MYHILL'S THEORY OF COMBINATORIAL FUNCTIONS

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Abstract. This is an expository account of the theory of combinatorial functions introduced by J. Myhill in 1958.

§1. Introduction. The English logician John Myhill (1923-1987) introduced certain functions which he called combinatorial [3]. He used these functions to study recursive equivalence types, but we conjecture that many logicians will find these functions of interest apart from their use in recursion theory. This is an expository account of some of Myhill's work. Combinatorial functions can have any finite number n of variables, but the most important cases are $n = 1$ and $n = 2$. We shall therefore restrict our attention to these two cases.

We write ε for the set of all nonnegative integers and ε^* for the set of all integers. Our starting point is the following theorem which is a special case of Newton's approximation theorem.

THEOREM T1. For every function $f:\varepsilon \to \varepsilon$ there is exactly one function $c:\varepsilon \to \varepsilon^*$ such that

(1) $$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i},$$ namely the function

(2) $$c_i = \Delta^i f(0) = \left[\Delta^i f(n)\right]_{n=0},$$ where $\Delta f(n) = f(n+1) - f(n)$.

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The function c_i related to $f(n)$ by (1) or equivalently by (2) is the function associated with $f(n)$.

ILLUSTRATION. Let $f(n) = a^n$, where $a \in \varepsilon$, $a > 0$. Then

$$\Delta f(n) = (a-1)f(n), \quad c_i = \Delta^i f(0) = (a-1)^i .$$

In particular, if $a = 2$, $f(n) = 2^n$ and $c_i \equiv 1$. We see therefore that the function associated with one of the most important functions of combinatorics, namely 2^n is extremely simple: it is identically one.

THEOREM T1*. For every function $f:\varepsilon^2 \longrightarrow \varepsilon$ there is exactly one function $c:\varepsilon^2 \longrightarrow \varepsilon^*$ such that

(3) $$f(m,n) = \sum_{i=0}^{m} \sum_{k=0}^{n} c_{ik} \binom{m}{i} \binom{n}{k} ,$$ namely the function

(4) $$c_{ik} = \Delta_m^i \Delta_n^k f(0,0) = \left[\Delta_m^i \Delta_n^k f(m,n)\right]_{m,n=0} ,$$ where

$$\Delta_m f(m,n) = f(m+1,n) - f(m,n), \quad \Delta_n f(m,n) = f(m,n+1) - f(m,n).$$

The function c_{ik} related to $f(m,n)$ by (3) or equivalently by (4) is the function associated with $f(m,n)$.

ILLUSTRATIONS. We are especially interested in functions from ε^2 into ε for which the associated functions only assume values ≥ 0 , e.g.,

(A) $m+n = 1 \cdot \binom{m}{1} + 1 \cdot \binom{n}{1}$, where $c_{01} = c_{10} = 1$, $c_{ik} = 0$, otherwise,

(B) $m \cdot n = 1 \cdot \binom{m}{1}\binom{n}{1}$, where $c_{11} = 1$, $c_{ik} = 0$, otherwise,

(C) $\frac{(m+n)!}{m!n!} = \binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \ldots + \binom{m}{\ell}\binom{n}{\ell}$,

where $\ell = \min(m,n)$, $c_{ik} = 1$, for $i = k$, $c_{ik} = 0$, otherwise.

SUMMARY. We shall give set-theoretic characterizations of the functions $f(n)$, $f(m,n)$ and their associated functions c_i, c_{ik} in case c_i, c_{ik} only assume values ≥ 0.

§2. Notations and terminology. Let a subcollection of ε be called a set and a collection of sets a class. We write o for the empty set, Q for the class of all finite sets, $|\alpha|$ for the cardinality of α, $\subset$ for inclusion and $\subset_+$ for proper inclusion. For sets $\alpha, \beta, \gamma, \delta$,

$(\alpha,\beta) \neq (\gamma,\delta)$ means: $\alpha \neq \gamma$ or $\beta \neq \delta$,

$(\alpha,\beta) \subset (\gamma,\delta)$ means: $\alpha \subset \gamma$ & $\beta \subset \delta$,

$(\alpha,\beta) \subset_+ (\gamma,\delta)$ means: $(\alpha,\beta) \subset (\gamma,\delta)$ & $(\alpha,\beta) \neq (\gamma,\delta)$.

We need a function which maps ε^2 one-to-one onto ε. For this purpose we use the function

$$j(x,y) = \frac{(x+y)(x+y+1)}{2} + x .$$

For every $n \in \varepsilon$ there is exactly one ordered pair (x,y) with $j(x,y) = n$. The functions $k(n)$ and $\ell(n)$ are therefore well-defined by the identity $j\big(k(n),\ell(n)\big) = n$.

