Ernst Schröder and the Distribution of Quantifiers

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The third volume of Schröder's *Vorlesungen über die Algebra der Logik (Exakte Logik)* is entitled *Algebra und Logik der Relative*; it presents the logic of relations in the framework of the algebra of logic. The only extant part of this volume is the „Erste Abteilung",[1] being the last part of the *Vorlesungen* published by Schröder himself: with 1895 as the year of its appearance, it precedes the second „Abteilung" of the second volume edited by Eugen Müller in 1905 (after Schröder's death in 1902), and no further „Abteilung" followed.

Among the twelve lectures which form the first „Abteilung", the second lecture provides "the formal foundations, in particular of the algebra of binary relatives". In § 3, Schröder introduces the "fundamental stipulations", including rules for the use of the signs $\sum$ and $\prod$ for product and sum ("which were omitted at the beginning", as Schröder says on p. 35). These signs correspond to our modern universal quantifier ($\bigwedge$ or $\forall$) and existential quantifier ($\bigvee$ or $\exists$), and propositional schemata containing them belong to "quantificational logic". But Schröder does not delimit such a separate domain; he regards $\prod_x Ax$ and $\sum_y By$ as generalizations of $A_1 A_2 \ldots A_m$ and $B_1 + B_2 + \ldots + B_n$, respectively, and treats propositional schemata containing them within his "propositional calculus" („Aussagenkalkül") without much ado.

[1] Schröder, Ernst: *Algebra und Logik der Relative, der Vorlesungen über die Algebra der Logik dritter Band. Erste Abteilung.* B.G.Teubner: Leipzig 1895. The reprint (as "second edition" by Chelsea Publishing Company (Bronx, N.Y. 1966) contains some changes (see below, note 7) and is conjoined with Ernst Schröder, *Abriß der Algebra der Logik. Bearbeitet im Auftrag der Deutschen Mathematiker-Vereinigung von Eugen Müller [Teil I und II]*, Leipzig 1909/1910. On Müller's editorial work on the logical parts of Schröder's scientific estate see Volker Peckhaus, „Karl Eugen Müller (1865–1932) und seine Rolle in der Entwicklung der Algebra der Logik", *History and Philosophy of Logic* 8 (1987), 43–56.

It is already in the realm of classical quantificational logic (and Schröder did not know of any other!) that we encounter the tricky structures of some propositional schemata in which quantifiers and statement connectives interact. I wish to report on an error in Schröder's exposition of the distribution of a quantifier preceding a conditional (Schröder: "Subsumtion"), an error partially — but only partially — corrected by himself.[2] The relevant passage reads as follows:[3]

> If A_u is independent of (constant with respect to) u, i.e.: if u does not occur in the proposition here figuring as a general term, we may omit as irrelevant (also in the formulae) the suffix u of the proposition A_u and indicate the latter by A alone. Then, of course, we have once more:
>
> $$\delta) \qquad \prod_u A = A \text{ and } \sum_u A = A\,.$$
>
> Likewise, if B_u is a proposition with respect to u, and B a proposition which is constant with respect to u, we have, moreover, the schemata:
>
> $$\varepsilon) \quad \prod_u (A \mathrel{\in\!\!\!\!\!-} B_u) = (A \mathrel{\in\!\!\!\!\!-} \prod_u B_u) \mid \prod_u (A_u \mathrel{\in\!\!\!\!\!-} B) = (\sum_u A_u \mathrel{\in\!\!\!\!\!-} B)$$
>
> which may be combined into the more general schema:
>
> $$\prod_u (A_u \mathrel{\in\!\!\!\!\!-} B_u) = (\sum_u A_u \mathrel{\in\!\!\!\!\!-} \prod_u B_u)$$
>
> or to the even more general one:
>
> $$\zeta) \qquad \prod_{u,v} \text{ sive } \prod_u \prod_v (A_u \mathrel{\in\!\!\!\!\!-} B_v) = (\sum_u A_u \mathrel{\in\!\!\!\!\!-} \prod_v B_v)\,,$$
>
> where the domain of v may differ from that of u.
>
> In analogy to these Peircean schemata, the following schemata (of mine) are valid
>
> $$\eta) \quad \sum_u (A_u \mathrel{\in\!\!\!\!\!-} B) = (\prod_u A_u \mathrel{\in\!\!\!\!\!-} B) \mid \sum_u (A \mathrel{\in\!\!\!\!\!-} B_u) = (A \mathrel{\in\!\!\!\!\!-} \sum_u B_u)$$

