Individuals and Extensional Logic in Schröder's „Vorlesungen über die Algebra der Logik“

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Ernst Schröder's *Vorlesungen über die Algebra der Logik* is usually thought of as the mathematical presentation of a formal calculus, and specifically of a "classical" Boolean logic.[1] It is usually not examined for its philosophical content, as we might a work by Peirce or Frege. Schröder was a practicing mathematician after all, and his influence on philosophical discussion, other than indirectly through later mathematical logic, seems to have been very small. Viewed from a strictly Continental perspective, Schröder the algebraist appears to stand more in the tradition of Grassmann's *Ausdehnungslehre* — especially when we see the development from the *Lehrbuch* and *Operationskreis* — than in the tradition of the philosophers of and reformulators of syllogistic theory, such as De Morgan, Peirce, and even Boole. Within the German mathematical academic hierarchy, the elementary *Operationskreis*, the strictly pedagogical *Lehrbuch*, as well as his position at the newly founded *Polytechnische Hochschule Karlsruhe* at the rim of the German-speaking world, do not seem to give him the weight to issue philosophical pronouncements that a Riemann, Helmholtz, Mach, Boltzmann, Poincaré or

[1]This is not to say it is altogether like Boole's own calculus. It is not equational, instead primarily using his subsumption sign $\subset\!\!\!\!=$ borrowed from his 1873 *Lehrbuch* (*VAL* I p. 140). It also uses the inclusive interpretation of union/or, +, rather than the peculiar non-exclusive (but not precisely inclusive) notion of Boole's works of both 1847 and 1854: namely, Boole left $A+B$ "undefined" when there were any members common to A and B. This feature, widely regarded as a defect, was corrected by Jevons in 1864 and independently by Peirce in 1867 with the inclusive interpretation, and independently by Schröder in the 1877 *Operationskreis*. In other respects, such as interpreting propositional logic as a class logic for periods of time in the manner of Boole's 1854 *Laws of Thought*, Schröder is truer to Boole than Peirce or Venn.

Kronecker would have had. The problematic professional status of the "mathematical logician" in the last half of the 19th century is a difficulty Schröder shared with Peirce, Frege, Cantor, and Dedekind.

Nevertheless, the introductions to the several volumes of the *Vorlesungen* (hereafter: *VAL*), as well as many notes and digressions in the text itself, sparkle with philosophical awareness and with a highly developed sense of the history of logic and philosophy. Other than De Morgan and Peirce, no other logicians of the 19th century were historically so sensitive as was Schröder — and certainly not Frege. Furthermore, although De Morgan and Peirce were historically self-conscious logicians, they tended not to present complete views on the history of logic and rarely even gave precise references. The ever-careful Schröder gave us both elaborate historical explanations of many terms and issues, and also one of the first bibliographies of the history of logic (specifically, symbolic logic) in the *VAL* I pp. 700-715 and *VAL* II 598-605.[2] We might consider him, with Venn, as the first historian of 19th century logic.[3]

There are many broader, philosophical respects in which we might consider Schröder's work. We might look at his conception of logic and its purposes (*VAL* I p. 1–4), at his comments on Idealist and Kantian conceptions (e.g., *VAL* I, p. 35), at his philosophy of language and semiotics (*VAL* I, pp. 38 *ff*.), especially in the light of his later work on signs and pasigraphy, and at the difficult problem of his conception of the relationship of logic and mathematics. To my knowledge, virtually nothing has been written on what Schröder had to say on these topics, although it has been a common — and I think, false — presumption among philosophical logicians that Schröder had at best unsophisticated views on these topics when compared with Frege or Russell.

Instead of addressing these formidable themes, I would here like to consider two related and also substantive philosophical questions: What does Schröder mean by an "individual" („Individuum")? This is a question whose answer is vital to his conceptions of extension (*Umfang*) and domain (*Gebiet*), which have individuals as their members or elements. Secondly and relatedly, why does Schröder choose to develop an extensional theory of logic more in than tradition of English logic than in the grand German intensional tradition of Leibniz, Lambert, Ploucquet, and his countryman and competitor, Gottlob Frege?

[2]Its only 19th century competitor to my knowledge would be Robert Blakey's much earlier *Historical Sketch of Logic from the Earliest Times to the Present Day* (London: H. Baillère, 1851). Later (1936) came Alonzo Church's "Bibliography of Symbolic Logic" in the *Journal of Symbolic Logic*.

[3]See Venn's 1894.

Extensional and Intensional Logics

In post-medieval logic, especially formal or symbolic logic, a broad methodological issue crucial to understanding the role of "individuals" is the question of whether logics are extensional or intensional with respect to the reference of their terms. Extensional logics have terms which refer, in their intended interpretation, to concrete, often material, objects and phenomena, or to collectives of these (classes or sets). Intensional logics[4] have terms which refer to properties or concepts. The contrast between intensional and extensional logics in the history of logic is not always sharply drawn. An exception is C.I. Lewis' *Survey of Symbolic Logic* (1918) in which the organizing theme in his historical presentation is precisely this distinction. His conclusion is that progress in logic arrived only when logic turned toward extensional systems.