We also need an effective enumeration without repetitions of the class Q of all finite sets. For this purpose we use the enumeration $\rho_0, \rho_1, \ldots,$ where

$$\rho_0 = o,$$

$$\rho_n = \left\{ \begin{array}{l} (a_1,\ldots,a_i), \text{ where } a_1,\ldots,a_i \text{ are the} \\ \text{distinct numbers such that } n = 2^{a_1}+\ldots+2^{a_i} \end{array} \right\}, \text{ for } n \geq 1 .$$

Let $r_n = |\rho_n|$, then r_n is an effectively computable function of n. We put $\nu_n = \{x \in \varepsilon | x < n\}$, for $n \in \varepsilon$. Thus $\nu_0 = o$ and $\nu_n = (0,\ldots,n-1)$, for $n > 0$. A mapping from Q into Q or from Q^2 into Q is called an operator.

DEFINITION. An operator Φ

from Q into Q is numerical, if $|\alpha| = |\beta| \Rightarrow |\Phi(\alpha)| = |\Phi(\beta)|$,

from Q^2 into Q is numerical, if $|\alpha| = |\gamma|$ & $|\beta| = |\delta| \Rightarrow |\Phi(\alpha,\beta)| = |\Phi(\gamma,\delta)|$.

DEFINITION. For a numerical operator Φ

from Q into Q, $f_\Phi(n) = |\Phi(\nu_n)|$ = the function induced by Φ,

from Q^2 into Q, $f_\Phi(m,n) = |\Phi(\nu_m,\nu_n)|$ = the function induced by Φ.

§3. Combinatorial operators. We shall introduce two special types of numerical operators, namely the combinatorial operators and the dispersive ones. They are poles apart, but intimately related.

DEFINITION. A numerical operator $\Phi: Q \longrightarrow Q$ is combinatorial, if for $\Phi^\varepsilon = \bigcup\{\Phi(\alpha) | \alpha \in Q\}$, there is a mapping $\Phi^{-1}: \Phi^\varepsilon \longrightarrow Q$ such that

$$x \in \Phi(\alpha) \Leftrightarrow \Phi^{-1}(x) \subset \alpha, \qquad \text{for } x \in \Phi^\varepsilon,\ \alpha \in Q .$$

A numerical operator $\Phi: Q^2 \longrightarrow Q$ is combinatorial, if for $\Phi^\varepsilon = \bigcup\{\Phi(\alpha,\beta) | \alpha,\beta \in Q\}$, there is a mapping $\Phi^{-1}: \Phi^\varepsilon \longrightarrow Q^2$ such that

$$x \in \Phi(\alpha,\beta) \Leftrightarrow \Phi^{-1}(x) \subset (\alpha,\beta), \qquad \text{for } x \in \Phi^\varepsilon,\ (\alpha,\beta) \in Q^2 .$$

Both in the one-variable case and the two-variable case, Φ^{-1} is called a quasi-inverse of Φ. It is not an ordinary inverse, since (i) Φ need not be one-to-one and (ii) while Φ maps Q into Q or Q^2 into Q, the domain of Φ^{-1} is a set of nonnegative integers.

DEFINITION. A function $f(n)$ [or $f(m,n)$] is combinatorial, if it is induced by some combinatorial operator from Q into Q [or from Q^2 into Q].

ILLUSTRATIONS. (A) Let $\Phi(\alpha) = \{x \in \varepsilon | \rho_x \subset \alpha\}$, then Φ maps Q into Q and $|\alpha| = n \Rightarrow |\Phi(\alpha)| = 2^n$. Thus Φ is a numerical operator which induces the function $f(n) = 2^n$. Also, $\Phi^{\varepsilon} = \varepsilon$ and

$$x \in \Phi(\alpha) \Longleftrightarrow \rho_x \subset \alpha, \qquad \text{for } x \in \Phi^{\varepsilon},$$

hence $\Phi^{-1}(x) = \rho_x$, for $x \in \Phi^{\varepsilon}$, is a quasi-inverse of Φ. We have proved that Φ is a combinatorial operator and $f(n) = 2^n$ a combinatorial function.

