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Review of

PHILIP EHRLICH (ED.), *REAL NUMBERS, GENERALIZATIONS OF THE REALS, AND THEORIES OF CONTINUA*

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This is Volume 242 of the *Synthese Library: Studies in Epistemology, Logic, Methodology, and Philosophy of Science.* Therein:

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E. W. Hobson, "On the Infinite and the Infinitesimal in Mathematical Analysis," presidential address to the London Mathematical Society (November 13, 1902), pp. 3-26. Reprinted from *Proceedings of the London Mathematical Society* 35 (1902), pp. 117-140.

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Giuseppe Veronese, "On Non-Archimedean Geometry," invited address to the Fourth International Congress of Mathematicians, Rome, 1908, pp. 169-187. Translated by Mathieu Marion with editorial notes by Philip Ehrlich.

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In his introduction, editor Philip Ehrlich asserts that the arithmetico-set-theoretic conception of real number due to Cantor and Dedekind supplanted two competing conceptions of magnitude — one Euclidean and geometrical and the other invoking infinitesimals. While the Cantor-Dedekind conception now forms a cornerstone of orthodox foundations of mathematics, the twentieth century has witnessed a number of important generalizations of the real number system and various alternative theories of continua. It is these generalizations and alternatives that form the subject matter of this anthology.

A useful vantage point from which to consider the need for an arithmetic account of the continuum, as perceived by Dedekind, say, is that of analytic number theory. During the first half of the nineteenth century, Dirichlet and others obtained surprising number-theoretic results using arguments involving continuous variables and the theory of limits. If, however, the latter were to turn out to rest ultimately upon the "geometrically evident," then the results of Dirichlet would depend upon the likes of spatial intuition. Dedekind's goal of establishing a truly "scientific foundation for arithmetic" was an effort to free the theory of the continuum, and hence the theory of limits, from any reliance upon geometrical intuition. By arithmetizing the continuum, Dedekind thereby secured Dirichlet's striking results concerning the primes.

Of course, in doing so, Dedekind laid himself open to attacks from those who argued against the philosophical cogency of any reduction of the continuous to the discrete. In the words of Du Bois-Reymond (1882), as quoted by Ehrlich: "No matter how dense a series of points may be, it can never become an interval, which always must be regarded as the sum of intervals between points." And Ehrlich's sympathies apparently lie with those such as Du Bois-Reymond, who challenged the new orthodoxy on philosophical grounds. More generally, Du Bois-Reymond and others were objecting to the new situation whereby the ordered field of real numbers was deemed an adequate replacement for two competing conceptions of the continuum, one geometrical and composed of infinitesimal line segments, the other analytical and composed of infinitesimal quantities of some sort.

E. W. Hobson's 1902 address consists, in large measure, of a historical account of ideas regarding the infinitely large and the infinitely small leading up to the Cantor-Dedekind philosophy of the continuum, which he wishes to defend. Hobson's address is useful in reminding us of the extent to which the ideas of Cantor concerning the transfinite had their origin in the study of real analysis. Its position at the front of this anthology also serves as another reminder of the degree to which the various theories of the continuum propounded in the more recent papers in this anthology are grounded in various informal conceptions — each of which amounts to a philosophy of the continuum — detailing the intuitive properties of $\mathfrak{R}$: its property of continuity, its unbounded character, its "consisting of one piece" (its unbrokenness or connectedness), and so forth. Any mathematical theory of the continuum must attempt to account for these intuitive properties, and Hobson's survey of the method of indivisibles, the method of infinitesimals, the introduction of fluxions, and so forth, places Cantor-Dedekind at the end of a long series of such attempts. As for the Cantor-Dedekind philosophy of the continuum itself, we might characterize it as follows, if only for the purpose of clarifying differences with views to be considered later.

(1) The Cantor-Dedekind philosophy of the continuum postulates a correspondence between the geometric linear continuum and the arithmetized real continuum.
(2) All the usual properties, *e.g.*, connectedness, of continuous phenomena may be reduced to corresponding properties of discrete phenomena. Related to this, there is nothing incoherent about discrete entities being used to model continuous ones.

Both (1) and (2) are mirrored in Cantor's and Dedekind's technical achievements. For, as is well known, assuming the Hausdorff-Wiener-Kuratowski definition of the ordered pair, Cantor and Dedekind showed how the notion of real number may be defined, set theoretically, in terms of the notion of natural number. Zermelo then took it a step further and showed how the latter notion is itself set-theoretic. Zermelo's own concepts- and objects-reductionism may thereby be seen as an extension of the Cantor-Dedekind philosophy of the continuum:

with regard to (2) in particular, continuous entities, like all mathematical objects, are ultimately sets, and their properties are set-theoretic properties (see [6]).

