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REVIEW

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These two volumes bring to a triumphant conclusion the Oxford edition of the *Collected Works* of Kurt Gödel. They contain some of the most significant of Gödel's correspondences with leading logicians and mathematicians, as well as letters to and from editors and other less familiar figures. Purely personal letters are excluded; a few letters to Gödel's mother are included because of the light they shed on his theological views. Letters in German are printed in the original, with facing English translations.

As in the earlier volumes of the series, the books contain an exemplary editorial apparatus. Both volumes contain complete individual calendars for each correspondent, including unprinted letters. Each correspondence is prefaced by an introductory note giving information on the correspondent, and on the background to the letters, both biographical and scientific. Most of these notes have been contributed by the editors, but Akihiro Kanamori and David Malament were each responsible for two, Michael Beeson, Moshé Machover and Jens Erik Fenstad wrote one each, while Øystein Linnebo contributed to a jointly written note.

The correspondence with Paul Bernays is one of the most extensive in this collection, extending from 1930 to 1975, just three years before Gödel's death in 1978. It takes up about half of the pages in the first volume. The letters begin with Bernays's reactions to the stunning news of the incompleteness theorems, news that travelled rapidly in the small informal network of mathematicians and philosophers interested in the foundations of logic.

Today, when Gödel's ideas and results of 1931 are so much part of the fabric of the way we think in logic, it is difficult to imagine ourselves back in a time when they were brand new, and hard to assimilate for logicians brought up in the older traditions. In the case of Finsler and Zermelo, their misunderstandings of what Gödel had achieved were rooted in their inability to understand the purely formal viewpoint essential to the incompleteness results. The same misunderstandings do not apply in the case of Bernays, Hilbert's most important collaborator in the formalist programme; his initial difficulties with Gödel's theorems have a more specific source. Around 1930, it was widely believed that Wilhelm Ackermann and John von Neumann had provided finitistic consistency proofs for first order Peano arithmetic in their papers of 1924 and 1927. In his early letters to Bernays, Gödel lays bare the true situation with wonderful incisiveness and clarity.

The second group of letters in the Bernays correspondence, from 1939 to 1942, centre around the proof of consistency for the continuum hypothesis and Bernays's axioms of set theory that Gödel took as a basis for an extended exposition of his consistency proof. When the correspondence resumed in 1956 after a fourteen year lapse, it ranged over a wide variety of logical and philosophical topics, including Gentzen's first (unpublished) consistency proof for number theory, Kreisel's work on characterizing finitist reasoning, and philosophy of mathematics, including the work of Wittgenstein and the neo-Friesian school of Leonard Nelson. The most persistent theme in this last series of letters is Bernays's ultimately abortive attempts at publishing a revised and expanded English translation of Gödel's functional interpretation of intuitionistic arithmetic, first published in *Dialectica* in 1958. The proposed translation is first mentioned by Bernays in a letter of 1965, and is discussed in almost all the later letters of the correspondence. In the end, the revised translation remained unpublished during Gödel's life, and was printed for the first time in 1990 [4, pp. 271-280].

Rudolf Carnap was an important early influence on Gödel; he was among the first to learn of the incompleteness results. Gödel attended Carnap's lectures on logic, and also read the manuscript on which the lectures were based [1], unpublished during Carnap's lifetime. This manuscript was written within the older tradition of Whitehead and Russell, in which all logical considerations are carried out within a fixed formal language. In particular, Carnap has no notion of logical validity in the modern sense; what takes its place in the manuscript is the internal concept of "formal implication" due to Russell. Gödel pointed out in the introduction to his dissertation that if we replace the notion of logical consequence with Russell's formal implication, then completeness is provable in a few easy steps, as Carnap noted (Goldfarb's introductory note to the correspondence with Herbert G. Bohnert contains an illuminating discussion of these issues).

Under the impact of Tarski's work, Carnap abandoned his earlier project, and started work on a new one that eventually became *Logische Syntax der Sprache.* In his letter to Bohnert of 1974, Gödel modestly disclaimed much influence on this book, but the correspondence belies this. While Carnap was in Prague writing it, he sent Gödel a draft, and in return incorporated important criticisms and suggestions from him, particularly in connection with his semantical definition of "analytic." Gödel's letter of 28 November 1932 is particularly significant as it contains details of the promised sequel to the incompleteness paper of 1931. His discussion of the construction of a truth definition for a language containing all finite types throws considerable light on the famous footnote 48a of the 1931 paper.