(B) Let $\Phi(\alpha,\beta) = \{2x \in \varepsilon | x \in \alpha\} \cup \{2x+1 \in \varepsilon | x \in \beta\}$, then Φ maps Q^2 into Q and

$$|\alpha| = m \;\&\; |\beta| = n \Longrightarrow |\Phi(\alpha,\beta)| = m + n .$$

Thus Φ is a numerical operator which induces the function $f(m,n) = m+n$. Note that $\Phi^{\varepsilon} = \varepsilon$. For $x \in \Phi^{\varepsilon}$,

$$x \in \Phi(\alpha,\beta) \Longleftrightarrow \left[x \text{ even } \& \tfrac{x}{2} \in \alpha\right] \vee \left[x \text{ odd } \& \tfrac{x-1}{2} \in \beta\right].$$

Put

$$\Phi^{-1}(x) = \begin{cases} \left(\left\{\frac{x}{2}\right\}, o\right), & \text{if } x \text{ is even},\\[2ex] \left(o, \left\{\frac{x-1}{2}\right\}\right), & \text{if } x \text{ is odd}, \end{cases}$$

then Φ^{-1} is a quasi-inverse of Φ. We have proved that Φ is a combinatorial operator and $f(m,n) = m+n$ a combinatorial function.

(C) Let $\Phi(\alpha,\beta) = j(\alpha\times\beta)$, then Φ maps Q^2 into Q and

$$|\alpha| = m \;\&\; |\beta| = n \Longrightarrow |\Phi(\alpha,\beta)| = m\cdot n .$$

Thus Φ is a numerical operator which induces the function $f(m,n) = m\cdot n$. Also, $\Phi^{\varepsilon} = \varepsilon$ and

$$j(x,y) \in \Phi(\alpha,\beta) \Longleftrightarrow j(x,y) \in j(\alpha\times\beta) \Longleftrightarrow x\in\alpha \;\&\; y\in\beta$$

$$\Longleftrightarrow \big((x),(y)\big) \subset (\alpha,\beta) .$$

Substitution of $k(n)$ for x and $\ell(n)$ for y yields

$$n \in \Phi(\alpha,\beta) \Longleftrightarrow \Big(\big(k(n)\big),\big(\ell(n)\big)\Big) \subset (\alpha,\beta) .$$

Hence Φ has a quasi-inverse. We proved that Φ is a combinatorial operator and $f(m,n) = m\cdot n$ a combinatorial function.

(D) Let $\Phi(\alpha,\beta) = \left\{j(x,y)\in\varepsilon \mid \rho_x\subset\alpha \;\&\; \rho_y\subset\beta \;\&\; r_x = r_y\right\}$, then

$$|\alpha| = |\gamma| \;\&\; |\beta| = |\delta| \Longrightarrow |\Phi(\alpha,\beta)| = |\Phi(\gamma,\delta)| .$$

Thus Φ is a numerical operator. Note that

$$\Phi^{\varepsilon} = \left\{j(x,y)\in\varepsilon \mid r_x = r_y\right\} = \left\{n\in\varepsilon \mid r_{k(n)} = r_{\ell(n)}\right\},$$

$$j(x,y)\in\Phi(\alpha,\beta) \Longleftrightarrow \rho_x\subset\alpha \;\&\; \rho_y\subset\beta \Longleftrightarrow (\rho_x,\rho_y)\subset(\alpha,\beta) ,$$

since the condition $r_x = r_y$ holds for all $j(x,y)\in\Phi^{\varepsilon}$. Hence

$$n\in\Phi(\alpha,\beta) \Longleftrightarrow (\rho_{k(n)},\rho_{\ell(n)})\subset(\alpha,\beta) , \qquad \text{for } n\in\Phi^{\varepsilon} .$$

Thus $\Phi^{-1}(n) = \{\rho_{k(n)}, \rho_{\ell(n)}\}$ for $n \in \Phi^{\varepsilon}$, is a quasi-inverse of Φ. We proved that Φ is a combinatorial operator. Also,

$$f_{\Phi}(m,n) = \left|\Phi\{\nu_m, \nu_n\}\right| = \left|\left\{j(x,y) \in \varepsilon \mid \rho_x \subset \nu_m \ \&\ \rho_y \subset \nu_n \ \&\ r_x = r_y\right\}\right|$$

$$= \left|\bigcup_{i=0}^{\ell} \left\{j(x,y) \in \varepsilon \mid \rho_x \subset \nu_m \ \&\ \rho_y \subset \nu_n \ \&\ r_x = r_y = i\right\}\right| ,$$

where $\ell = \min(m,n)$. It follows that

$$f_{\Phi}(m,n) = \sum_{i=0}^{\ell} \left|\left\{j(x,y) \in \varepsilon \mid \rho_x \subset \nu_m \ \&\ \rho_y \subset \nu_n \ \&\ r_x = r_y = i\right\}\right|$$

$$= \sum_{i=0}^{\ell} \binom{m}{i}\binom{n}{i} = \frac{(m+n)!}{m!n!} = \binom{m+n}{m} = \binom{m+n}{n} .$$

We proved that $(m+n)!/m!n!$ is a combinatorial function.

