranged to form a well-ordered set, although it is true that in Grundlagen he didn't use the term "set" to refer to this totality.

What is more, the paragraph quoted above counts against Hallett's thesis which in any case has little textual support and amounts only to two passages. One of them is a note in Grundlagen and the other belongs to "Mitteilungen zur Lehre vom Transfiniten" (1887-8)[22]. In the first of these texts, Cantor states very clearly that the Absolute cannot be mathematically characterized, an idea designated by Hallet as the "principle of Absolute infinity."[23] Cantor affirms that the succession of the number classes has no end but that this does not bring us any closer to the Absolute. He maintains that we cannot understand the Absolute, only recognize it. In this regard he says:

> "The absolutely infinite sequence of numbers therefore seems to me in a certain sense a suitable symbol of the Absolute."[24]

And in an extract from Mitteilungen we can read:

"The *transfinite* with its plenitude of formations and forms necessarily indicates an Absolute, a 'true infinite' whose magnitude is capable of no increase or diminution, and is therefore to be looked upon quantitatively as an absolute maximum."[25]

The statement that the endless series of cardinal numbers may be considered in some sense as a symbol of the Absolute does not seem to me to affirm a strong relation between the two domains. From the terms used by Cantor in this note in Grundlagen we cannot conclude that he thought of the Absolute as some kind of insuperable limit of the transfinite numbers in a mathematical sense. As I see it, Cantor is trying to allay suspicions on the part of theologians against his number theory. His words imply that the theory of transfinite numbers is not dangerous to Roman Catholic dogma, but rather eminently manifests the power of God.

Moreover, the text from Mitteilungen continues as follows:

"The latter [the Absolute] in some way transcends human understanding and eludes mathematical calculation; on the other hand, the transfinite not only fills the extensive realm of possibility of the understanding of God, but also offers a rich domain, forever growing in ideal research, and also in my opinion to a certain extent in the created world and in different relationships with reality and reaches existence to manifest the excellence of the Creator even more strongly [...] than would have occurred in a simple 'finite world'."[26]

So the simplest interpretation of the two texts cited by Hallett in support of his theory is, in my view, that Cantor thought that the infinity of domains of abstract and concrete entities shows the magnificence of God, in the sense in which the infinity of Creation manifests the omnipotence of the Creator. It has nothing to do with the mathematical device of limitation of size.

3. Grundlagen and Beiträge: the naive conception

It is neither my belief, nor was it Cantor's intention, that the first theory should be seen as an example of the iterative conception; this is clear from the letter sent by the author to Young in 1907, mentioned above. The reasons are as follows:

1. The first theory is contradictory: as we have seen, Cantor explicitly says that the multiplicity of all cardinal numbers constitutes a well ordered set[27].

This reason corresponds to point (ii) above and counts against Wang's interpretation of the paradoxes as products of misunderstandings, since it was Cantor himself who used the term "set" applied to the totalitity of powers. However, the following reason strikes me as even more convincing:

2. In reply to point (i) there is the question of why the solution that Cantor arrived at does not particularly resemble Zermelo's theory. If the concept in Beiträge is iterative, it would be reasonable to expect that Cantor should state this subsequently. Far from doing any such thing, in his letters to Dedekind and Jourdain, he outlines a theory of limitation of size, making a sharp distinction between collections which are sets and those which are not. The limiting criteria of this distinction employ the ordinal numbers: all collections of this size are inconsistent, and therefore are not sets.

In 1904, in a letter to Jourdain[28], Cantor replies to a problem that Russell poses in The Principles of Mathematics, concerning Cantor's theorem. Russell's difficulty lies in the apparent contradiction that exists between Cantor's affirmation that there can be no cardinal number higher than all others, and the fact that some collections appear to contain all possible elements, such as the set of all things. If Cantor had in mind something like the IC, the expected reply to Russell's objection would be that the set of all things does not exist, since it cannot be formed on any level: in this conception the elements of a set must be formed in a previous phase to that of the set, and the set of all things must belong to itself, which is impossible in the IC. Cantor's answer, however, is completely different: his theorem cannot be applied to the collection of all things since this last is not a set, but rather an inconsistent multiplicity like the totality of ordinal numbers; that is, a multiplicity that cannot be considered as a unit without giving rise to contradictions. In this reply, there is nothing that may be interpreted within the iterative conception. Also, in the letter to Young of 1907 he criticizes Burali-Forti's assumption that transfinite ordinal numbers form a set rather than an inconsistent multiplicity as he affirms he has believed at least since Grundlagen.

In conclusion, the difference between sets and inconsistent multiplicities falls into what Russell called theories of limitation of size[29]: there exists a limit to the size that a set can have. Russell refers to certain processes in which the totality of objects generated thereby can be considered as the final limit process. In fact this is what occurs in the so-called Cantor and Burali-Forti paradoxes: certain numbers are obtained, an aleph and an ordinal respectively, which should be the last transfinite numbers. Russell suggests that when faced with a process of this type, the best thing is to suppose that the union of all the individuals generated thereby does not form a set. Thus there exists no set either of all the alephs or of all the ordinals. The totality of all these individuals only produces inconsistent multiplicities, in Cantor's terminology. The strategy of limitation of size only makes sense *a posteriori*, that is, once the contradictions are known.