[2] An earlier German version of the present study, more elaborate because of cross-references to my lecture course on Formal Logic, was presented for discussion in the History of Logic Colloquium at the University of Erlangen-Nürnberg on 27 October 1987. The English version has profited from valuable suggestions by Thony Christie and Volker Peckhaus.

[3] op.cit., 39*f*.

and may be combined into the more general

$$\sum_u (A_u \Subset B_u) = (\prod_u A_u \Subset \sum_u B_u)$$

as well as into the even more general schema

$$\theta) \qquad \sum_{u,v} \text{ sive } \sum_u \sum_v (A_u \Subset B_v) = (\prod_u A_u \Subset \sum_v B_v) \, .$$

Specializing to $A = 1$ in ε) and η) (and changing the remaining B to A) or to $B = 0$, one gets the schemata:

$$\iota) \quad \begin{array}{l|l} \prod_u (A_u = 1) = (\prod_u A_u = 1) & \prod_u (A_u = 0) = (\sum_u A_u = 0) \\ \sum_u (A_u = 0) = (\prod_u A_u = 0) & \sum_u (A_u = 1) = (\sum_u A_u = 1) \end{array}$$

of which the first and the last are uninformative in view of the "specific principle" of the propositional calculus,

$$(A = 1) = A \, ,$$

whereas the other two have plenty of applications.

Finally we must mention, because of its frequent application, the schema:

$$\kappa) \qquad \prod_u (A_u \Subset B_u) \Subset \left\{ \begin{array}{c} (\prod_u A_u \Subset \prod_u B_u) \\ (\sum_u A_u \Subset \sum_u B_u) \end{array} \right\} \Subset \sum_u (A_u \Subset B_u) \, .$$

Here it is sufficient to take one (or the other) of the two middle schemata (the two subsumptions), be it as a thesis (assertion, conclusion) or as an hypothesis (presupposition, condition).

The question is how to distribute a quantifier in front of a classical conditional to the latter's components, including the case that one of the components is not a propositional function but a pure propositional schema. It will be convenient to list the most important formulae investigated by Schröder in modern notation, using our standard variables and quantifiers and replacing Schröder's subsumption sign " $\Subset$ " by the classical conditional "$\supset$" between propositional functions, and by the implication sign "$\prec$" between propositional schemata, and his equality sign by the classical sign " $\succ\!\prec$ " of equivalence, i.e. of bidirectional implication. Formulae which Schröder places on the left or right side of the same line, will be distinguished by subscripts "1" or "2" added to their designations. Then, first, we have

$$\varepsilon_1: \quad \bigwedge_x .A \supset Bx. \;\succ\!\prec\; A \supset \bigwedge_x Bx$$

$$\varepsilon_2: \quad \bigwedge_x .Ax \supset B. \;\succ\!\prec\; \bigvee_x Ax \supset B\,.$$

Schröder claims that ε_1 and ε_2 may be combined into the "more general schema"

$$\bigwedge_x .Ax \supset Bx. \succ\!\prec \bigvee_x Ax \supset \bigwedge_y By\,.$$[4]

This equivalence, inserted by Schröder between ε and ζ without designation, may be split up into its two parts (the implications from left to right and from right to left, respectively) which we designate as ε_3 and ε_4:

$$\varepsilon_3: \quad \bigwedge_x .Ax \supset Bx. \quad \prec \bigvee_x Ax \supset \bigwedge_y By$$

$$\varepsilon_4: \quad \bigvee_x Ax \supset \bigwedge_y By \;\prec\; \bigwedge_x .Ax \supset Bx.$$