German logics have been notoriously intensional, across a wide swath of history: Leibniz, Lambert, Ploucquet, Euler, even Frege. Rare exceptions include one proposal of von Holland (in a letter to Lambert), occasional diagrams, and in the late 19th century the later work of Schröder, and the set-theorists, such as Cantor. The question of the extensionality of the work of early set-theorists, such as Cantor and Dedekind, is however complicated, since their particular interest was almost exclusively with sets whose members are mathematical objects (e.g., numbers or points), rather than with more ordinary concrete entities. However, mathematical objects may be concrete or not, depending on precisely how they are defined. They are certainly not concrete entities of a "usual" sort, i.e. objects of sense experience, such as tables or chairs. (But such a mathematically-inclined set theory could be extensional, if, for example we define the number 1 as the set of all unitary sets — perhaps under some restriction). Boole's logic is however quite clearly extensional: its terms refer to classes of concrete entities, including physical objects.[5] This results in some curious, although harmless differences in the resulting calculuses. For Leibniz, for example, a universal affirmative, "All A's are B's" would become $B = AB$, meaning that the (class of) concepts in B are identical with the (class of) concepts in both A and B. Term A embraces

[4]The use of "intensional" in this context is not to be understood precisely in the modern sense as logics in which truth is not preserved by the substitution of co-extensional terms. A better term for this class of logics might be "conceptual" logics, if the contrast with extensional were not already so traditional in centuries of the history of logic.

[5]That is, in the 1854 *Laws of Thought*. In the 1847 *Mathematical Analysis of Logic* the terms seem to refer to "selecting" operators, or functions that are applied the universe. Thus the term 'A' that may in fact serve to designate (extensionally) concrete apples is actually a function or operator that "selects" apples from the universe. The ***result*** of applying this function are the concrete apples — and Boole there demurs from collectivizing them, e.g., the class of all apples.

the "larger" concept, since, according to his theory of truth, B is included in A. Hence the "intersection" of B and A is just A, since A (the more inclusive concept) excludes nothing. However, Boole would formulate the same categorical, extensionally, as: $A = AB$. This is interpreted as meaning: the class *of things* that are A are the same as the class of things that are both A and B. It is B in this interpretation that is "larger" or more inclusive. The general relationship between the extension and intension of a word was widely noted in the early 19th century (and earlier), and described in the doctrine of "inverse proportionality": generally speaking, the larger the extension the smaller the intension, and *vice versa*.

Thus although the *formal* calculus Leibniz gave, and the one that Boole later gave, are similar if not identical, the accompanying intended semantics are quite different. The validation of Barbara, for example, would be represented by quite different series of equations — but both series would use the same algebraic techniques. This has not always been made clear.

	Leibniz (intensional)		Boole (extensional)	
1. All A's are B's	$B = AB$		$A = BA$	
2. All B's are C's	$C = BC$		$B = CB$	
So, All A's are C's	$C = AC$		$A = AC$	
PROOFS	$C = (AB)C$	1,2	$A = (CB)A$	1,2
	$C = A(BC)$	Assoc.	$A = C(BA)$	Assoc.
	$C = AC$	2	$A = CA$	1
			$A = AC$	Comm.

Seen in this broad perspective, Frege is a traditional German logician, opposing the newer British extensional logics, with their probable roots in English attractions to nominalism/empiricism going back to Ockham. The historical predilection and unity of English logic is forcefully made by Venn's term "material logic." (We should also note an exception: W.S. Jevons, whose logic is, almost uniquely in the English-language tradition, intensional.)

However, although *Principia Mathematica* pays tribute to Frege in diverse ways, it is quite clear that it, following the preferences of Russell's earlier *Introduction to Mathematical Logic*, is an extensional logic in the tradition of Boole, Peirce, Schröder, Peano, and Cantor. (Russell presumably believed that, among other things, the parodoxes were endemic to intensional logics like Frege's, but could be avoided in extensional systems. His attitude in still other works toward Meinong shows signs of a more general disdain for intensional theories — again, all quite typical of the English tradition.) When one adds to this the powerful influence of Schröder and Peano, and the separate influence of Cantor, on figures in early twentieth century German logic, set theory, and foundations of mathematics such as Zermelo, Löwenheim, and Skolem — in which Frege's role was very small — one sees that extensional logics were almost completely dominant by 1910. Extensional logics continued this dominance at least until Church's "Logic of Sense and Denotation," and the revival of Frege and Meinong in the 1960's. Intensional logics are now, once again, well represented. It is true that this discussion focuses attention upon the "analytic" movements in the English-speaking world, and upon strictly mathematical logic in the German- and English-speaking world. Still represented in German philosophical circles throughout this period were various deeply traditional (e.g., B. Freytag-Löringhoff) and intensional logics. They had however very little influence outside of, or even inside of Germany, since formal (and especially symbolic) logic was so little cultivated in the highest circles of German philosophy in the 19th century and in the 20th until Heinrich Scholz.