The most well-known alternatives to Cantor-Dedekind are the various constructivist conceptions of the continuum. Nearly one-quarter of Ehrlich's anthology is given over to Douglas Bridges' description of just one of these conceptions, namely, that of Bishop. Bridges' discussion is very clear and completely self-contained, so that it will make an excellent, relatively rapid introduction to the constructivist school of Bishop and his students for those who lack the time to work through [1]. Along the way, Bridges distinguishes Bishop's brand of constructivism from the recursive constructivist school associated with Markov, as well as from the sort of constructivism associated with Brouwer. Very briefly, Markov is described as committed to the thesis that every sequence of natural numbers is recursive. Brouwer adheres to two principles that ensure strong continuity properties of functions on intervals of the real number line. Bridges does not argue for his own particular brand of constructivism, however. Nor does he indicate what form such arguments would take. Rather, he is content here to investigate the conception of the continuum that follows from not adopting any of the principles adhered to by competing schools.

It is remarkable that the statements of many of Bridges' theorems generally differ not at all from their classical counterparts — something that is frequently pointed out with respect to Bishop's writings as well. On the other hand, Bridges' definitions of central notions such as function, empty set(s), countable set, and so on differ decisively from those of classical mathematics. Typically, the effect of this is that, whereas the classical proof of a given proposition may be completely trivial, the constructivist proof of that proposition will be nontrivial. If one grants that the meaning of proposition P is revealed only by its proof, then it follows that the meaning of P within the constructivist setting will differ considerably from its meaning within the classical setting. For the classically trained reader, Bridges' proofs, although not really hard to follow, frequently have a quite unpredictable character. This is often due to Bridges' use of the following constructively acceptable replacement (4.9v) for Trichotomy: for any reals x and y, if $x > y$, then for any real z, either $x > z$ or $z > y$.

Bridges' assertion on page 31 that "every proof of a proposition P within [Bishop's constructive mathematics] is ... a proof of P in classical mathematics" must be construed accordingly. What must be meant here, of course, is that the constructive proof of P, without any modification whatever, becomes a classical proof of P', where P' is the result of building into P the constructive understandings of the various concepts such as empty set, function, countable set, and so on that occur within it. To give an example, consider the proposition P, stating that a nonempty set is countable if and only if it is the range of a function with domain $\mathcal{N}$ (pp. 38-39). Within this proof, the notions of nonempty set, countable set, and function are given their constructivist readings. In particular,

(1) A set is nonempty provided we can construct a member of that set.
(2) A function with domain A and range B is an algorithm that associates a unique member of B with any member of A.
(3) A set A is countable if there exists a function mapping a decidable subset of N onto A.

Explicitly incorporating these understandings into the statement of the proposition, one obtains the following proposition P': where the notions nonempty set, function, and countable set are understood as in (1)–(3), we have that a nonempty set is countable if and only if it is the range of a function with domain $\mathcal{N}$. Now Bridges' proof is acceptable, even within classical mathematics, as a proof of P'.

Turning specifically to the real number system, what is the constructivist philosophy of the continuum in Bishop's sense? Reals are defined as Cauchy sequences of rationals and a notion of equality is introduced that is an equivalence relation. Bridges' discussion focuses on a variety of classical properties of the reals that do not hold in the constructivist context. Among these are:

(1) For all reals x and y, we have that $xy = 0$ implies that either $x = 0$ or $y = 0$.
(2) For any real x and y, either $x > y$ or $x = y$ or $x < y$ (Trichotomy).
(3) For any real x, either $x \geq 0$ or $x \leq 0$.
(4) For any real x, if the assumption that $x \leq 0$ leads to contradiction, then $x > 0$.

Bridges' method consists in showing that each of (1) through (4) entails one of three classical principles concerning binary sequences that is not acceptable to constructivists. This gives a welcomed unity to his development of the constructivist theory of the continuum. The classical principles in question are:

- The Limited Principle of Omniscience (LPO): if (a_n) is a binary sequence, then either there exists n such that $a_n = 1$ or else $a_n = 0$ for all n.

- The Lesser Limited Principle of Omniscience (LLPO): if (a_n) is a binary sequence containing at most one term equal to 1, then either $a_n = 0$, for all even n, or else $a_n = 0$, for all odd n.
- Markov's Principle (MP): if, for a binary sequence (a_n), it is impossible that all the terms equal 0, then there exists (that is, we can find) a value m such that $a_m = 1$.

The last of these principles, we are told, is accepted by the school of recursive constructive mathematics but not by either Brouwer or Bishop. Those who reject it base their criticism upon the fact that no bound is provided in advance on the number of terms that must be examined before a 1 will be found. Later on, the following additional, classically acceptable but constructively unacceptable principles are cited:

- The Principle of Finite Possibility (PFP): to each binary sequence (a_n) there corresponds a binary sequence (b_n) such that, for all n, $a_n = 0$ if and only if $b_m = 1$ for some m.
- The Weak Limited Principle of Omniscience (WLPO): if (a_n) is a binary sequence, then either, for all n, we have $a_n = 0$ or else it is not the case that, for all n, we have $a_n = 0$.

Bridges, following Bishop, rejects the usual formulation of the Axiom of Choice (AC) and presents the proof of Goodman and Myhill that AC implies the Law of Excluded Middle. On the other hand, Bridges accepts the so-called Principle of Dependent Choice (DC) as given on page 41.