The letters to and from Alonzo Church consist of four relatively brief exchanges, beginning with a discussion of Church's attempt at foundations for logic based on the concept of function. Church hoped that his system could evade Gödel's incompleteness result. As the editors note (Vol. IV, 362), he was right, though for an unwelcome reason; his axioms proved to be inconsistent. Perhaps the most interesting letter in this group was written by Gödel in response to Church's request for information about his unpublished independence results of 1942 (Church was to give a short talk on the work of Paul Cohen on his being awarded a Fields medal in 1966). Gödel replied that he could reconstruct his proof of the independence of the axiom of constructibility in type theory with the axiom of choice, but not his independence proof for the axiom of choice itself.

The correspondence with Paul J. Cohen unfortunately lacks Cohen's own letters, since he did not grant permission for their publication. However, their content is fairly clear from Gödel's own letters. Gödel's first letter of June 20 1963 is notable for its warmth and generosity. Cohen evidently felt himself under some strain, as he strove for acceptance of his great advances in set theory. He must have gained assurance and strength from Gödel's remark that he had just achieved "the most important progress in set theory since its axiomatization." Later letters from Gödel bring up the question of Hausdorff's conjectures concerning the ordering of numerical functions by rates of growth. These played a key role in Gödel's own abortive attempts at settling the Continuum Hypothesis [5, pp. 420-425].

The two letters to and from Jacques Herbrand are of great historical interest. The fundamental issues of the letters are the extent of finitistic methods, and the impact of Gödel's incompleteness theorems on Hilbert's consistency program. Herbrand, like von Neumann, did not see how it was possible for there to be an "intuitionistic" proof that is not formalizable in the system of the 1931 incompleteness paper (Herbrand apparently does not distinguish "finitist" from "intuitionistic"). On the other hand, he expresses scepticism that an exact description of intuitionistic proofs is possible, since we can always diagonalize out of any given family of constructive functions. Gödel in reply follows the cautious approach of his published paper concerning the implications for the Hilbert program, and even goes so far as to conjecture that there might exist a finitary proof not formalizable in *Principia Mathematica*. It is clear that he changed his mind on this point, since in his 1933 lecture to the Mathematical Association of America [5, pp. 45-53], he sharply distinguishes finitist from intuitionistic proofs, and states that all finitist proofs can be carried out in a fixed, restricted formal system.

In addition to their foundational interest, the letters are of importance in the historical development of the idea of computable function. Gödel in his 1934 Princeton lectures had claimed in a footnote that the central idea of his definition of general recursive function had been suggested to him by Herbrand in correspondence. When van Heijenoort queried Gödel about this (in a letter of 1963), Gödel replied that he could no longer find Herbrand's letter, but that he distinctly recalled that Herbrand had made the suggestion exactly as stated in the lecture notes. When Herbrand's letter was rediscovered by John Dawson in 1986, it became clear that Gödel was over-generous in acknowledging Herbrand's influence.

Herbrand, like Gödel in his 1934 lectures, defines a notion of computable function in terms of an equational calculus, by means of which a new function f is defined implicitly in terms of given functions. However, Gödel introduces two very significant alterations. First, he gives a precise set of rules for deriving numerical equations; second, he requires that for a given vector $k_1, \ldots, k_n$ of numerals, exactly one equation of the form $f(k_1, \ldots, k_n) = m$ is derivable (where m is a numeral). By contrast, Herbrand does not state precise derivation rules. Even more significantly, he requires that the uniqueness of the function f implicitly defined by the equations be demonstrable "by means of intuitionistic proofs." It is this last restriction that makes Herbrand's definition quite different from Gödel's. His definition depends on the vague notion of "intuitionistic proof," and so can not be considered mathematically precise. Even worse, any attempt to make the concept of "intuitionistic proof" precise will result in a proper subclass of computable functions. Herbrand was quite well aware of this last consequence, as he emphasizes in his letter to Gödel. Consequently, the terminology "Herbrand-Gödel" for the class of functions defined in the 1934 lectures, introduced by Kleene in his well known paper of 1936, must be considered a misnomer, even though sanctioned by Gödel's own remarks.