One of the great curiosities of the history and philosophy of logic, given this obvious clash of approaches between extensional and intensional treatments, is why arguments for the superiority of one approach over the other were not clearly advanced. Partly, I suspect, they involved deep issues of philosophical framework as well as philosophical, even cultural, presuppositions and predispositions. Some of these factors may have approached being unconscious. Partly, too, English and German logics simply did not take much note of each other and, when they did, tended to minimize the differences. Thus, once Leibniz's calculus became known following Couturat's edition of his work in 1901, he was widely heralded as having anticipated Boole's logic. So too, because of the enthusiastic endorsement of Russell, and widespread ignorance about the German logical tradition, Frege was widely regarded as a founder of twentieth century logic — even though that logic has been almost universally extensional.[6]

The somewhat undeveloped criticism of C.I. Lewis toward intensional logics confirms, and contributed to, this attitude:

> Whoever studies Leibniz, Lambert and Castillon cannot fail to be convinced that a consistent calculus of concepts in intension is either immensely difficult or, as Couturat has said, impossible ... The more serious difficulty is that a calculus of "concepts" is *not* a calculus of things *in actu* but only *in possibile*, and in a rather loose sense of the latter at that ... That the long period between [Leibniz] and De

[6] A perspective on Frege's work that sees him — properly, I think — in the tradition of Leibniz (*via* Trendelenburg) is to be found in Hans Sluga's *Gottlob Frege*. However, I think even he does not appreciate the full extent to which Frege is part of a long-standing German intensional, mathematical tradition with a penchant for 2-dimensional representations of our ideal thought.

> Morgan and Boole did not produce a successful system of symbolic logic is probably due to the predilection for this intensional point of view. It is no accident that the English were so quickly successful after the initial interest was aroused; they habitually think of logical relations in extension ... [C.I. Lewis, *A Survey of Symbolic Logic*, p. 35].

John Venn had much earlier announced the death of intensional logic, writing in 1894: "I think it may be said that the true intensive view is practically abandoned now, though verbally it is from time to time espoused" [Venn 1894/1971, p. 453 note 1]. He dismisses Jevons, and quotes in his support from Schröder's review of Frege's *Begriffsschrift* (1879): „In diesem gänzlichen Absehen vom Inhalte der Begriffe liegt nun allerdings eine Einseitigkeit."

Four hypotheses might explain more precisely why one style of logic is to be preferred to another. First, there is the metaphysical preference or background theory. Favoring extensionality, perhaps one does not believe that there *exist* with any certainty concepts or properties, but only concrete, even material, entities (nominalism, materialism). This seems to be part of Lewis's argument. Or perhaps, favoring intensionalism, one believes that there are properties/concepts, but that the status of mind-independent *concreta* are problematic (Platonism, Idealism). Either position implicitly endorses the view that logic is to portray our metaphysics, or even that logic is "formal metaphysics." This view is suggested by Quine, although perhaps it would be better to say that he maintains that metaphysics constrains one's logic — what one can quantify over — rather than metaphysics defining logic's whole *raison d'être*. Second, one could believe (whatever one's metaphysical position) that logic's primary purpose lies in its pedagogical value in improving our thoughts. Whether there are any *concreta* beyond our thoughts and thought patterns is a nice metaphysical question, but the only things we should be trying to affect and improve are our thoughts. Hence if our logical purpose is primarily pedagogical, logical language should be focused upon those thoughts rather than "beyond." This is an argument for intensional logic. There is a variant of this pedagogical position that supports extensional logic. Namely, focusing upon a logic of (our own) concepts directs thought toward what is most idiosyncratic and non-public. By addressing ourselves to public objects of awareness, we are more likely to be able to communicate logical assistance, rather than by dealing with incommensurate private concepts.

Third, and this is the position of Lewis, Russell, and probably many others, one might believe that a formal/symbolic intensional logic is technically impossible, or that it is especially cumbersome. This is a difficult argument to deal with here, since it would require us to look at all possible extensional and intensional logics, and examine the technically best of each. Since the work of Church and the serious reexamination of Frege's logic, I do not believe that this position can be seriously maintained; I suppose it is still possible, in the manner of Quine and others, to maintain that intensional logics are especially complicated or have technical flaws (such as the incompleteness of a full second-order logic). Earlier writers have been mislead, I think, into thinking that because the early logics of Lambert et al. were deficient, and paradoxes plagued Frege's intensional logic (and no one could then see how to avoid them — or seriously tried), then all intensional logics are flawed. This conclusion does not follow, however. It is a faulty generalization.

Finally, and this cuts harshly against extensional logics (and still might), it has been thought at various points in the history of logic that it is impossible to deal with terms of infinite extent, or to contrast terms with identical extensions, e.g., contradictory terms. We can distinguish them, and consider them, in thought but only as thoughts. For example, I can quite adequately conceive of what it is to be an atom of hydrogen but, I cannot contemplate or manipulate the class of all atoms of hydrogen.[7] In other words, my mind can contemplate those concepts that it takes to constitute the concept of hydrogen (at least to some degree) but only to a far less satisfactory extent can my mind contemplate the class of all hydrogen atoms in the universe. Hence intensional logics seem to be preferable for dealing with expressions whose extensions are conceptually intractable. Certainly until the techniques of Bolzano were applied to infinite collections by Dedekind and Cantor, it appeared that "closed infinities" were not contemplatable or manipulable by finite minds. *Impossibilia* constitute a more severe threat to extensional logic, since what appear to be quite distinct thoughts (e.g., round-squares vs. ovular-rectangles) collapse into co-extensional entities. Yet another problem in this general family of difficulties is the chronic gap that emerges between epistemological and metaphysical considerations. Namely, many if not most terms will have extents — refer to individuals somewhere in the universe — of which we, perhaps necessarily, have no idea whatsoever. They have extensions that reach out beyond our cognitive realm. Modern semantics since Frege has tended toward "semantic realism", that the extension of linguistic expressions is independent of the ability of the user's mind to grasp this extent, or even of the ability of any finite mind to contemplate in detail this far-reaching extension (other than by using referential tools themselves). Whether this is a tolerable or desirable situation is highly dependent upon one's theory of