Bridges shows that the set of reals is complete in the sense that it contains a limit for each of its Cauchy sequences. A discussion of open and closed sets of reals is provided, where a set is closed provided that it contains all its cluster points essentially. In the constructivist setting, we have (1) the complement of an open set is closed but (2) from the fact that the complement of S is closed, it does not follow that S itself is open. Further, (3) the complement of a closed set may not be open and (4) from the fact that the complement of S is open, it does not follow that S itself is closed. Most strikingly, we have (5) the union of two closed sets is not in general closed. Again, in the case of each of (2)–(5), Bridges' proof consists of an appeal to the unacceptability of either LPO or LLPO. For example, regarding (5), he shows that if taking unions were to preserve closedness, then LLPO would follow.

One of the classical analyst's tools is the rule that every nonempty set of reals that is bounded above possesses a least upper bound or supremum. In the constructivist setting, this Least-Upper-Bound Principle implies LPO. A constructive substitute is available in the form: *Let S be nonempty and bounded above. Then S has a supremum if and only* *if, for all $x, y \in \mathfrak{R}$ with $x < y$, either y is an upper bound of S or there exists z in S with $z > x$.* A corresponding result regarding infima is available as well.

The important notion of locatedness is introduced on page 58: where X is a set of reals, a nonvoid subset S of X is *located in* X if, for each x in X, we have that $\inf\{|x - y| : y \in S\}$ exists. In the absence of the classical mathematician's Greatest-Lower-Bound Principle, not every bounded set S of reals is going to be located (in $\mathfrak{R}$) in this sense. So when is such an S located? Bridges shows that it is necessary and sufficient that S be totally bounded in the sense that, for arbitrary $\varepsilon > 0$, there exists nonvoid finite $E \subseteq S$ such that, for each $x \in S$, there exists $y \in E$ with $|x - y| \leq \varepsilon$. (Subset E is termed a [finite] ε-approximation to S.) Further, it is shown that if every nonvoid closed set of reals were to be located (in $\mathfrak{R}$), then LPO would follow.

Bridges' Theorem 8.2 amounts to a constructive version of Cantor's Theorem and at the same time attempts to clarify a certain misconception. Namely, it is often said that the constructive continuum is countable. However, this can only mean that it is countable when viewed classically. Within the constructivist setting, the continuum is uncountable, which is what Theorem 8.2 says in effect. (This sort of situation will be familiar to any reader who has considered the so-called Skolem Paradox.) Bridges' proof of Theorem 8.2 is related to the notion of the ternary (base-3) expansion of a real number. As it turns out, in the constructivist setting, not every real number has a ternary expansion. (Supposing the contrary commits us to LLPO.) Moreover, even if two reals x and y do happen to possess expansions, sum $x + y$ may yet have none, which explains why the constructivist, in developing a theory of the continuum, cannot focus exclusively upon reals with computable expansions.

In the classical setting, every pointwise continuous function f is uniformly continuous as well: the character of its continuity does not vary throughout its entire domain. Bridges shows that, in the constructivist setting, matters are very different. Namely, assuming the Church-Markov-Turing Thesis whereby every partial number-theoretic function is partial recursive, there exist pointwise continuous functions that are not uniformly continuous. The final three sections of Bridges' article investigate constructive concepts of locatedness, density, convexity, and connectedness for subsets of $\mathfrak{R}$. With respect to convexity in particular, Bridges shows how a single classical concept yields a variety of classically equivalent, but constructively nonequivalent, concepts, *viz.*, convexity, weak convexity, paraconvexity, weak paraconvexity, and ultraweak paraconvexity. For example, any weakly paraconvex set turns out to be ultraweakly paraconvex, even constructively. But the converse implies LPO and hence is constructively unacceptable. Bridges' point regarding the branching of a single classical concept into a plurality of nonequivalent constructive concepts is illustrated again in the case of connectedness. Classically, the connected subsets of $\mathfrak{R}$ are all and only the intervals. But connectedness, in the constructivist setting, spawns three distinct properties, of which it can be said constructively only that the first (C-connectedness) implies the second (O-connectedness), which in turn implies the third (connectedness) (Theorem 14.9). For instance, it is shown that, assuming LPO to be false, there exists a connected set that is not O-connected (Theorem 14.17). Further, any interval has all three properties (Theorem 14.13). On the other hand, if any of the three properties should turn out to characterize precisely the intervals, then the three properties would coincide, contradicting Theorem 14.17.

The great value of Bridges' contribution to this anthology lies in the clarity of his exposition and in his habit of repeatedly showing how the constructivist's rejection of various classical principles can serve as the methodological basis for the investigation of the properties of the continuum. No doubt, Bridges' unified presentation — always showing how our classical preconceptions conflict with the constructivist's rejection of certain classical principles — will enable many readers to come away with a much clearer idea of just what constructivism is. Just as valuable is his having clearly exposed a certain methodology that effectively likens mathematics to the empirical sciences. Namely, imagine the situation whereby a constructivist proceeds to develop the theory of the continuum in a way that involves showing that every nonvoid closed set of reals is located (see above). Later on, someone recognizes that this would imply LPO, at which point the constructivist reexamines the proof in order to uncover some constructively unacceptable step. If the variety of constructivism at issue is consistent, then there will be such a step, and identifying it may necessitate other changes in the theory. Something like this occurs in the experimental sciences. Imagine laboratory results that conflict directly with some well-established set of equations. In such a case, the scientist would presumably look back over the results in question in order to discern some faulty procedure or inexact measurement and would no doubt repeat the experiment. This sort of analogy suggests that the methodology of mathematics is not so different from that of the empirical sciences. But the broader analogy is to the methodology of the systematic philosopher or, more generally, to that of any rational individual attempting to render a belief set consistent.