THEOREM T2. For a combinatorial operator Φ of one variable,

$$\alpha \subset \beta \Longrightarrow \Phi(\alpha) \subset \Phi(\beta) , \qquad \text{for } \alpha, \beta \in Q .$$

PROOF. Let Φ^{-1} be a quasi-inverse of Φ. Assume $\alpha \subset \beta$, then

$$\left.\begin{array}{r} x \in \Phi(\alpha) \Longrightarrow \Phi^{-1}(x) \subset \alpha \\ \alpha \subset \beta \end{array}\right\} \Longrightarrow \Phi^{-1}(x) \subset \beta \Longrightarrow x \in \Phi(\beta) .$$

COROLLARY. Every combinatorial function of one variable is monotone increasing.

PROOF. For a combinatorial operator Φ of one variable,

$$m \le n \Longrightarrow \nu_m \subset \nu_n \Longrightarrow \Phi(\nu_m) \subset \Phi(\nu_n)$$

$$\Longrightarrow \left|\Phi(\nu_m)\right| \le \left|\Phi(\nu_n)\right| \Longrightarrow f_{\Phi}(m) \le f_{\Phi}(n) .$$

This corollary tells us that some of the simplest functions of combinatorics are not combinatorial. Let e.g., $f(t) = \binom{5}{t}$. Then $f(t)$ is not monotone increasing, hence not combinatorial (i.e., in the sense of Myhill).

THEOREM T2*. For a combinatorial operator Φ of two variables,

$$(\alpha,\beta) \subset (\gamma,\delta) \Longrightarrow \Phi(\alpha,\beta) \subset \Phi(\gamma,\delta) , \qquad \text{for } (\alpha,\beta),\ (\gamma,\delta) \in Q^2 .$$

The proof is similar to that of T2. As a corollary we now obtain the statement that a combinatorial function of two variables is monotone increasing in each variable. Thus $\binom{n}{k}$ is not a combinatorial function of two variables (i.e., in the sense of Myhill).

If Φ is a combinatorial operator we define

$$\Phi_o(\alpha) = \Phi(\alpha) - \bigcup\left\{\Phi(\gamma)\,|\,\gamma \subset_+ \alpha\right\} , \qquad \text{for } \alpha \in Q .$$

We claim that for a combinatorial operator Φ,

(5) $$\Phi_o(\alpha) = \left\{x \in \Phi^\varepsilon\,|\,\Phi^{-1}(x) = \alpha\right\} , \qquad \text{for } \alpha \in Q ,$$

(6) $$\Phi^{-1}(x) = \bigcap\left\{\alpha \in Q\,|\,x \in \Phi(\alpha)\right\} , \qquad \text{for } x \in \Phi^\varepsilon ,$$

(7) Φ uniquely determines its quasi-inverse and vice versa.

Re (5). $x \in \Phi_o(\alpha) \Longleftrightarrow x \in \Phi(\alpha)\ \&\ x \notin \bigcup\left\{\Phi(\gamma)\,|\,\gamma \subset_+ \alpha\right\}$

$\Longleftrightarrow x \in \Phi(\alpha)\ \&\ (\forall\gamma)\left[\gamma \subset_+ \alpha \Longrightarrow x \notin \Phi(\gamma)\right]$

$\Longleftrightarrow \Phi^{-1}(x) \subset \alpha\ \&\ (\forall\gamma)\left[\gamma \subset_+ \alpha \Longrightarrow \text{not } \left\{\Phi^{-1}(x) \subset \gamma\right\}\right]$

$\Longleftrightarrow \Phi^{-1}(x) = \alpha .$

Re (6). For $x \in \Phi^\varepsilon$,

$$\Phi^{-1}(x) = \bigcap\left\{\alpha \in Q \mid \Phi^{-1}(x) \subset \alpha\right\}, \qquad \text{since } \Phi^{-1}(x) \in Q ,$$

$$= \bigcap\left\{\alpha \in Q \mid x \in \Phi(\alpha)\right\} .$$

Re (7). Φ uniquely determines Φ^{-1} by (6). Moreover,

$$\Phi(\alpha) = \left\{x \in \Phi^{\varepsilon} \mid x \in \Phi(\alpha)\right\} = \left\{x \in \Phi^{\varepsilon} \mid \Phi^{-1}(x) \subset \alpha\right\} ,$$

hence Φ^{-1} uniquely determines Φ.

The definition of Φ_o and the relations (5), (6), (7) can be generalized to operators of two variables.