[7] I assume in this argument that this class is imaginably large — unable for an individual to contemplate "clearly and distinctly" its size, in the manner of Descartes' 1000-sided polygon — or even infinitely large.

the purpose of one's logic and semantics. If the goal is to reflect and tutor the ideal reasoning of finite minds, then one surely would prefer an intensional logic, or an extensional logic in which the maximum extension is limited by the cognitively graspable world.

Schröder's Logic: Individuals and Extent („Umfang")

The issues of "individuals" and collections of individuals such as classes or sets, arise primarily in extensional logics. In fact, discussions of individuals and the relationship of individuals to collectives is symptomatic of extensional logic. It is true that Leibniz's own philosophical theory contains a notion of a kind of "individual" that has the key characteristic, typical of classical definitions of individuality and particularity, of being completely defined with respect to every property/concept. This is his notion of a "complete individual concept," such as that of Julius Caesar. This complete individual concept is the logical correlate of the monad itself. But it is a concept nonetheless, and so is not concrete. It is a universal (concept) that is as finely individuated as possible. Rarer in this tradition is the notion of "basic" concepts or properties. Lambert and Leibniz seem often to suggest that concepts can be indefinitely decomposed and analyzed — although typically not by finite minds. (Leibniz suggests that some concepts that are human beings' own artificial creations, such as numbers, might be "adequately" grasped, to use his epistemological term. But this suggests that the wider range of "natural" concepts based upon sense experience are not humanly adequately analyzable.) Such basic concepts, out of which all other concepts are constituted, would have at least a kind of particularity, although they too would not be concrete.

Schröder's early *Operationskreis des Logikkalkuls* of 1877 is an elegant, concise formulation of the Boolean calculus. It shows no appreciation of the revisions and improvements to Boole's logic by Jevons and Peirce and so is primarily derived from Boole's work. But in the introduction, Schröder carefully pays homage to Robert Grassmann, and with this, to the Grassmannian tradition of the *Ausdehnungslehre* and German formal logic.[8] Furthermore, the language shows far more sensitivity than an English-speaking logician's of the period typically did to the issues of extensional vs. intensional logic. It

[8] p. v: „Der veränderte Standpunkt bedingte, dass, während von den durch Boole aufgestellten Sätzen ein grosser Theil beibehalten werden konnte, doch die Beweise durchweg durch ganz andere ersetzt werden mussten, deren einige man auch ähnlich bei Rob. Grassmann finden wird."

is one of my chief points that Schröder had a thorough grasp of the German intensional tradition (unlike the English-language logicians of the 19th century), that his acceptance of the non-German extensional perspective was a difficult one, and that his position is extremely nuanced and respectful *vis-à-vis* the older German intensional tradition.

We see this care immediately in the language of the opening pages of the first chapter.

> Gegenstand der logischen Operationen sind Buchstaben, welche — in dem genannten ersten Theile[9] — als Klassensymbole zu bezeichnen sind. Unter einem Buchstaben, wie a, verstehen wir nämlich hier stets eine *Klasse* oder *Gattung* von Objekten des Denkens. Der sprachliche Ausdruck einer solchen ist in der Regel ein *Gemeinname* und gibt zugleich Veranlassung zur Bildung eines *Begriffes*, in welchem wir uns die wesentlichen Merkmale, die allen zu der Gattung gehörenden Individuen gemeinsam sind, zusammengefasst denken. Im Gegensatz zu diesen Merkmalen, dem sogenannten „Inhalte" des erwähnten Begriffes, stellt dann die Klasse selbst dessen „Umfang" vor, sodass wir in Gestalt dieser Klassensymbole in der That mit den hinsichtlich ihres Umfanges dargestellten Begriffen rechnen werden.

This passage signals to the German reader, in terms that would not have been necessary in an English work, the contrast of Schröder's work in being an extensional logic of individuals, classes or kinds (*Gattungen*), and extension (*Umfang*), as opposed to an intensional logic of characteristics/concepts composing the "intension" (*Inhalt*). Two other subtleties of this passage are worth calling attention to. First, Schröder uses the expression "concept" (*Begriff*) as his basic semantic bearer of interest, e.g., in speaking of the "content of the concept." This contrasts with the older medieval tradition of using concrete linguistic entities, namely, (spoken, written, and mental) terms.[10] It signifies an intrusion of the German Rationalist, and perhaps specifically Kantian, tradition into Schröder's logic.[11] Second, and more importantly, he does not speak of the extension of a term being the class of things of which the term is true, or to which it refers. Instead, they are a class *of objects*

[9] Boole's notion of "primary" vs. "secondary" (hypothetical, propositionally-complex) judgments.