Schröder claims that the still more general propositional schema (in which the indices x and y may range over different domains) is also valid:

$$\zeta: \quad \bigwedge_x \bigwedge_y .Ax \supset By. \succ\!\prec \bigvee_x Ax \supset \bigwedge_y By\,.$$

Immediately after their presentation, Schröder (p. 39, middle) ascribes the "preceding" propositional schemata to C.S. Peirce (which so far I have been unable to verify). He complements them by "analogous" ones discovered by himself, first, corresponding to the formulae ε, the following schemata η which we will once more list separately as η_1 and η_2:

$$\eta_1: \quad \bigvee_x .Ax \supset B. \;\succ\!\prec\; \bigwedge_x Ax \supset B$$

$$\eta_2: \quad \bigvee_x .A \supset Bx. \;\succ\!\prec\; A \supset \bigvee_x Bx\,.$$

[4] Instead of "x" and "y", Schröder uses only a single variable "u", presumably because of his habit to use different variables (as in formula ζ shortly afterwards) to indicate that they may refer to different domains. By contrast, I shall restrict myself to a single domain, using different variables in order to signalize that different replacements are admitted.

They, too, may according to Schröder be combined into a "more general" propositional schema,[5]

$$\bigvee_x .Ax \supset Bx. \asymp \bigwedge_x Ax \supset \bigvee_y By\ .$$

As before, we list the two directions separately:

$$\begin{array}{llll} \eta_3: & \bigvee_x .Ax \supset Bx. & \prec & \bigwedge_x Ax \supset \bigvee_y By \\ \eta_4: & \bigwedge_x Ax \supset \bigvee_y By & \prec & \bigvee_x .Ax \supset Bx. \end{array}$$

In like manner as before, we are told that the more general equivalence may be further generalized to

$$\theta: \qquad \bigvee_x \bigvee_y .Ax \supset By. \asymp \bigwedge_x Ax \supset \bigvee_y By\ .$$

Taking $B = 0$ in ε_2, we get (keeping Schröder's " $\not\in$ " for the conditional because of its particular suggestiveness in the resulting formula)

$$\bigwedge_x .Ax \not\in 0. \asymp \bigvee_x Ax \not\in 0\ ;$$

so that, since in the algebra of logic "$a = 0$" does the job of our "$\neg a$", we have (as the second and third formula in Schröder's ι)

$$\bigwedge_x \neg Ax \asymp \neg \bigvee_x Ax$$

and (in the same manner from η_1)

$$\bigvee_x \neg Ax \asymp \neg \bigwedge_x Ax\ .$$

Finally, Schröder presents as κ a branching sequence of propositional schemata, which he declares to be "of the most frequent use",

[5]Cf. footnote 4.

$$\kappa: \qquad \bigwedge_x .Ax \supset Bx. \prec \left\{ \begin{array}{l} \bigwedge_x Ax \supset \bigwedge_y By \\ \bigvee_x Ax \supset \bigvee_y By \end{array} \right\} \prec \bigvee_x .Ax \supset Bx.^6$$

Schröder remarks that of the two formulae inside the braces "only the one (or the other) needs to be taken" (p. 40, line $10f.$), but what he wishes to point out is only that the "upper" chain of implications and the "lower" chain are both valid; he does not wish to claim that the upper and the lower formula inside the braces are equivalent (which indeed would be wrong — neither of the two implies the other one, as is easily shown).

But what has been said so far, does not yet describe our "relevant text" (quoted above) completely. Anyone using, instead of the rare first edition of Schröder's Vorlesungen, the reprint edition published by the Chelsea Publishing Company in 1966,[7] will not find in it the text reproduced in English translation above. Rather, after the line with the schemata ε (our ε_1 and ε_2) we read:

> which may be combined into the more general schema
>
> $$\zeta) \qquad \prod_{u,v} \text{ sive } \prod_u \prod_v (A_u \in B_v) = (\prod_u A_u \in \prod_u B_v) \,,$$
>
> where the domain of v may differ from that of u.