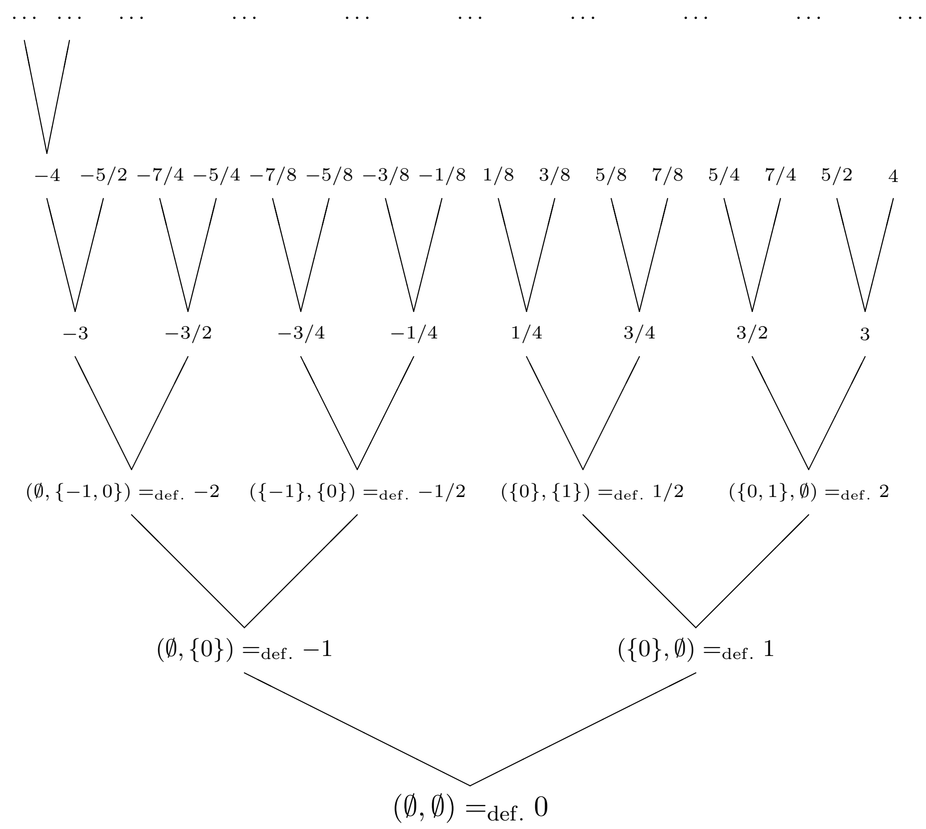
In his brief contribution to this volume, J. H. Conway looks back at his construction of the reals in [2]. He describes that construction only in broad outline, providing few details and no proofs. Nonetheless, it is a useful and engaging introduction to an important alternative theory of real numbers in the tradition of Cantor and Dedekind. Real numbers turn out to be a subclass of the so-called surreal numbers, where a surreal number is anything of the form (X_L, X_R) with X_L and X_R sets of numbers and $x > y$ for no $x \in X_L$, $y \in X_R$. (It is easiest to think of (X_L, X_R) as the ordered pair $\langle X_L, X_R \rangle$, although Conway himself urges that surreals be construed as "bisets" characterized by both left- and right-membership.) If X_L and X_R are permitted to both consist solely of rationals such that $X_L \cap X_R = \emptyset$ and $X_L \cup X_R = \mathcal{Q}$, then we have the Dedekind reals. On the other hand, X_L and X_R may contain previously constructed surreals starting with $0 =_{\text{def.}} (\emptyset, \emptyset)$. It is this, together with the peculiar semantics of quantification over the empty domain, that gives Conway's theory its novel features. We mention the following.

(1) $(\{0, 1, 2, \ldots\}, \emptyset)$, which Conway calls ω, satisfies the definition of surreal given above. Moreover, it can be shown that $\omega > n$ for any natural number n in the sense that is given to the relation $>$. Thus there exist surreals that are infinite.
(2) Conway, with some misgivings, suggests that real numbers be taken to be finite surreals of the form $(\{x-1, x-1/2, x-1/3, \ldots\}, \{\ldots, x+1/3, x+1/2, x+1\})$, where x is a (finite) surreal number.
(3) Analogous to (1), there exist surreals that are infinitesimal. Take, for example, the surreal defined by $(\{0\}, \{1, 1/2, 1/3, \ldots\})$, which is smaller than any positive real.