[10] Boole retains this linguistic flavor in both *MAL* and the *Laws of Thought*, speaking explicitly of language and of signs.

[11] Although Boole's own language, as well as the language of other early-19th century English logicians, with the exception of De Morgan, occasionally also show the distinctive and usually unhelpful influence of Kant's terminology.

of thought (*eine Klasse ... von Objekten des Denkens*).[12] These objects are thus not external (to the mind), concrete, possibly physical individuals in the sense of Venn's somewhat naive "material logic". Furthermore, there is the possibility that they do not embrace all individual objects of possible thought (e.g., by anyone anywhere), but the objects of thought of the person using the calculus at the time. Schröder's expression is in fact slightly more suggestive of this, and is more precisely translated as "objects of thinking."[13]

Schröder's precision of language is thus striking, I think, and pinpoints him rather precisely as trying, even at this early point in his career, to stand in both the English extensional and the German intensional traditions. His 1877 logic is neither precisely an intensional logic, identifying terms with classes of properties/concepts or with something like the far more mysterious and ill-defined Fregean senses (*Sinne*), nor a traditional English extensional logic, identifying terms with "classes" of (concrete, physical) "things." Instead, he seems to construe logic as dealing with classes of *our concepts of* individual things — but not further construing our concepts of individual things as themselves classes or logical composites of properties or general concepts, in the Leibnizian tradition.

The „Vorlesungen“, Volume I

The lengthy, historical and philosophical remarks that constitute the Introduction (*Einleitung*) to the *VAL* Vol. I, as well as discussions in Lecture I, and the later Lecture 22 in Vol. II that is completely devoted to the topic of "individuals," exhibit a similarly complex attitude to the issue of extensional logics and to the exact nature of the "individuals" that constitute the basic elements of his semantic interpretation of the calculus. It is also rather clear that by this point, he had studied Peirce's remarks on individuals; Peirce's views are metaphysically extremely complicated and revolve around older medieval distinctions that few in the 19th century would have been able to appreciate.[14]

[12] J. Lüroth rather massively distorts this point in his *Lebenslauf* of Schröder at the beginning of the posthumously-published complete Vol. II, p. x, where he writes of the *Operationskreis*: „Er operiert mit *Klassen von Dingen*, d.h. mit der Gesamtheit aller Dinge, die gegebene Merkmale gemein haben.“ These remarks are truer to English Boolean logics of the period than to Schröder's language.

[13] Boole speaks of "conceivable classes of objects" [*MAL*, p. 60 (original p. 15)], but not of the conceivability or mental nature of the objects constituting these classes. His casualness about an "object" is greater than is possible in the German tradition, e.g., whether it is a physical, sensed, mental, or ideal "object".

[14] On Peirce's views, see my "Peirce's Philosophical Conception of Sets."

Schröder describes the "essence" (*Wesen*) of a concept (*Begriff*) as its intension (*Inhalt*): this intension forms the shared characteristics of the things that are referred to with a common noun (p. 83). Importantly, and with some historical precedent, he distinguishes between the "actual" (*faktische*) intension and a concept's "ideal" (*ideale*) intension. The ideal intension includes all of the characteristics held in common by the referred-to things, even if the totality of these characteristics could never be fully grasped („niemals vollständig auszudenken möglich [ist]"). The actual intension consists of those characteristics that are contemplated when the term is created („bei seiner Bildung reflektirt wurde"). He does not consider the complicated cases of terms whose intension initially is "erroneously conceived," or whose intension is revised — cases constantly pondered in the philosophy of science (e.g., phlogiston). The totality (*Gesamtheit*) or class (*Klasse*) of the gathered-together (*zusammengefassten*) individuals under the common name constitute (*bezeichnet*) the extension (*Umfang*) of the associated concept. In a digression (indicated in smaller type, as was usual in the *VAL* — one of Schröder's inventions that deserved to be followed as an alternative to footnotes, and perhaps constitutes an anticipation of hypertext), he describes the intension and extension of the concept, "material substance." Its intension includes extention, having volume, spatial localizability, inertia, and so on. Its extension includes "every body, every part of every body, and every group of bodies in the universe." This last phrase is interesting, because the extension can apparently include collective composites of bodies („Gruppen von Körpern") that are distinct from the bodies in this group but are also in the extension. Schröder then describes how it is possible to define (*bestimmen*) a concept in one of two ways: by giving its intension, or its extension.