And likewise, the text following line η (i.e., after our two schemata η_1 and η_2) reads:

> which may be combined into the more general
>
> $$\theta) \qquad \sum_{u,v} \text{ sive } \sum_u \sum_v (A_u \in B_v) = (\sum_u A_u \in \sum_v B_v) \,.$$

In other words: previous to line ε and to line η the immediately preceding two lines have been omitted altogether and the talk of "more general" combined schemata now refers to the schemata ζ and θ which Schröder had originally introduced as "still more general" only in a second step. These deviations are explained by the information printed on the back of the title pages of each of the three volumes:

[6]Cf. footnote 4.

[7]Schröder, Ernst: *Vorlesungen über die Algebra der Logik (Exakte Logik)*, I–III. Second Edition. Chelsea Publishing Company: Bronx, New York 1966 (reprint of the *Vorlesungen* and of the *Abriß* mentioned in footnote 1, with alterations reported on the back of the title pages of all three volumes; Schröder's corrigenda have been integrated, as will be discussed presently).

> The present work is a reprint of the first edition of Vorlesungen ueber die Algebra der Logik, in which the errata noted have been corrected and various supplementary remarks by the author have been incorporated into the text and to which has been added, as an appendix to Volume III, the work Abriss der Algebra der Logik, by E. Mueller.

In order to incorporate part of the corrigenda (or more precisely: of the "correcta") into the text, some of the corrigenda („Berichtigungen") listed in the first edition were omitted, and others transferred to other places. Volume I of the reprint edition includes neither pages IX–XII with the table of contents of the announced second volume nor the "corrigenda", and in the first Section („Erste Abteilung") of Volume II we miss pages VIII–XIII including the pages with "further corrigenda and additions to the first volume" („Weitere Berichtigungen und Nachträge zum ersten Bande"); those of Schröder's statements from these two parts which were not incorporated into the text, were contracted in the reprint edition into three newly added pages 719–721 (the statement of a correction concerning the second motto on the title page of Volume I being kept despite the fact that the title page of the reprint edition is already corrected as indicated by Schröder). The three corrigenda to Volume II („Berichtigungen zum zweiten Bande") listed on the uncounted page following page XIII of the „Erste Abteilung" of Volume II, appear in the reprint edition as distributed to page 95 (shifting the first two lines of page 95 back to the bottom of page 94), page 227 (with a linguistic distortion of the original wording), and page 216 where plenty of space was available after the last sentences of the nineteenth lecture.

Moreover, the Roman pagination of the second volume was changed for the reprint edition, since pages III–XIX are filled by Lüroth's obituary of Ernst Schröder, which originally had its place (with the same Roman page numbers) in the second part of volume II as edited by Eugen Müller after Schröder's death. Lastly, the reprint of Volume III does not reproduce pages VII–VIII of the original with „Berichtigungen zu Bd. 3, I". The second of these corrigenda reads as follows:

> On page 39, line 15 and 16 from above, and line 10 and 9 from bottom *should be cancelled*: the unnumbered two schemata preceding ζ) and θ) are *wrong* and should be replaced by κ) on page 40.
>
> (Seite 39 ist Z. 15 und 16 v.o. sowie Z. 10 und 9 v.u. *zu tilgen*: [die nicht chiffrierten beiden Schemata über ζ) und θ) sind *falsch* und durch κ) auf S. 40 vertreten zu denken.])[8]

[8]The brackets are Schröder's.

Comparison shows that exactly the four lines mentioned in Schröder's corrigendum were cancelled in the reprint edition, which permits us to terminate the philological part of our investigation. Still before us, however, is an evaluation of Schröder's treatment of the distribution of quantifiers over conditionals. I will restrict myself to the following three questions:

1. Are the two propositional schemata ζ and θ, left untouched by Schröder, valid?

2. Are the four propositional schemata ε_3, ε_4, η_3, and η_4, which were withdrawn by Schröder, really "wrong" (i.e., invalid)?

3. May the propositional schemata withdrawn by Schröder indeed be "replaced" („vertreten") by the chain κ of implications?