§4. Dispersive operators. We now define a second type of numerical operator.

DEFINITION. A numerical operator $\Psi : Q \longrightarrow Q$ is *dispersive*, if it maps distinct sets onto disjoint sets, i.e., if

$$\alpha \neq \beta \Longrightarrow \Psi(\alpha) \cap \Psi(\beta) = o .$$

A numerical operator $\Psi : Q^2 \longrightarrow Q$ is *dispersive*, if

$$(\alpha,\beta) \neq (\gamma,\delta) \Longrightarrow \Psi(\alpha,\beta) \cap \Psi(\gamma,\delta) = o .$$

We do not call a function c from ε into ε dispersive, if it is induced by some dispersive operator of one variable. For let c be any function from ε into ε. Define

$$\Psi(\alpha) = \left\{j(x,y) \in \varepsilon \mid \rho_x = \alpha \;\&\; y < c_{r(x)}\right\}, \qquad \text{for } \alpha \in Q, \text{ then}$$

$$r_x = i \Longrightarrow \Psi(\alpha) = \left\{j(x,y) \in \varepsilon \mid \rho_x = \alpha \;\&\; y < c_i\right\} \Longrightarrow |\Psi(\alpha)| = c_i ,$$

hence Ψ is a numerical operator which induces the function c. Moreover, $\alpha \neq \beta$ implies that $\Psi(\alpha)$ and $\Psi(\beta)$ are disjoint, hence Ψ is dispersive. Thus every function from ε into ε would be dispersive.

§5. The natural one-to-one correspondence.

THEOREM T3. There is a natural one-to-one correspondence between the family of all combinatorial operators of one variable and the family of all dispersive operators of one variable. It can be described as follows:

$$\Phi_0(\alpha) = \Phi(\alpha) - \bigcup\left\{\Phi(\gamma) \mid \gamma \subset_{+} \alpha\right\}, \qquad \text{if } \Phi \text{ is combinatorial,}$$

$$\Phi_\Psi(\alpha) = \bigcup\left\{\Psi(\gamma) \mid \gamma \subset \alpha\right\}, \qquad \text{if } \Psi \text{ is dispersive.}$$

Moreover, $\left(\Phi_\Psi\right)_0 = \Psi$.

THEOREM T4. A function

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}$$

is combinatorial iff $(\forall i)\left[c_i \geq 0\right]$. If f is combinatorial, f is induced by the combinatorial operator Φ and c by the dispersive operator Φ_0, where

$$\Phi(\alpha) = \left\{j(x,y) \in \varepsilon \mid \rho_x \subset \alpha \;\&\; y < c_{r(x)}\right\},$$

$$\Phi_0(\alpha) = \left\{j(x,y) \in \varepsilon \mid \rho_x = \alpha \;\&\; y < c_{r(x)}\right\}.$$

We shall prove a part of T4, namely:

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i} \ \& \ (\forall i)\left[c_i \geq 0\right] \Longrightarrow f(n) \text{ combinatorial.}$$

For assume the hypothesis. If $|\alpha| = n$ we have

$$\begin{aligned}
\left|\Phi(\alpha)\right| &= \left|\left\{j(x,y) \in \varepsilon \,|\, \rho_x \subset \alpha \ \& \ y < c_{r(x)}\right\}\right| \\
&= \left|\bigcup_{i=0}^{n} \left\{j(x,y) \in \varepsilon \,|\, \rho_x \subset \alpha \ \& \ r_x = i \ \& \ y < c_i\right\}\right| \\
&= \sum_{i=0}^{n} \left|\left\{j(x,y) \in \varepsilon \,|\, \rho_x \subset \alpha \ \& \ r_x = i \ \& \ y < c_i\right\}\right| .
\end{aligned}$$

Since $|\alpha| = n$, the condition $\rho_x \subset \alpha \ \& \ r_x = i$ is satisfied by $\binom{n}{i}$ values of x; also, $y < c_i$ holds for c_i values of y. Hence

$$|\Phi(\alpha)| = \sum_{i=0}^{n} c_i \binom{n}{i} = f(n) .$$

Thus Φ is a numerical operator which induces f. Moreover,

$$j(x,y) \in \Phi(\alpha) \Longleftrightarrow \rho_x \subset \alpha, \qquad \text{for } j(x,y) \in \Phi^{\varepsilon} ,$$

$$n \in \Phi(\alpha) \Longleftrightarrow \rho_{k(n)} \subset \alpha, \qquad \text{for } n \in \Phi^{\varepsilon} .$$

Hence Φ is a combinatorial operator and f a combinatorial function.