In connection with extensional definitions of terms, Schröder remarks (p. 85) that divisions of the extension always lead back to („führt in letzter Instanz (zuguterletzt) immer auf ... zurück") individuals, described as things which are not further divisible, *with respect to the extension* (*dem „Umfange" nach*). The characteristic of indivisibility is one of the standard metaphysical characteristics of individuals, the other being a concreteness or well-definedness with respect to every property that we will shortly see Schröder (following Peirce) making a great deal of. The added qualification of indivisibility "with respect to the extension" is not trivial, because Peirce, Peirce's student O.H. Mitchell, and Schröder considered entities to have individuality or indivisibility only from one perspective; from another perspective (Peirce/Mitchell: "respect of extension") what had been considered individual can be considered plural.[15] Both Peirce and Schröder used this technique to accomplish in

[15]See my "O.H. Mitchell's Life and Logical Work" and "Peirce's Philosophical Conception of Set."

their "single-rank" class theory what is done in set theory with multiple-type or multiple-rank sets of sets, and Schröder developed a formal account of this theory in his domain calculus.[16]

Schröder then states an issue that seems initially to favor an intensional logic. He suggests that extensional definitions of concepts cannot be completely stated when the extension is indefinitely large (*unbegrenzt viele Individuen*) or when it is open (*offen*) — apparently thinking in this last case of individuals in the extension of which we might not now be aware, or which swell and shrink according to the number of individuals at the moment of utterance ("human beings on the earth"). He explicitly mentions as problematic the unbounded row of individuals we call the natural numbers. (Although references to Dedekind's 1888 *Was sind und was sollen die Zahlen?* abound in the *VAL* I, mainly to its initial pages, there seems to be here no appreciation on Schröder's part of "extensional" ways of dealing with infinite classes. This was to change in later writings, especially with his later „Über zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze" of 1896 and the third volume of *VAL.*) There is a grasp of recursive extensional definitions of number in the manner of Peirce and Peano. Schröder also considers the points on the interior of an ellipse to be well-defined.

There follow murky passages (pp. 85–86) in which Schröder argues that extensional definitions of such unbounded or open concepts are possible: we give the method of producing or identifying (*Umfangsangaben*) the members of the extension. Schröder does not however consider Frege's profound underlying observation that to specify individuals by such methods is in fact to give an intensional definition, not the extensional definition one thinks one is giving. The more modern point would be that a description of an algorithm (e.g., in a computer language) is intensional, and distinct such intensional specifications of a concept may be extensionally identical (e.g., of adding).

Schröder finally (p. 99) endorses extensional over intensional logic in emphatic terms: to construct an intensional logic is like building the roof before one builds the house. He also waxes polemical calling intensional logic's defenders its *Verfechter* and (p. 101) refers to the neo-Hegelian attacks on extensional logic (dry, boring, unfruitful; a dead formalism, and empty schematism) as in Germany but a "fashion" among haughty philosophers („besonders in Deutschland bei geistreichen Philosophen Mode geworden — und neuerdings in verstärktem Maasse" with the footnote: „Es würden sich eine Menge Citate beibringen lassen; ich halte mich aber durch das ‚nomina sunt odiosa' gerechtfertigt, wenn ich ... abstehe, solche Beispiele anzuführen

[16]See A. Church, "Schröder's Anticipation of the Simple Theory of Types" (1939) and my evaluation of Church's claims in my *Development and Crisis in Late Boolean Logic: The Deductive Logics of Peirce, Jevons, and Schröder* (1978) pp. 246 *ff*.

...Selbstverständlich indess sind zu obigem auch erfreuliche Ausnahmen zu konstatiren.")

His main argument against intensional logics is that we use concepts whose intensions are ill-defined. Thus when we use the complement of human being, non-human-being, as is common in logic, its intension includes a diverse grab-bag of properties. It would be (using Lotze's joke) an eternal exercise to identify these and would include such concepts as triangle, melancholy (*Wehmut*) and sulfuric acid (*Schwefelsäure*). Here, however, Schröder seems to have neglected his own distinction between actual and ideal intensions, since ideal intensions are more often than not ungraspable by finite minds in any case. Schröder's objection perhaps could be stated as seeing an asymmetry: the complements of actual intensions will be ideal rather than actual. He does not consider the possibility that "not-human" has a higher-order intension that is not simply the complement of its intension conceived as a class of properties, or De Morgan's subtle point that complements of classes — including, presumably the class of properties constituting an intension — are always taken relative to a restricted, contemplated universe of discourse, rather than to a class of "all properties."

In any case, Schröder regards this objection as devastating against intensional logics. The extension of such complementary terms clearly "exists," he believes. In other words, the extension is well-defined even when its intension in ungraspable. But Schröder's assertion is not itself unproblematic. If we understand extensions as classes of individuals-as-conceived (*Objekte des Denkens*), then it is hardly clear whether or not large, or infinite, classes of individuals can be individually contemplated — except perhaps through the *methods* of identifying or producing them. A great deal then turns on the nature of these individuals, and specifically, whether they are really, mind-independently existing *concreta* in the casual English way of talking, or whether they are "thought objects" in the language of the *Operationskreis* that is often repeated in the Introduction to *VAL* I as well. Namely, the question becomes whether they are individually, "ideally" conceivable as objects and specifically, individuals as *conceived*, by any mind at any time — in which case we might permit indefinitely large classes of such entities — or whether the classes of these entities must themselves be conceivable (in the manner of Boole's language), or whether all such individuals must be actually conceived by an individual at a time, which would rule out "large" extensions just as handily as Lotze's joke does "large" intensions. Of particular concern when we consider individuals is how, especially in large classes, they are individuated — i.e., how they are theoretically or conceptually to be distinguished from one another. This question brought Peirce to criticize Dedekind and Cantor, and to raise issues close to considerations governing the Axiom of Choice in set theory.[17]