For the *first question*, we return to Schröder's statement that the propositional schemata ζ and θ are "even more general" than those which immediately precede them and were rejected in Schröder's corrections. Two remarks will facilitate the full understanding of the subsequent test of these schemata for validity. *Firstly*, we may wonder why Schröder adds to his comments on ζ the words, "where the domain of v may differ from that of u", whereas there is no similar addition to schema θ. I assume that Schröder wished to admit the possibility of two different domains for u and v also in the latter case, quite analogous to the former. If this is so, it remains obscure in both cases (and at any rate in the first case) whether Schröder took this admissibility of two different domains as evidence for the greater generality of the propositional schemata involved. In this respect it is useful to remember that quantification over several variables is already more general than quantification of a single variable, since different variables may be replaced by different letters even if the latter are taken from the same domain. E.g., substitution of t for x and y in $\bigwedge_x \bigwedge_y .Ax \supset By.$ and substitution of t for z in $\bigwedge_z .Az \supset Bz.$ both yield $At \supset Bt$, but substitution of t for x and of u for y in the first formula yields $At \supset Bu$, a formula which can in no way be derived from the second formula.

Secondly, we are puzzled, in view of the degrees of generality just mentioned, by the fact that Schröder keeps ζ and θ, but rejects the formulae immediately preceding them — for the latter proceed from ζ and θ by identification of u and v (or more precisely: by substituting the same letters for u and v in every free occurrence of them). If the more specific formulas are "wrong" as claimed by Schröder, the more general ones cannot be valid. Obviously we need a systematic examination of their validity. As all of our candidates are equations (i.e., equivalences), we again split them up into implications which are easier to test by the dialogue method. The necessary test dialogues run as follows:

ζ, case I:

	$\bigwedge_z . Bz \vee \neg Bz.$	0		
	$\bigwedge_x \bigwedge_y . Ax \supset By.$	1	$\bigvee_x Ax \supset \bigwedge_y By$	
	?, $\bigvee_x Ax$	2	$\bigwedge_y By$	
	t?	3	t?0	$\boxed{Bt}$ +
	$Bt \vee \neg Bt$	4	?	
$\neg Bt$	Bt	5	?2 | Bt	−
	Au	6	u?1	
	$\bigwedge_y . Au \supset By.$	7	t?	
	$Au \supset Bt$	8	?, Au [from 6]	
	Bt	9	?5, Bt	
	‡			

ζ, case II:

	$\bigvee_x Ax \supset \bigwedge_y By$	1	$\bigwedge_x \bigwedge_y . Ax \supset By.$	
	t?	2	$\bigwedge_y . At \supset By.$	
	u?	3	$At \supset Bu$	
	?, At	4	?1, $\bigvee_x Ax$	$\boxed{Bu}$ +
?	$\bigwedge_y By$	5	At [from 4] | u?	
	Bu	6	Bu	−
	‡			

θ, case I:

$\neg \bigvee_z . Az \supset Bz.$	0	
$\bigvee_x \bigvee_y . Ax \supset By.$	1	$\bigvee_z . Az \supset Bz.$
?	2	?1
$\bigvee_y . At \supset By.$	3	?
$At \supset Bu$	4	$At \supset Bt$
?, At	5	?4, At
Bu	6	?0, $\bigvee_z . Az \supset Bz.$
?	7	$Au \supset Bu$
?, Au	8	Bu [from 6]
‡		

θ, case II:

$\bigvee_z . Az \supset Bz.$	1	$\bigvee_x \bigvee_y . Ax \supset By.$	
?	2	?1	
$At \supset Bt$	3	$\bigvee_y . At \supset By.$	
?	4	$At \supset Bt$	
?, At	5	?3, At	$\boxed{Bt}$ +
Bt	6	Bt	−
‡			

The examination has shown that all of the four implications tested are classically valid, and both directions of ζ and the direction from left to right in θ even effectively valid. *Thus Schröder was right in not withdrawing these schemata.*

We now turn to the *second question* which, in view of the earlier remark, has become even more interesting by the result just obtained. We once more use the dialogue method for testing ε_3, ε_4, η_3, and η_4.