Along the way, Conway makes a number of interesting logical points. It turns out that, under the proposed definition of $=$ for surreal numbers, the equivalence class of any real x under $=$ will be a proper class in any model of ZF. This is unfortunate to the extent that it is natural to think of the continuum as the collection of these equivalence classes. Still, Conway asserts that his approach affords the simplest construction of the reals "from a purely logical point of view." Part of what this involves apparently is a definition of multiplication that enables us to avoid the usual splitting into cases according to the signs of the factors. (Conway makes much of the pedagogical value of his definition of multiplication for surreal numbers appearing on page 96 [see below].) Also, it is noted that the theory of reals that emerges from the notion of surreal number entails a rather rigid ordering of both construction and proof. For example, 1/4 must be constructed before either 3/8 or 1/3, and $0 \leq 0$ must be proved before any propositions concerning other numbers can be proved. Finally, in considering briefly an alternative construction of the surreals as binary sequences (cf. [3]), Conway remarks that if we want to define the ordinal numbers as certain surreals, "then we should not make the definition of the surreals depend on some previously defined notion of ordinal number." This remark suggests the possibility of predicative versions of this alternative construction.

Conway's invention of surreal numbers is a reflection of the extent to which Hilbert's axiomatic conception, making conceptual independence paramount and relativizing notions of mathematical truth and existence to particular models, has become the presiding *Zeitgeist* of twentieth-century mathematics. Knuth's [5], novelistic in character, brings home the point by portraying Conway as a god figure whose creative activity appears to be restricted to number systems. (The same thing could be said of the results presented in the papers of Keisler and Klaua [see below]).

Philip Ehrlich's own paper in his anthology presents, in some detail, his alternative tree-theoretic account of Conway's theory of surreal numbers. He begins by briefly recounting the classical presentation as contained in [2] and [5]. As mentioned above, we take $x = (X_L, X_R)$ to be a surreal provided $X_L \not\geq X_R$, *i.e.*, provided no member of X_L is $\geq$ any member of X_R. Even though $\geq$ has yet to be defined, our understanding of quantification already renders $0 =_{\text{def.}} (\emptyset \,|\, \emptyset)$ a surreal number. For the case where left- and right-sets are nonempty, one stipulates that $x \leq y$ provided $X_L \not\geq y$ and $x \not\geq Y_R$, *i.e.*, provided no member of X_L is $\geq y$ and no member of Y_R is $\leq x$. From this minimal beginning, one generates everything. Relation $\leq$ is shown to be transitive. Further, for arbitrary $x = (X_L, X_R)$, we have that every member of X_L is $< x$ and that x is $<$ every member of X_R, where $x < y$ provided $x \leq y$ but not vice versa. Relation $\leq$ is seen to totally order No, the class of all surreals. Moreover, if $Y_L < x < Y_R$, then $x \equiv Y_L \cup X_L, X_R \cup Y_R)$, where we are writing $a \equiv b$ just in case both $a \leq b$ and $b \leq a$. Addition is defined by $x + y =_{\text{def.}} ((X_L + y) \cup (Y_L + x), (Y_R + x) \cup (X_R + y))$ and is shown to have all the usual properties. Subtraction is defined by $x - y =_{\text{def.}} x + (-y)$, where $-y = (-Y_R, -Y_L)$. Similarly, subtraction is shown to be well-behaved, *e.g.*, $x - x \equiv 0$ and $(x + y) - y \equiv x$. It can be shown that, given any number y, if x is the first number created with the property that $Y_L < x < Y_R$, then $x \equiv y$. Finally, multiplication is defined by $xy =_{\text{def.}} ((X_L y + xY_L - X_L Y_L) \cup (X_R y + xY_R - X_R Y_R), (X_L y + xY_R - X_L Y_R) \cup (X_R y + xY_L - X_R Y_L))$. And we have that $xy = yx$, $0y = 0$, $1y = y$, and $-(xy) = (-x)y$. Further, multiplication is associative and distributes over addition.



Ehrlich's goal is to describe the complete binary tree $\langle No, <_S \rangle$, where $x <_S y$ provided x is a child of y in that tree (see figure). (An infinite binary tree is complete if every node has two children and if every infinite path starting at the root possesses one successor node.) Thus, 1/4 is said to be simpler than 1/8 but not simpler than 5/4, although both 1/8 and 5/4 appear at a level of the tree that is higher than the level at which 1/4 appears. Ehrlich defines a surreal x to be an ordered pair (X_L, X_R) with $X_L \cap X_R = \emptyset$ such that (1) if $y \in X_L \cup X_R$, then y is itself of the form $\langle Y_L, Y_R \rangle$, where $Y_L \cap Y_R = \emptyset$ and $Y_L \subseteq X_L$ and $Y_R \subseteq X_R$, and (2) there exists a well-ordering $<_x$ of $X_L \cup X_R$ such that, for each $y \in X_L \cup X_R$, we have that $Y_L \cup Y_R = \{z \in X_L \cup X_R \mid z <_x y\}$. (The advantage of Ehrlich's approach is that a surreal is identified with a particular pair of sets of surreals, whereas, following Conway's own approach, a surreal would be an equivalence class of pairs of sets of surreals, as was noted above.) With respect to $\langle No, <_S \rangle$, Ehrlich proves, in effect, that (1) root $\langle\emptyset,\emptyset\rangle$ is a number, that (2) for any number x in the tree, both children of x are numbers, and, finally, that (3) any infinite path P within the tree is associated with a number whose left-(right-)set is the sumset of all the left-(right-)sets of the numbers along P (Theorem 1.1). Each member of No then comes to be associated with a unique binary sequence ("genetic code") in the obvious way (*cf.* [3]).