Schröder considers briefly the issues of extension, intension, and individuality in Lecture 1 of *VAL* I (pp. 130–131), but his considerations there are more technical and practical than theoretical and philosophical. Later in the first lecture (p. 144), he argues that an extensional interpretation of subsumption is dictated by common usage, and that an alternative intensional interpretation is clumsier (*schwerfälliger*). Following a lengthy digression on the logical and psychological content (*Gehalt*) of concepts, he announces (p. 147): Here, we claim, one can always grasp the subject and predicate as classes, and the logical content (*Gehalt*) of the judgment be completely restated by interpreting it as the assertion [*Versicherung*, with the English word in parentheses]: the subject class is wholly contained in the predicate class. He then indicates that this use of 'class' [*Klasse*] is not to be interpreted too narrowly — namely, it should be permitted to admit [in the manner now familiar since the widespread introduction of set theory] classes with only one individual, or even with no individuals. (He uses analogies with "classes" in the sense of schools and classrooms to make his point, a double meaning that works in both English and German, pointing out that a scheduled "class" could have no students registered for it.) He then says, rather unfortunately:

> Beyond this we will not further discuss what we would understand under "class" and under an "individual". Everyone understands what is meant, when one speaks of the class of mammals ... [namely] arbitrary objects of thought as individuals unified ("brought together") as a class.
>
> Im übrigen wollen wir, was unter einer „Klasse" und was unter einem „Individuum" zu verstehen sei, zunächst nicht weiter erörtern. Jedermann versteht, was gemeint ist, wenn man spricht von der Klasse der Säugetiere ... [aus den Betrachtungen der Einleitung:] ***Wir sind im stande irgend welche Objekte des Denkens als „Individuen" zu einer „Klasse" zu vereinigen („zusammenzufassen").***[18] [*VAL* I p. 147 *f.*]

A bit puzzling to the modern reader is however his next comment: And also an individual can be described as a class that contains only this individual itself. Every thought-of-thing (*Gedankending*) can be stamped/converted (*gestempelt*) to such an individual. This conflation of an individual and its

[17]This is discussed at length in my "Peirce's Philosophical Conception of Set."

[18]The language was apparently in common usage, since Cantor's later definition in 1895 was of a set (*Menge*) as a „Zusammenfassung von bestimmten wohlunterschiedenen Objecten unsrer Anschauung oder unsreres Denkens zu einem Ganzen" [Cantor 1895]. In 1914 Hausdorff's was: „Eine Menge is eine Zusammenfassung von Dingen zu einem Ganzen, d.h. zu einem neuen Ding" (Hausdorff 1914, p. 1).

unit class is here either outright confusion or an enlightened suggestion similar to Quine's well-known proposal for his system, New Foundations.[19] He promises to consider the precise notion of an individual in Lecture 22, but then notes — and this employs his notion, borrowed from Peirce, of individuals being classes, or classes being individuals from different perspectives (individuations):

> But also every such class, which itself encompasses a set of individuals, can further be regarded as a thought-of-thing, namely as also itself an "individual" (in the broader sense, for example, relative to higher classes). When we talk however about an individual "in the absolute (narrower) sense," then we understand by it an object of thought, whose name is treated as a proper name und not as a common name (compare Section B of our Introduction).
>
> Auch jene Klasse aber, die selber eine Menge von Individuen umfasst, kann wieder als ein Gedankending und demgemäss auch als ein „Individuum" (im weiteren Sinne, z.B. „relativ" in Bezug auf höhere Klassen) hingestellt werden. Wenn wir jedoch von einem Individuum „im absoluten (engeren) Sinne" reden, so verstehen wir darunter ein Objekt des Denkens, dessen Name als ein Eigenname und nicht als ein Gemeinname gehandhabt wird (vergl. den Teil B unsrer Einleitung).

This notion is apparently applied in the posthumously-published section of Part II of *VAL* II (p. 461 *ff*.). It had been noted by N. Wiener in his dissertation (defending Schröder against Russell) and by Church.[20]

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The promised discussion of individuals in Lecture 22 (*VAL* II part I, 1891) is however somewhat disappointing. It collapses the question of the nature of an individual in the philosophical sense with that of a "point" (*Punkt*) in the domain-calculus (*Gebietekalkul*). The most obvious definition would have been to say something like i is an individual if and only if

$$x \neq 0 \not\in (x \subset i = 0)$$

or more precisely:

$$\prod_x x \neq 0 \not\in (x \subset i = 0)$$

[19]Peirce is clearer about the distinction, but also considers the possibility — in a clearer way than Schröder — of conflating the two in a formal system.

[20]See [Grattan-Guinness 1975] on Wiener and [Church 1939].