ε_3 :

$\bigwedge_z .Az \supset Bz.$	1	$\bigvee_x Ax \supset \bigwedge_y By$	
$?,\ \bigvee_x Ax$	2	$\bigwedge_y By$	
$t?$	3	$?2$	$\boxed{Bt}$ +
Au	4	$u?1$	
$Au \supset Bu$	5	$?,\ Au$ [from 4]	
Bu	6	‡	

ε_4 :

	$\bigvee_x Ax \supset \bigwedge_y By$	1	$\bigwedge_z .Az \supset Bz.$	
	$t?$	2	$At \supset Bt$	
	$?, At$	3	$?1,\ \bigvee_x Ax$	$\boxed{Bt}$ +
$?\rfloor$	$\bigwedge_y By$	4	At [from 3] | $t?$	
	Bt	5	Bt	$\boxtimes$ −
	‡			

η_3 :

$\bigvee_z .Az \supset Bz.$	1	$\bigwedge_x Ax \supset \bigvee_y By$	
$?,\ \bigwedge_x Ax$	2	$\bigvee_y By$	
$?$	3	$?1$	$\boxed{B/./}$ +
$At \supset Bt$	4	$?2,\ t$	
At	5	$?4,\ At$	
Bt	6	Bt	$\boxtimes$ −
‡			

η_4 :

	$\bigwedge_x .Ax \vee \neg Ax.$	00		
	$\neg \bigvee_z .Az \supset Bz.$	0		
	$\bigwedge_x Ax \supset \bigvee_y By$	1	$\bigvee_z .Az \supset Bz.$	
	$?$	2	$?1,\ \bigwedge_x Ax$	
$u?$	$\bigvee_y By$	3	$u?00$ $\boxed{Au}$ +	$?$
$Au \vee \neg Au$	Bt	4	$?$	$At \supset Bt$
$\neg Au$	$?,\ At$	5	$?0,\ \bigvee_z .Az \supset Bz.$	Bt [from 4]
	$?$	6	$Au \supset Bu$	
	$?,\ Au$	7	$?5,\ Au$	
	‡			