For the complete proofs of T3 and T4 see [2, P12, P13 and P20]. We now state the two-variable analogues of T3 and T4.

THEOREM T3*. There is a natural one-to-one correspondence between the family of all combinatorial operators of two variables and the family of all dispersive operators of two variables. It can be described as follows:

$$\Phi_o(\alpha,\beta) = \Phi(\alpha,\beta) - \bigcup\left\{\Phi(\gamma,\delta) \mid (\gamma,\delta) \subset_+ (\alpha,\beta)\right\}, \text{ if } \Phi \text{ is combinatorial,}$$

$$\Phi_\Psi(\alpha,\beta) = \bigcup\left\{\Psi(\gamma,\delta) \mid (\gamma,\delta) \subset (\alpha,\beta)\right\}, \qquad \text{if } \Psi \text{ is dispersive.}$$

Moreover, $\left(\Phi_\Psi\right)_o = \Psi$.

In the next theorem we need a function which maps ε^3 one-to-one onto ε. The function $j_3(x,y,z) = j[j(x,y),z]$ is such a function.

THEOREM T4*. A function

$$f(m,n) = \sum_{i=0}^{m} \sum_{k=0}^{n} c_{ik} \binom{m}{i} \binom{n}{k}$$

is combinatorial iff $(\forall i)(\forall k)\left[c_{ik} \geq 0\right]$. If f is combinatorial, f is induced by the combinatorial operator Φ and c by the dispersive operator Φ_o, where

$$\Phi(\alpha,\beta) = \left\{j_3(x,y,z) \in \varepsilon \mid \rho_x \subset \alpha \ \&\ \rho_y \subset \beta \ \&\ z < c_{r(x),r(y)}\right\},$$

$$\Phi_o(\alpha,\beta) = \left\{j_3(x,y,z) \in \varepsilon \mid \rho_x = \alpha \ \&\ \rho_y = \beta \ \&\ z < c_{r(x),r(y)}\right\}.$$

§6. <u>Interpretation of the function c_i</u>. Suppose

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}, \qquad \text{where } (\forall i)\left[c_i \geq 0\right]$$

and $f(n)$ can be defined conceptually, say as $|F_n|$, for some family F_n of entities associated with a finite set of cardinality n, say α. Then c_i can be interpreted as the contribution to $f(n)$ made by each i-element subset of α. If for example, $f(n) = 2^n$ so that $f(n)$ is the number of subsets of an n-element set α, each i-element subset of α with $0 \leq i \leq n$ contributes $c_i = 1$ element to $f(n)$. We now discuss two more examples of this type, namely:

(A) $f(n) = n! = \#$ of permutations of an n-set,

(B) $f(n) = n^p = \#$ of functions from a p-set into an n-set.

In order to associate numerical operators with the functions f and c we use appropriate Gödel numberings of the entities under consideration.

Re (A). $\mathcal{S}$ = the family of all permutations of ε,

$$\pi p = \{x \in \varepsilon \mid p(x) \neq x\}, \quad \mathcal{P} = \{p \in \mathcal{S} \mid \pi p \in Q\}.$$

The family $\mathcal{P}$ consists therefore of all permutations of ε which leave almost all elements of ε fixed. It is readily seen that there is an effectively computable function $p_m(x)$ of two variables such that $\mathcal{P}$ is enumerated without repetitions in the sequence $p_0(x), p_1(x), \ldots$ of functions from ε into ε. Define

$$\Phi(\alpha) = \left\{m \in \varepsilon \mid \pi p_m \subset \alpha\right\}, \qquad \text{for } \alpha \in Q,$$

then $|\alpha| = n$ implies $|\Phi(\alpha)| = n!$ Hence Φ is a numerical operator which induces the function $f(n) = n!$ Note that $\Phi^{\varepsilon} = \varepsilon$ and that $\Phi^{-1}(m) = \pi p_m$ is a quasi-inverse of Φ. Hence Φ is a combinatorial operator and $n!$ a combinatorial function. Using (5) we see that

$$c_i = \left|\Phi_o(\nu_i)\right| = \left|\left\{m \in \varepsilon \mid \pi p_m = \nu_i\right\}\right|$$

$= \#$ of <u>derangements</u> (permutations without fixed points) of ν_i.

Henceforth we write d_i for the number of derangements of an i-set. Thus

$$(8) \qquad n! = \sum_{i=0}^{n} d_i \binom{n}{i} .$$

Relation (8) reflects the fact that every permutation p of an n-set, say α, is characterized by a derangement of some subset of α, namely the set of all nonfixed points of p.