Regarding the placement, within this anthology, of the two articles on surreal numbers, Conway's article describes the real number system as a proper part of the system of surreal numbers. To this extent, his theory of the continuum is an *alternative* to Cantor-Dedekind. On the other hand, the existence of infinite reals, infinitesimals, and their reciprocals indicates that Conway's theory of real numbers itself constitutes a *generalization* and *extension* of the theory of real numbers, which explains the placement of Ehrlich's own article under the eponymous heading.

Gordon Fisher's contribution is largely given over to an account of Giuseppe Veronese's construction of a nonarchimedean linear continuum as carried out in the latter's *Fondamenti di geometria* (1891). By all accounts, that work is difficult in the extreme. Contemporary estimations of its value varied widely, however, as Fisher recounts. Peano's review was scathing — perhaps due to Veronese's obvious interest in epistemological issues. (The present reviewer found this aspect of Veronese's thought, as described by Fisher, quite intriguing although nonetheless perplexing.) Fisher reports Hilbert's favorable reaction as well as Veronese's influence on algebraist Hans Hahn, who, in his 1907 article "Über die nichtarchimedischen Grössensysteme," set himself the task of giving an algebraic account of Veronese's "intuitive continuum." Fisher's article actually begins with a discussion of Hahn's reconstruction. Since readers will unquestionably find Hahn's treatment clearer than Veronese's own discussion, this seems a reasonable choice.

Hahn takes a *nonarchimedean system* $\mathcal{S}$ (of *quantities* or *magnitudes*) to be a linearly ordered group in which the archimedean axiom may fail to hold. (The archimedean axiom asserts that, for any two positive a and b with $a < b$, there exists natural number n with $na > b$.) Hahn was able to show that the quantities of such a system $\mathcal{S}$ may be partitioned into equivalence classes, within each of which the archimedean axiom does hold. Moreover, since the equivalence classes of $\mathcal{S}$ are themselves linearly ordered, it becomes possible to assign to $\mathcal{S}$ a certain *class type*, namely, the Cantorian order type of the set of its equivalence classes. It then turns out that an *archimedean system* is just a nonarchimedean system of class type 1. The complex numbers can be ordered in such a way as to yield a nonarchimedean system of class type 2, and the generalization to hypercomplex numbers having n basis elements yields a nonarchimedean system of class type n. Indeed, Hahn goes on to extend this process so as to obtain nonarchimedean systems of arbitrary class type.

In his review of Hilbert's *Foundations of Geometry*, Poincaré makes it clear that he regards Hilbert's nonarchimedean geometry — again, the axiom of Archimedes fails — as the most original part of that work. For this purpose Hilbert had introduced a system of nonarchimedean numbers consisting of all series of the form

$$a_m\,t^m \;+\; a_{m-1}\,t^{m-1} \;+\; a_{m-2}\,t^{m-2} \;+\; \ldots \;+\; a_0\,t^0$$

where m is any integer and where the coefficients $a_m, a_{m-1}, a_{m-2}, \ldots$ are real. The sign of such a nonarchimedean number is taken to be the sign of its leading coefficient. The arithmetic operations on these entities are defined in the usual way and, for two such nonarchimedean numbers r_1 and r_2, we have $r_1 < r_2$ provided $r_2 - r_1$ has positive sign. It follows that t itself is greater that any real x. Moreover, for any real x, there exist infinitely many nonarchimedeans that are at once less than x and greater than any real y with $y < x$.

A further generalization of the reals is needed for Hilbert's nonpascalian geometry wherein multiplication is not commutative. Take the nonpascalian (and nonarchimedean) numbers to be those of the form

$$T_m\,s^m \;+\; T_{m-1}\,s^{m-1} \;+\; T_{m-2}\,s^{m-2} \;+\; \ldots \;+\; T_0\,s^0$$

where m is any integer and where the coefficients $T_m, T_{m-1}, T_{m-2}, \ldots$ are nonarchimedeans. Arithmetic operations are the usual ones except that $s \cdot t = -t \cdot s$. The sign of such a nonpascalian number is that of T_m, and order is imposed in analogy with the nonarchimedean system. It turns out that nonarchimedean t exceeds any real x and that nonpascalian s exceeds any nonarchimedean.