That is: it is false that there is a non-empty x such that x is a proper part of i — i.e., i is indivisible. Schröder also distinguishes between individuals and the empty domain by requiring: $i \neq 0$ (*VAL* II p. 320). This could be more cleverly stated as:

$$\prod_x x \Subset i \Subset (x = i + x = 0)$$

That is, domain x's subsumption in i, implies that domain x is identical to i or is the empty domain. Schröder in fact derives this on p. 325. (These formulas use the strange device, common in Peirce and Schröder, of using $\Subset$ and 0 autonomously as both inclusion and the material conditional (for '$\Subset$') and as both signifying the empty domain and indicating the negation of an assertion (for '$= 0$'). Schröder alone distinguished positive propositional assertion from the universal domain, 1, by using $\dot{1}$.) We might also compare Peirce's 1880 (*CP* 3.216): A is an individual just when "$\bar{A} \prec 0$, but such that if $x < A$ then $x \prec 0$" which Schröder in fact refers to and translates into his system (p. 326).

Schröder gives however a far more complicated definition, initially stating that i, an individual or point, is such that:

$$(xy = 0) \Subset [(ix \neq 0)(iy \neq 0)] = 0$$

translated as:

> For all x and y, if x and y are mutually exclusive domains, then it is false that the intersections of x and i, and y and i both contain something,

then later:

$$(i \neq 0) \prod_{x,y} [(xy = 0) \Subset (ix \neq 0)(iy \neq 0 \Subset 0)] = \dot{1}$$

translated as:

> i is not empty and, for all x and y, if x and y are mutually exclusive [= their intersection is null] then it is propositionally true [$\dot{1}$] that it is false that both the intersections of i and x, and i and y, are non-empty.

Later, the complete freedom of range of the mutually exclusive x and y are decreased, and Schröder argues that it is sufficient that x and y are complements with respect to the considered universal domain, 1 (p. 325). That is, 'y' becomes replaced with the complement of x, '$1 - x$'.

Freed of what to us is Schröder's cumbersome notation and formalism, this condition requires that no individual straddles two mutually exclusive domains („[das Individuum/der Punkt] kann nicht in zwei getrennte (disjunkte) Gebiete zugleich hineinragen", p. 320). But this seems a far more complicated one than the simple condition that an individual cannot itself be divided (its „Unteilbarkeit," which Schröder also acknowledges). Why did he require this condition to be made? To what philosophical issue, or what historical tradition, was he appealing?

The rest of the lecture amounts to a formal examination of the implications of this condition, and equivalents to it. As nearly as I can determine, he offers no further explanation as to why exactly this condition for individuality is desirable. As a hypothesis I can offer the following. He mentions Peirce's definition of individuality (p. 320 and a precise reference on p. 326 to Peirce's 1880 "On the Algebra of Relatives"). Peirce had a notion, stated as early as his 1870 *CP* 3.96, and repeated throughout many publications that an "absolute" individual is well defined with respect to every conceivable property. For every property, an individual either clearly has the property, or lacks it. That is, it has a specificity or well-definedness (what some would describe as metaphysical "concreteness"). He also uses a more limited notion of an individual being distinguished from other individuals in a contemplated universe of discourse (rather than absolutely): some property or relation distinguishes it from all other individuals in that universe. He later uses this notion to criticize both Dedekind and Cantor.[21]

Now an arbitrary property would amount to a domain, when interpreted in Schröder's calculus, and to say that an indvidual is such that, for every property, it either has that property or lacks it, is to say that, for every domain, it is either in that domain or in its complement.[22] The additional condition of indivisibility in fact follows from this. Consequently, I propose that Schröder simply borrows Peirce's often-stated desideratum of well-definedness for individuals, and formulates it in the domain calculus. The contribution Schröder makes is then to show that indivisibility — supposedly the more basic requirement — follows from this more complicated condition.

I do not know why Schröder thinks this specificity is desirable or necessary for individuals. There is some explanation for this in historical-metaphysical terms in Peirce's works but not, so far as I can see, in Schröder's. Possibly, he is here just following Peirce's lead.

[21]This is developed in my "Peirce's Philosophical Conception of Set."

[22]I do not have an explanation, however, of why Schröder would first consider a definition based on two domains whose only restriction is that they are mutually exclusive rather than complementary with respect to the universe: complementarity implies mutual exclusivity, but not *vice versa*.

Conclusion

The conceptual clarity of the notion of an individual, and of the "collectives" made up of these individuals, is of crucial importance for the intellectual integrity and understandability of extensional logic. "Extensional logic" in this broad sense includes set theory, the algebra of logic, as well as first-order predicate logic as it is commonly practiced and interpreted. Unfortunately, I am not sure that, other than Peirce and Schröder, extensional logicians of the late 19th century or, for that matter of the 20th, have worried much about this issue — in other than simple-minded terms, in for example simply assuming there are of course "individuals" out there and we can somehow mentally collect (zusammenfassen) them. Objections to the notion of a "set" are fairly widespread — and increasing. (See for example [Black 1971] and [Hallett 1984].) I do not believe however that it has yet fully dawned on would-be extensional logicians that the notion of an "individual" (or however one describes the ultimate members of sets[23] or the objects of reference) is itself problematic. Even to see that our extensional house is not in order, it is necessary, I believe, to return to the reflections of philosophical logicians such as Peirce and Schröder, who worried more than we typically have in the 20th century about basic notions in extensional logic.

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