We may state the result of our examination as follows. Schema ε_3,

$$\bigwedge_x .Ax \supset Bx. \prec \bigvee_x Ax \supset \bigwedge_y By \ ,$$

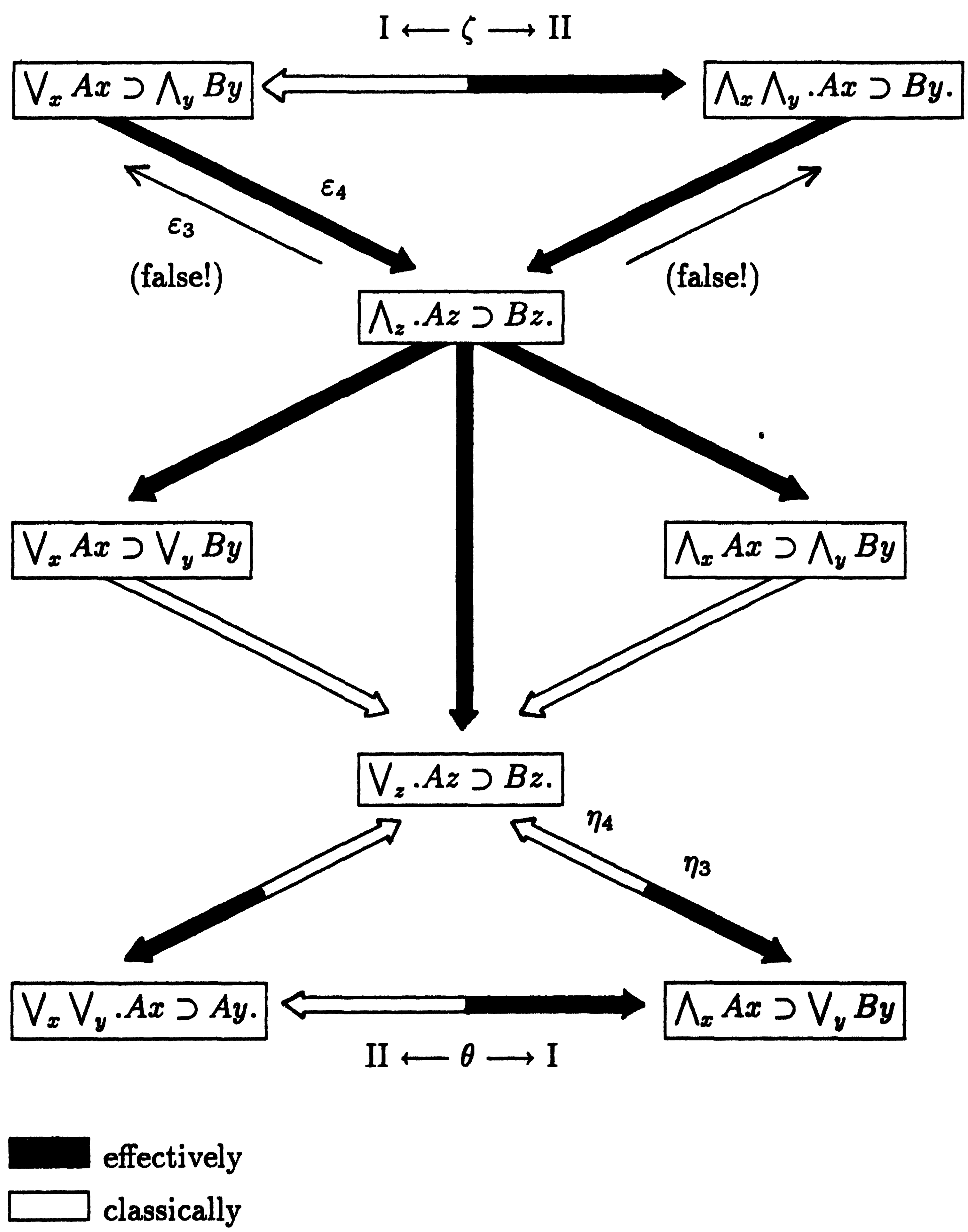
is invalid, as may also be shown by a counterexample: Take the natural numbers as domain, and $4 \mid x$ as Ax, $2 \mid y$ as By. Then, since every multiple of 4 is also a multiple of 2, the antecedent of the implication comes out true. But as there certainly exists a multiple of 4 (i.e., $\bigvee_x Ax$ is true), whereas not all natural numbers are multiples of 2 (so that $\bigwedge_y By$ is false), the succedent of the implication and therefore the entire implicational schema ε_3 comes out false.

Schema η_4 is classically valid, and the schemata ε_4 and ε_3 are even effectively valid. If we return from our implications to Schröder's equations, combining two corresponding implications, we may state our result as follows: *The schema listed immediately before line ζ is invalid, because the direction from left to right does not hold; the schema listed before line θ, however, is valid and Schröder was mistaken in rejecting it as "wrong" in his corrigenda list.*

W.V. Quine, who treats the distribution of quantifiers at some length in his *Mathematical Logic*,[9] lists the equivalence combining η_3 and ε_4 as Theorem *142 and mentions Schröder as its discoverer as well as Schröder's mistake as to ε_3 (the validity of ε_4 is also stated).

The answer to the remaining *third question* is easy. Schröder's schema κ is valid, as may be verified by standard methods. But since in each of the formulae ε_4, η_3 and η_4 one side of the implication contains a universal as well as an existential quantifier, whereas in the chain κ of implications the middle formulae (with the distributed quantifier) contain only homogeneous quantifiers, the answer to our third question is: *The implications assembled in Schröder's κ may not be "substituted" for („vertreten") Schröder's formulas ε_4, η_3 and η_4 in the sense of replacing them.* A more detailed survey of the relations involved is given by the following diagram, suggested by Pierre Leich in 1986:

[9]Quine, Willard Van Orman: *Mathematical Logic*. W.W. Norton: New York 1940; revised edition published by Harvard University Press: Cambrigde, Mass. 1951 and in a paperback edition by Harper & Row: New York/Evanston 1962 (§ 20. Distribution of Quantifiers, pp. 105–109; theorem *142 on page 106).