The correspondence with Karl Menger spans the years from 1931 to 1972, with some notable gaps. The early letters have to do with the incompleteness theorems, and Gödel's participation in Menger's mathematical colloquium. Later letters are concerned with Gödel's work in set theory, his visit to Notre Dame, and Menger's work in geometry. The early letters from Menger are remarkable for their informal and jocular tone; the later letters are more reserved, apparently reflecting a cooling in their relationship.

The brief exchange of letters with Emil Post is confined to 1938 and 1939. Post encountered Gödel at a meeting of the American Mathematical Society, and subsequently wrote him three letters detailing his own anticipation of Gödel's results in the 1920s, while expressing his own admiration of Gödel's own achievement. He rather touchingly apologizes for his "egotistical outbursts" on their encounter; Gödel responded briefly with reassuring and friendly remarks. Post's own letters make clear his error in delaying publication of his own results. He had hoped to give a kind of philosophical analysis proving the complete generality of his own notion of formal system. Gödel on the other hand concentrated on proving very concrete and specific results about a particular axiomatic system, while leaving somewhat indefinite the extent to which they applied in general (recall Church's initial scepticism on this point). Their full generality emerged subsequently.

Gödel admired the work of Abraham Robinson, and wished him to be his successor at the Institute for Advanced Study, a plan made impossible by Robinson's premature death in 1974. The logical content of their letters mostly centres around the power of non-standard analysis. As in the case of von Neumann, Gödel sent his friend a letter on logical topics as he was dying, trying to take his mind off his terminal illness.

The correspondence with Alfred Tarski stands out for the warmth of friendship exhibited in it. Tarski is unique among the German-speaking logicians in these volumes in being addressed by the familiar form (even Paul Bernays remains "Sie" to the very end, although a 1928 letter to Herbert Feigl uses the familiar form). Only five letters appear here, since a great deal of their correspondence was purely personal. They touch on topics such as intuitionistic propositional logic, Gödel's contribution to the Princeton Bicentennial Conference in 1946, and Scott's proof that $V = L$ is incompatible with the existence of a measurable cardinal.

The correspondence with Stanisław Ulam is sparse and sporadic, touching on problems of set theory, including measurable cardinals, the consequences of the axiom of constructibility for the theory of projective sets and Cohen's independence results. As Kanamori points out in his introductory note, Ulam's second letter throws considerable doubt on Kreisel's claim [3, p. 197] that it was Ulam, not Gödel, who noticed that $V = L$ implies the existence of a non-measurable $\mathbf{\Sigma}^1_2$ set of reals.

The correspondence with John von Neumann is one of the most remarkable in this collection. It opens with von Neumann's letter of 20 November 1930 excitedly reporting his independent discovery of the second incompleteness theorem (by a direct argument, rather than by formalizing the first incompleteness theorem). Gödel must surely have breathed a sigh of relief at being able to tell von Neumann in reply that he had already presented the second incompleteness theorem to the Vienna Academy of Sciences on 23 October 1930 (the incompleteness paper itself was submitted on 17 November 1930, so Gödel came close to being anticipated in his great discovery). Von Neumann's third letter to Gödel, of 12 January 1931, is also fascinating. He not only sketches a treatment of the second theorem in terms of formal provability conditions, but also claims that he has a decision procedure for the variable-free fragment of modal provability logic (containing only propositional constants). Thus it seems that von Neumann in 1931 already had a solution to the 35th of Harvey Friedman's one hundred and two problems [2]; George Boolos in 1976 gave the first published solution.

Later letters in the von Neumann correspondence are concerned with details of the consistency proof for the generalized continuum hypothesis. Perhaps the most astonishing letter in the collection, however, is Gödel's last letter to von Neumann, written at a time when the latter was dying of cancer. It appears to have been intended to take von Neumann's mind off his illness, and to cheer him up. In it, Gödel raises a problem having to do with the complexity of computations, at a time when no other logicians seem to have been concerned with such questions. More specifically, he asks whether there could be a feasible algorithm for questions of the form: "Does there exist a proof of length n of formula ϕ of the lower predicate calculus?" Formulated in a sufficiently general way, this turns out to be very closely related to the famous P versus NP problem of computer science. It is astounding that Gödel was thinking about such problems as early as 1956.