Re (B). Let $X_r = (x_1,\ldots,x_r)$, for $r \geq 1$ and

$$j_1(x_1) = x_1,\ j_2(x_1,x_2) = j(x_1,x_2),\ j_{r+1}(X_{r+1}) = j\left[j_r(X_r),\ x_{r+1}\right] .$$

For every $r \geq 1$ the function j_r maps ε^r one-to-one onto ε. Thus the functions $k_1(m),\ldots,k_r(m)$ are well-defined by the identity

$$j_r\left[k_1(m),\ldots,k_r(m)\right] = m .$$

Define $\pi = (1,\ldots,p)$ and for $\alpha \in Q$,

$$\Phi(\alpha) = \left\{j_p(x_1,\ldots,x_p)\,|\,x_1,\ldots,x_p \in \alpha\right\} , \qquad \text{hence}$$

$$\Phi(\alpha) = \left\{x\,|\,k_1(x),\ldots,k_p(x) \in \alpha\right\} .$$

Thus $|\alpha| = n$ implies $|\Phi(\alpha)| = n^p$, hence Φ is a numerical operator which induces the function n^p. Also, $\Phi^\varepsilon = \varepsilon$ and for $x \in \Phi^\varepsilon$,

$$x \in \Phi(\alpha) \iff \left\{k_1(x),\ldots,k_p(x)\right\} \subset \alpha .$$

Thus Φ has a quasi-inverse, namely $\Phi^{-1}(x) = \left\{k_1(x),\ldots,k_p(x)\right\}$. It follows that Φ is a combinatorial operator and n^p a combinatorial function of n, for every $p > 0$ (also for $p = 0$, since $n^0 = 1$). Using (5) we see that

$$c_i = \left|\Phi_0(\nu_i)\right| = \left|\left\{x \in \varepsilon \,\middle|\, \left(k_1(x),\ldots,k_p(x)\right) = \nu_i\right\}\right|$$

$$= \text{\# of surjective functions from } \pi \text{ into } \nu_i .$$

Henceforth we write $su(i,p)$ for the number of functions from π onto ν_i, i.e., from a p-set onto an i-set. Thus

$$n^p = \sum_{i=0}^{n} su(i,p) \binom{n}{i} . \tag{9}$$

Relation (9) reflects the fact that every function f from a p-set <u>into</u> an n-set, say α, is a function from that p-set <u>onto</u> some subset of α, namely the range of f.

The well-known formulas for d_i and $su(i,p)$, namely

$$d_i = i! \sum_{t=0}^{i} (-1)^t \frac{1}{t!} , \tag{10}$$

$$su(i,p) = \sum_{t=0}^{i} (-1)^t \binom{i}{t} (i-t)^p , \tag{11}$$

readily follow from formula (2) of section 1. For let $Ef(n)$ denote the function $f(n+1)$ of n and $If(n)$ the function $f(n)$ itself. Write 1 for I, then $E - 1 = \Delta$. If we substitute

$$(E - 1)^i = \sum_{t=0}^{i} \binom{i}{t} E^{i-t} (-1)^t$$

for Δ^i in (2), we obtain

(12) $$c_i = \sum_{t=0}^{i} (-1)^t \binom{i}{t} f(i-t) .$$

Taking $c_i = d_i$ and $f(n) = n!$ in (12) yields (10), while taking $c_i = su(i,p)$ and $f(n) = n^p$ in (12) yields (11).

§7. Composition. Let $f(n)$ and $g(n)$ be combinatorial functions, say,

(13) $$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i} , \qquad g(n) = \sum_{i=0}^{n} d_i \binom{n}{i} ,$$

where $c_i, d_i \geq 0$, for $i \geq 0$. Then $c_i + d_i \geq 0$, for $i \geq 0$, hence the function $f(n) + g(n)$ is also combinatorial. We now consider the following two questions:

(A) Is the function $f(n) \cdot g(n)$ combinatorial?

(B) Is the function $f(g(n))$ combinatorial?

The answer to each question is "Yes." Combinatorial operators enable us to prove this without the algebraic complications which arise from substitution involving the expressions listed in (13). Let the functions $f(n)$ and $g(n)$ be induced by the combinatorial operators Φ and Ψ respectively. Suppose Φ^{-1} and Ψ^{-1} are the respective quasi-inverses of Φ and Ψ.

Re (A). Put $\chi(\alpha) = j[\Phi(\alpha) \times \Psi(\alpha)]$, for $\alpha \in Q$. We claim that for $\alpha, \beta \in Q$,

(i) $\chi(\alpha) \in Q$,

(ii) $|\alpha| = |\beta| \Longrightarrow |\chi(\alpha)| = |\chi(\beta)|$,

(iii) $f_\chi(n) = f(n) \cdot g(n)$,

(iv) χ has a quasi-inverse.