Hourya Sinaceur's historical article aims to show how the real algebra of Artin and Schreier (1926) enabled mathematicians to achieve the long-standing goal of rendering the linear continuum numerical. Sinaceur places these authors and their work within the context of the Hilbert school, by which is meant Hilbert's emphasis upon the axiomatic method and finiteness theorems. A class of algebraic structures — the real closed fields — is defined, of which the reals turn out to be a model. Since Skolem-Löwenheim implies that the reals of such models are not determined even up to isomorphism, it is natural to ask in precisely what sense the reals have thereby been characterized. According to Sinaceur, the innovation of Artin and Schreier rests in the point of view whereby the reals are taken to have been successfully characterized just because the various versions of $\mathfrak{R}$ are elementarily equivalent, *i.e.*, they model the same first-order sentences. Although Sinaceur does not herself do so, it is interesting to contrast this approach with that of Zermelo during the same period. As a metaphysical realist, he views axiomatic set theory as describing a determinate, independently existing reality — a platonist realm of sets. So the question arises: How after all does set theory, given the model-theoretic conception, which Zermelo by this point in time has made very much his own, determine its subject matter as the *sets*? A certain common expectation with respect to the semantics of formal systems expresses itself as a demand for categoricity. The Skolem Paradox gives an unfavorable ruling in the case of the axioms of first-order ZF: they fail to determine their subject matter even up to isomorphism. Zermelo's idea in the late twenties is to move to a second-order formalism so as to obtain several results that together give something approximating categoricity (cf. [7] and [4]). That Zermelo could not have been satisfied with elementary equivalence would appear to stem from his foundationalism — in particular, from his foundationalist aspirations for axiomatic set theory. At the very least, the contrast between Artin and Schreier, on the one hand, and Zermelo, on the other, shows again the diversity of views held by those belonging to the Hilbert school.

Sinaceur sees Artin and Schreier as at least "approximating" Hilbert's finitist method to the extent that they provide a countable algebraic description of the continuum, *i.e.*, a countable set of first-order statements. But, again, this description does not characterize $\mathfrak{R}$ even up to isomorphism, so in what sense has the linear continuum been captured really?

Finally, a small point. Sinaceur asserts that, because two of Artin and Schreier's axioms are in fact first-order schemata, Hilbert's axiomatic method is only being approximated. It is true enough that Hilbert called for "finite" axiomatizations. However, it is also clear that the distinction between an axiom and an axiom schema was not always significant in this regard. Consider that Hilbert plainly regarded Zermelo's 1908 axiomatization as being quite on the mark, despite its incorporation of an axiom schema (*Aussonderung*). Viewed in this light, Sinaceur's qualification is probably unnecessary.

H. Jerome Keisler's article is a very interesting, introductory treatment of that extension of the continuum known as the *hyperreal line*. Like Conway's surreals, the hyperreals include both infinite and infinitesimal numbers. However, in the case of the hyperreal numbers, it is model-theoretic, rather than algebraic, ideas that drive the theory. Significantly, the hyperreal number system has the same first-order properties as the real number system. It is best known from the work of A. Robinson, who used it as the basis for his development of non-standard analysis.

An element of an ordered field is said to be infinitesimal if its absolute value is less that $1/n$ for every n. Infinitesimals are a feature of any ordered field having at least one positive infinite element — any nonarchimedean ordered field. Any element x of such a field possesses a *monad* of elements that differ from x by an infinitesimal amount. Similarly, for arbitrary element x, the *galaxy* of x consists of those elements differing from x by a finite amount. The monad of 0 is thus the infinitesimals, and its galaxy comprises the finite elements of the field.

Keisler motivates the introduction of hyperreals by citing a deficiency in Tarski's celebrated completeness result for the first-order theory of real closed ordered fields. Namely, that result applies only to first-order formulas built up from predicates $=$ and $\leq$, function symbols $+$ and $\cdot$, and constants 0 and 1. The goal is to extend this result to the full range of first-order properties of the reals. The hyperreals are constructed explicitly as an ultrapower of the real number system. The fundamental features of the hyperreal number system are expressed by (1) the Transfer Principle — the analog of Tarski's theorem — stating that the hyperreal number system satisfies the same first-order formulas as does the real number system and (2) the Saturation Principle stating that any countable decreasing chain of nonempty internal sets of hyperreals has nonempty intersection, where the internal sets are those whose first-order properties are those of corresponding sets of reals. This leads to an informative presentation of the notion *hyperfinite grid*. Relative to a given positive infinite H, this might take the form

$$\{-H, -H+1/H, -H+2/H, \ldots, -2/H, -1/H, 0, 1/H, 2/H, \ldots$$
$$\ldots, H-2/H, H-1/H, H\}$$

so that, by the Transfer Principle, every real number is infinitesimally close to some element of the grid.

At the end of his article, Keisler turns to foundational issues. Whereas in Zermelo set theory one can prove the existence of a unique real line, one can prove the existence of the hyperreal line in ZFC but not its uniqueness. This leads Keisler to consider alternative set theories that enable one to prove uniqueness. The first of these alternatives is based on the *superstructure approach* of Robinson and Zakon. In this theory, both the real line and the hyperreal line can be proved to be unique.

The same is true of Nelson's Internal Set Theory, which is the other set theory considered.

Keisler contrasts what he terms the platonistic approach with a pragmatic approach to theorizing about the geometric line. Following the former approach involves the search for some mathematical structure exhibiting properties attributed to the geometric line, which is assumed to exist and to be accessible to us. The pragmatic approach consists in proposing mathematical structures useful in explaining natural phenomena or obtaining mathematical results. Keisler suggests that both platonistic and pragmatist justifications for the study of the hyperreal number system are available. The article closes with a brief review of applications within economics and physics and of the fruitfulness of hyperfinite computations for probability theory and analysis.