On the other hand, the IC imposes no restrictions on the size of sets, but rather on how they are formed. If Cantor's theory were iterative, it would not be necessary to limit the size of existing sets, since the "sets" responsible for the problems would quite simply not appear. Thus the IC proposes a consistent theory of sets, defensible even if the contradictions had never occured. In the limitation of size one finds no intuitive thesis concerning sets which is not based on the elimination of the paradoxes. Moreover, in 1899 Cantor explicitly formulated an axiom of replacement:

> "Two equivalent multiplicities are either bothe 'sets' or both inconsistent"[30]

This formulation is the basis of the theories of limitation of size (every multiplicity which is equivalent to a set is e set) while it is not easily justifiable from the iterative conception (two equivalent multiplicities do not necessarily have to share the same fate - one may exist if its elements precede the multiplicity in some specified way, while the other may be rejected if it does not fulfill these conditions).

For all these reasons, it is plain that in his correspondence, Cantor presents a system of limitation of size for set theory. This fact leads me to believe that in Beiträge there is a naïve conception in which Cantor found contradictions[31]. If, from the beginning, the author had envisaged the IC, these contradictions would not have occurred; and, if indeed he had discovered the IC after becoming aware of them, how can one explain that in 1899, only two years after the publication of the second part of Beiträge, he was quite clearly in favour of the strategy of limitation of size, which intuitively is much less justifiable than IC? It is more plausible to suppose that in Beiträge he presents a contradictory theory the scope of which he subsequently limited. By this I am not suggesting that the characterization of set which appears in this work does not in any way suggest the IC, but this is due to its ambiguity which makes it very easy to see in it ideas that appeared later. Far from proving the IC, the echoes here and elsewhere in his works of the iterative affirmation that a set is preceded by its elements merely show that in Cantor's mind sets are objects remarkably similar to groups of physical objects: a flock of sheep, a constellation of stars or a pack of cards, to the extent that in his theory the empty set does not exist. This is the intuitive origin of the notion of set, but once one introduces and accepts the conception that there are sets whose elements are abstracts, the idea that the elements pre-exist the set disappears, and we are left with the thesis that any group of objects is an irreproachable candidate for the title "set."

NOTES

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1. H. Wang (1974) p. 541
2. Cantor (1932), p. 204
3. Ibid., p. 282
4. Gödel (1947) pp. 474-5
5. Ibid., p. 475
6. H. Wang, op. cit., p. 541
7. Ibid., p. 538
8. M. Hallett (1984) p. 9
9. A. Fraenkel (1932) p. 470
10. I. Grattan-Guinness (1971a) p. 365
11. J. Dauben (1979) p. 241
12. H. Meschkowski (1967) p. 144
13. *loc. cit.*
14. *loc. cit.*, Meschkowski does not mention Bernstein. He just says: "*Cantor* hatte sie [die erste Cantorsche Antinomie] im Jahre 1895 entdeckt und in einem Brief *Hilbert* mitgeteilt."
15. Moore & Garciadiego (1981) p. 338
16. Ph. Jourdain (1904) cited by Moore & Garciadiego (1981) p. 338
17. Fraenkel (1932) p. 470, footnote
18. Cantor (1932) p. 204
19. This letter can be found in Moore & Garciadiego (1981), p. 342, English translation, and p. 345, the original German version.
20. Cantor (1890). In Cantor (1932) pp. 278-81.
21. Cantor (1932) p. 280.

22. Cantor (1887-8). In Cantor (1932) pp. 378-439. Hereafter referred to as Mitteilungen.
23. M. Hallet (1981) p. 7
24. Cantor (1932) p. 205. The translation I quote is Hallett's, Hallett (1981) p. 42.
25. Cantor (1932) p. 405. Hallett's translation in Hallett (1981) p. 44.
26. Cantor (1932) pp. 405-6
27. Cantor (1890). In Cantor (1932) p. 280
28. This letter can be found in I. Grattan-Guinness (1971b) p. 119
29. B. Russell (1907)
30. Cantor (1932), p. 444
31. In the definition found in Beiträge, Cantor points out that the elements of a set are objects "of our intuition or of our thought." Here there is no attempt to restrict the scope of this definition, since in an unpublished text of 1913(Meschkowski, 1967, p. 114), Cantor explicitly states that everything that exists may be an object of our thought, "Jedes Seiende kann Gegenstand unsres Denkens sein." The reference to intuition and thought means rather that, in sets, not even their elements -- qua elements of a set -- are spacial-temporal objects, but, instead, exist as abstract objects.

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Is Cantorian set theory an iterative conception of set?

Abstract

The aim of this paper is to argue against the view that Cantor's is an iterative conception of set. I shall distinguish between the theory found in Grundlagen and Beiträge, which I call the "first theory," and that expounded in Cantor's correspondence with Dedekind and with Jourdain. I consider that Cantor's first theory encloses a naïve and unrestricted concept of set, and that the set-theoretical paradoxes *do* therefore follow from it. In this sense, I do not support the idea that Cantorian set theory is an iterative conception of set, as maintained by Boolos, Parsons and Wang, among others, or Hallett's interpretation, which considers it from the outset as a theory of limitation of size.

Is Cantorian set theory an iterative conception of set?

Resumen

El objetivo del presente artículo es argumentar en contra de la idea de que la concepción de los conjuntos que aparece en Grundlagen y Beiträge es una concepción iterativa. Denomino a la teoría que se encuentra en estas dos obras "la primera teoría" para distinguirla de la que aparece en la correspondencia con Dedekind y Jourdain. En mi opinión, la primera teoría es una teoría ingenua que incorpora una concepción irrestricta de lo que es un conjunto y que, por tanto, es inconsistente. Dicho de otro modo, las paradojas conjuntistas se siguen de la primera teoría y no son meros productos de una deficiente interpretación de las ideas de Cantor. Mi punto de vista no coincide, pues, ni con el de los que, como Boolos, Parsons y Wang, sostienen que Cantor mantuvo una concepción iterativa ni con la tesis, defendida por Hallett, de que Cantor presenta desde el principio de su obra una teoría de la limitación del tamaño.