Statements (i)-(iii) are almost immediate. As far as (iv) is concerned, we observe that $\chi^{\varepsilon} = j\left[\Phi^{\varepsilon} \times \Psi^{\varepsilon}\right]$, since each side is a subset of the other side. In the proof that the right side is included in the left side we use the fact that if $k(x) \in \Phi(\alpha)$ and $\ell(x) \in \Psi(\beta)$, then $k(x) \in \Phi(\alpha \cup \beta)$ and $\ell(x) \in \Psi(\alpha \cup \beta)$ by T2. Define

$$\chi^{-1}(x) = \Phi^{-1}[k(x)] \cup \Psi^{-1}[\ell(x)] , \qquad \text{for } x \in \chi^{\varepsilon}, \text{ then}$$

$$x \in \chi^{\varepsilon} \Longrightarrow k(x) \in \Phi^{\varepsilon} \;\&\; \ell(x) \in \Psi^{\varepsilon} \Longrightarrow$$

$$\Phi^{-1}[k(x)] \in Q \;\&\; \Psi^{-1}[\ell(x)] \in Q \Longrightarrow \chi^{-1}(x) \in Q .$$

Thus χ^{-1} is a well-defined mapping from χ^{ε} into Q. Moreover, for $x \in \chi^{\varepsilon}$ and $\alpha \in Q$,

$$x \in \chi(\alpha) \Longleftrightarrow k(x) \in \Phi(\alpha) \;\&\; \ell(x) \in \Psi(\alpha) \Longleftrightarrow$$

$$\Phi^{-1}[k(x)] \subset \alpha \;\&\; \Psi^{-1}[\ell(x)] \subset \alpha \Longleftrightarrow \chi^{-1}(x) \subset \alpha .$$

This completes the proof of (iv) and thereby of the affirmative answer to (A).

Re (B). Put $\chi(\alpha) = \Phi\Psi(\alpha)$, for $\alpha \in Q$. We claim that for $\alpha, \beta \in Q$,

(i) $\chi(\alpha) \in Q$,

(ii) $|\alpha| = |\beta| \Longrightarrow |\chi(\alpha)| = |\chi(\beta)|$,

(iii) $f_{\chi}(n) = fg(n)$,

(iv) χ has a quasi-inverse.

Statements (i) and (ii) are immediate, while (iii) follows from

$$|\Psi(\nu_n)| = g(n) \Longrightarrow |\Phi\Psi(\nu_n)| = |\Phi(\nu_{g(n)})| = fg(n) = f_{\chi}(n) .$$

As far as (iv) is concerned, we note that

$$x \in \chi^{\varepsilon} \Longrightarrow \Phi^{-1}(x) \text{ defined } \& \ \Phi^{-1}(x) \subset \Psi^{\varepsilon} .$$

For assume the hypothesis, say $x \in \Phi\Psi(\alpha)$, where $\alpha \in Q$. Then $\Phi^{-1}(x)$ is defined and $\Phi^{-1}(x) \subset \Psi(\alpha)$, hence $\Phi^{-1}(x) \subset \Psi^{\varepsilon}$. Define

$$\chi^{-1}(x) = \bigcup \left\{ \Psi^{-1}(y) \mid y \in \Phi^{-1}(x) \right\}, \qquad \text{for } x \in \chi^{\varepsilon}, \text{ then}$$

we have for $x \in \chi^{\varepsilon}$ and $\alpha \in Q$,

$$x \in \chi(\alpha) \Longleftrightarrow x \in \Phi\Psi(\alpha) \Longleftrightarrow \Phi^{-1}(x) \subset \Psi(\alpha) \Longleftrightarrow$$

$$(\forall y)\left[y \in \Phi^{-1}(x) \Longrightarrow y \in \Psi(\alpha)\right]$$

$$\Longleftrightarrow (\forall y)\left[y \in \Phi^{-1}(x) \Longrightarrow \Psi^{-1}(y) \subset \alpha\right]$$

$$\Longleftrightarrow \chi^{-1}(x) \subset \alpha .$$

This completes the proof of (iv) and thereby of the affirmative answer to (B).

We mention another theorem of this type: if $f(m,n)$ is a combinatorial function of two variables and $g(m), h(n)$ are combinatorial functions of one variable, then $f[g(m),h(n)]$ is a combinatorial function of two variables. All these theorems are special cases of a general theorem [5, Prop. (6.4), p. 381] which deals with the composition of combinatorial functions of any number of variables.

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