Hao Wang was perhaps the most important of Gödel's interlocutors in later life. Their correspondence began in 1967 with an enquiry of Wang about the relationship of Skolem's work of 1922 to his own completeness theorem. Gödel points out that although the essential mathematical ideas of his own proof are already present in Skolem's paper, Skolem failed to establish the result. Gödel attributes this to the contemporary prejudices against non-finitary reasoning, and his own successes, here and in his later work, to his own objectivistic conceptions. Later letters are concerned with setting up the rather odd arrangement whereby Gödel published some of his most interesting philosophical ideas and remarks in Wang's 1974 book *From Mathematics to Philosophy*. From the point of view of philosophy of mathematics, this is one of the most intriguing group of letters in the collection.

The second volume ends with the correspondence with Ernst Zermelo, one that Gödel must have found very frustrating because of Zermelo's inability to understand the basic ideas of the incompleteness theorems. Zermelo did not seem to be capable of grasping the basic ideas of formal syntax and formal derivations, but instead inveighed against the "finitistic prejudice" of contemporary mathematicians, telling Gödel that his proof contained a gap. Gödel patiently replied with a letter in which he tried to dispel Zermelo's misunderstandings, but received a response exhibiting still further muddle and confusion, on which Gödel broke off the correspondence. In his letter of 11 September 1932 to Carnap, he alludes to Zermelo's "nonsensical criticism" of his paper (in his younger days, Gödel was noticeably less guarded in his utterances than later in life).

In addition to the major correspondences discussed above, the volumes also contain briefer interchanges with many other figures in logic and philosophy, including Heinrich Behmann, William Boone, J. Richard Büchi, Burton Dreben, Paul Finsler, Arend Heyting, Karl Popper, Bertrand Russell, Thoralf Skolem and Jean van Heijenoort. A major logical correspondent of Gödel not represented in these volumes is Georg Kreisel (since he did not allow publication of his letters). In his case, the correspondence was entirely one-sided, except for one letter from Gödel (not printed), but it is a pity that a significant source of information on Gödel's interests is missing.

The volumes also contain numerous letters that throw light on Gödel's biography and philosophical interests. An unusual item on the philosophical side is the fairly lengthy correspondence with Gotthard Günther, who was interested in the German idealist tradition, but also in modern mathematical logic. Gödel shows considerable patience with his rather strange interpretations of logical ideas.

Gödel was a notoriously difficult person for editors, and a good part of the two volumes is taken up with the protracted negotiations that he often inflicted on them. Paul Arthur Schilpp was perhaps the most successful in dealing with this temperamental author, and he succeeded in extracting two of Gödel's best essays for the Russell and Einstein volumes in his Library of Living Philosophers, though he failed in the case of the Carnap volume. His letters show that he succeeded by a judicious mixture of flattery and firm persuasion. Ernest Nagel, however, failed to convince Gödel to allow a translation of his 1931 incompleteness paper to be published as an appendix to his popular exposition co-authored with James R. Newman [6] and their interchanges ended in acrimony and recriminations. The editors (Vol. V, 142) note that a letter from Wilson Follett at New York University Press is the only document known to them in which someone "really chews Gödel out."

Gödel guarded his privacy jealously, and projected a severe and forbidding appearance to the outside world. His letters to those whom he admitted to his inner circle, however, present a very different person, warm and generous with both his friendship and with suggestions and ideas for work in logic and philosophy. The correspondences with Bernays, Boone, Cohen, Feigl, Robinson and Tarski all contribute in different ways to form a picture of the Kurt Gödel that his friends knew.

When the editorial project began, there were plans to publish some of the unpublished work in Gödel's notebooks on logic and philosophy, mostly written in Gabelsberger shorthand. However, these did not bear fruit, although Volume V contains a complete listing of the collection of Gödel's papers at Princeton. As a consequence, Gödel's unpublished 1942 proof of the independence of the axiom of constructibility in the theory of types, discussed in his 1966 letter to Church, as well as in a letter of 1967 to Wolfgang Rautenberg, is still a mystery. It is to be hoped that further work in Gödel's papers will clear up this remaining historical puzzle.

These are beautifully produced volumes, books that anybody interested in the history of logic in the twentieth century will wish to own, read and reread.

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