In view of the importance of elementary quantification, it is astonishing that most of the standard textbooks of logic appear overladen with tautologies from propositional logic while treating multiple quantification (if at all) rather sketchily and incompletely. The most remarkable exception is Joseph

Dopp's *Leçons de logique formelle*[10] of 1950, which in its third volume contains a very detailed treatment of the distribution of quantifiers and, among others, lists the schemata corresponding to Schröder's formulae ε_4, η_3/η_4, ζ, and θ as theorems 455, 434, 921, and 924. As further exceptions I have found Quine's already mentioned *Mathematical Logic*, Hans Reichenbach's *Elements of Symbolic Logic*[11] of 1947, Heinrich Scholz's *Lectures*[12] of 1949 and — as its offspring — Scholz's and Hasenjaeger's *Grundzüge der mathematischen Logik*[13] of 1961.[14] To put it shortly: Even the classical validity of our critical formulae seems to have been investigated only in these few classical treatises. Small wonder then, that the more complicated situation arising after sharpening classical to effective validity has (to my knowledge) never been presented and discussed in the logical literature up to now.

There remains the question why errors like those we have analyzed could be committed by such an outstanding logician as Ernst Schröder. If I may venture upon a conjecture, it seems to me that the treatment of quantification within the algebra of logic, i.e. in the framework of a logic of classes and of propositions, barred or at least impeded a clear insight into the intricate matter, and that it was the deductive approach with its explicit concern for

[10]Dopp, Joseph: *Leçons de logique formelle* I–III. Éditions de l'Institut Supérieure de Philosophie: Louvain 1959.

[11]Reichenbach, Hans: *Elements of Symbolic Logic*. Macmillan: New York 1947. In § 25 Schröder's formulas ε_4, η_3 and η_4 appear in a list of "Formulas concerning fusion or division of operanda" as 10e and 10c. The list has a predecessor in a related assembly in the second section „Einführung in die Logistik" in Reichenbach's *Wahrscheinlichkeitslehre. Eine Untersuchung über die logischen und mathematischen Grundlagen der Wahrscheinlichkeitsrechnung* (A.W. Sijthoff: Leiden 1935), 17–52 with the mentioned list on p. 37*f*.; but this earlier list does not yet include the formulas involving the inversion of quantifiers. (The chapter appeared separately in a French translation as *Introduction à la logistique*, Hermann: Paris 1939).

[12]Scholz, Heinrich: *Vorlesungen über Grundzüge der mathematischen Logik. Teil I*. Aschendorffsche Verlagsbuchhandlung: Münster (Westf.) 1949 [Ausarbeitungen mathematischer und physikalischer Vorlesungen, hg. v. Heinrich Behnke, Band VI. The published text was, according to the information given on the back of the title page, prepared for print by „Heinrich Scholz, mit Unterstützung durch Gisbert Hasenjaeger"].

[13]Scholz, Heinrich†/Gisbert Hasenjaeger: *Grundzüge der mathematischen Logik*. Springer: Berlin/Göttingen/Heidelberg 1961.

[14]Distribution of quantifiers restricted to simple quantification is also treated by Stephen Cole Kleene in his *Mathematical Logic* (John Wiley & Sons: New York/London/Sydney 1967, p. 128, formulas *99° and *99b corresponding to Schröder's η_4 and ε_4, and by Joseph Dopp (apart from the extensive exposition in his earlier work mentioned in note 10) in his *Formale Logik* (Benziger: Einsiedeln/Zürich/Köln 1969, p. 193 („3. Gesetze für die Distribution eines Quantifikators über die Elemente einer elementaren Funktion", formulae 35 and 45 corresponding to Schröder's η_3 and η_4 or ε_4, respectively). I have not consulted Dopp's *Notions de logique formelle*, I–III (Louvain and Paris 1950, [2]1967) for the present study.

conditionals and implications that made a clarification possible. Obviously this conjecture is rooted in the assumption that Schröder would not have committed his errors if he had been forced to obtain the pertinent formulae by a step-by-step („lückenlos“ in Frege's sense) deduction starting from perspicuous quantificational axioms — and such a conjecture does, of course, not admit of any “proof”.