Dieter Klaua's brief article recounts efforts — his own and that of others — to develop a generalization of the rational and real number systems based upon ordinals. An addition and a multiplication that are commutative and absorption-free are obtained, following Hessenberg, by adding and multiplying the Cantor normal forms of ordinals in a manner suited to polynomials. In that case, for α an arbitrary ordinal, ω_α is closed under $+$ and $\cdot$. $\mathcal{N}_\alpha = \omega_\alpha$ is defined as the set of *natural ordinal numbers of type* ω_α, ordered by membership and otherwise structured by the Hessenberg operations. The set $\mathcal{Z}_\alpha$ of *integer ordinal numbers of type* ω_α is obtained from $\mathcal{N}_\alpha \times \mathcal{N}_\alpha$ by taking equivalence classes under difference equality $=_{\mathrm{D}}$ and using the induced ordering and operations. Next, the set $\mathcal{Q}_\alpha$ of *rational ordinal numbers of type* ω_α is obtained from $\mathcal{Z}_\alpha \times (\mathcal{Z}_\alpha \setminus \{0\})$ by taking equivalence classes under quotient equality $=_{\mathrm{Q}}$. By isomorphic embedding, we have $\mathcal{N}_\alpha \subseteq \mathcal{Z}_\alpha \subseteq \mathcal{Q}_\alpha$, and $\aleph_\alpha$ is the cardinality of all three sets. Klaua stresses that the transfinite rational domains $\mathcal{Q}_\alpha$ differ in essential ways from their classical counterpart $\mathcal{Q}$: whereas not every ω-sequence of members of $\mathcal{Q}$ contains a convergent subsequence, Sikorski showed long ago that, in $\mathcal{Q}_\alpha$, every bounded ω_α-sequence does possess a convergent ω_α-sequence (generalized Bolzano-Weierstrass Theorem for $\mathcal{Q}_\alpha$).

A metric on $\mathcal{Q}_\alpha$ is available, and the collection of all open subsets of $\mathcal{Q}_\alpha$ constitutes a topological space (of type ω_α) in a recognizable sense. Transfinite versions, for $\mathcal{Q}_\alpha$, of well-known properties of classical real analysis are then seen to hold. For example, analogous to the Heine-Borel Theorem, one has the following: If $M \subseteq \mathcal{Q}_\alpha$ is a closed, bounded set and, for every $x \in M$, we have that $U(x, \varepsilon_x)$ is an open ball with center x, then there is a subset $X \subseteq M$ with $M \subseteq \bigcup_{x \in X} U(x, \varepsilon_x)$ and $card(X) < \aleph_\alpha$.

As for the real extension of $\mathcal{Q}_\alpha$, one defines $\mathcal{R}_\alpha$ as the set of all cuts (A, B) in $\mathcal{Q}_\alpha$ such that A has no maximum element. One obtains a total ordering of $\mathcal{R}_\alpha$ by setting $A < B$ with $A = (A_1, A_2)$ and $B = (B_1, B_2)$ provided that $A_1 \subset B_1$. The elements of $\mathcal{R}_\alpha$ are the real ordinal numbers of type ω_α in the most general sense, Klaua tells us. This is because isomorphic embedding gives $\mathcal{Q}_\alpha \subseteq \mathcal{R}_\alpha$ and yet the arithmetic properties of $\mathcal{R}_\alpha$ are not the ones we seek. For instance, for $\gamma \in \mathcal{R}_\alpha$, we can show only that $\gamma - \gamma \leq 0$ unless $\gamma \in \mathcal{Q}_\alpha$. This situation leads Klaua to define the set $\mathcal{R}^*_\alpha \subseteq \mathcal{R}_\alpha$ having algebraic and topological properties superior to that of $\mathcal{R}_\alpha$, *e.g.*, $\mathfrak{R}$ is isomorphically embedded in $\mathcal{R}^*_\alpha$.

In an appendix, Klaua provides axioms for ZFC that enable one to define the rank relation ("a is of lower rank than b") without presupposing ordinals.

With the exception of the Hobson, Poincaré, and Veronese pieces, none of the articles in this collection has appeared previously. (The article by Veronese appears here for the first time in English translation.) The articles by Bridges and Keisler could serve as introductions to intuitionistic analysis and the theory of hyperreal numbers, respectively. The same cannot be said of the two papers on the theory of surreal numbers included here, which clearly presuppose considerable familiarity with the surreal numbers. Ehrlich's anthology contains very few typographical errors. It includes a name index but lacks a subject index, which some readers might have found helpful.

Ehrlich has brought together some valuable work on issues of great interest to logicians and philosophers of mathematics. This work reveals once again the extent to which the various players, in presenting whatever theory of the continuum, have been motivated by basic philosophical attitudes regarding infinity and mathematical existence generally. This motivation is, of course, well known in the case of disagreements between platonists and constructivists. The value of Ehrlich's anthology lies in reminding us that other, less well-known, or even forgotten, theories of the continuum were likewise motivated by views that were equally philosophical. In particular, his anthology gives tangible form to one's sense that metaphysics, and, in particular, ontology, has made a difference for mathematical analysis. Turning matters around, these papers collectively embody the efforts of a group of mathematicians to give scientific, technical answers to what are, essentially, philosophical questions concerning the nature of the continuum[1].

[1]The reviewer wishes to thank Jane Stanton for editorial assistance.

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