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Review of
LEO CORRY, *MODERN ALGEBRA AND THE RISE OF MATHEMATICAL STRUCTURES*

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This book is a detailed historical account of the development, in one area of mathematics, of the idea of mathematical 'structure'. For Leo Corry, the key to understanding this development, and perhaps the reason it has until recently received too little attention from historians, is that it was characterized not so much by the usual expansion a new idea brings to the body of knowledge, as by a fundamental change in viewpoint in the practice of mathematical research as a whole. As Corry presents it, the subject has been waiting for the historian willing to treat it within this larger context, going beyond the usual recital of mathematical results in chronological sequence.

Since an analysis of the rise of structure even in algebra alone would be a formidable task, Corry restricts his study to the theory of ideals, as being both typical and of interest in its own right. This is the subject of the first two-thirds of the book, in which Corry describes how a nascent structural point of view gradually emerged in the research of Dedekind, Hilbert, Fraenkel, and others, until developing into a truly comprehensive, 'modern' approach to ideal theory with Emmy Noether and her associates in the 1920s. The publication in 1930 of Van der Waerden's widely influential textbook *Moderne Algebra*, inspired to a great extent by Noether's work, is, for Corry, the watershed event in the rise of mathematical structure, signaling not just a significant change in the content of algebra as compared to the focus of research in previous decades, but the birth of a new consensus as to what algebra as a discipline henceforth would be. This consensus would determine the proper objects of study, the legitimate open questions, and the appropriate methodologies for solving them. Such matters, having to do less with the mathematical content of algebra than with the discipline of algebra *qua* discipline, constitute what Corry throughout his book refers to as the 'image' of mathematical knowledge held by professional mathematicians at any given time. The term 'image of knowledge' is due to Yehuda Elkana ([7]), whose cultural approach to the philosophy of science has clearly influenced Corry's slant on the history of mathematics.

What is meant by 'structure'? In its everyday use among professional mathematicians, the term 'structure' is an informal one which for that very reason seems to resist precise definition. Consequently, when we try to explain to a non-mathematician what 'structure' means, we usually find ourselves falling back on giving a potentially wide range of examples centered mostly on mathematical practice: perhaps examples of how mathematicians investigate the properties of certain mathematical objects by studying mappings from those objects to other objects, or examples of the types of larger questions mathematicians typically address in investigating these properties. In the study of algebraic systems such as groups, rings, and fields, these questions often deal with how the subsystems of a given system are interrelated, or how systems can be constructed from simpler systems or characterized as subsystems of larger systems — in all cases ignoring, for the most part, the nature of the elements of the sets underlying the systems involved. In any event, it is more than likely that our informal attempt to define structure will merge into an attempt to describe a *structural* approach, emphasizing mathematical *practice* rather than content. A philosopher of mathematics with a Kuhnian bias could therefore argue that the informal notion of structure is so very closely tied to mathematical practice that no attempt to characterize the notion with any semblance of completeness can succeed without considering how that practice has evolved over the last 150 years. This in fact is Corry's main contention. He wishes to take "this specific, historically conditioned image of mathematical knowledge," which it is the purpose of his book to describe, "as implicitly defining the idea of mathematical structure" (p. 8). The book is therefore an attempt not merely to show how the concept of structure evolved, but to demonstrate that the concept itself is not strictly mathematical at all in the narrow sense, deriving its true meaning only as an 'image' of mathematical knowledge inseparable from the context of its historical evolution.

Thus, over several chapters of the book, Corry analyzes in detail the decisions algebraists made over the years, from the late nineteenth century until the 1930s, about which questions were worth investigating and which methods were appropriate for attacking them, looking everywhere for evidence of a real change in point of view from the 'classical' to the 'modern'. For this he relies upon public statements algebraists sometimes made about their discipline as well as upon the content of their mathematical work. Corry does not go so far as to call the rise of structure a scientific 'revolution' in Thomas Kuhn's sense of the word, but while Corry prefers Elkana's terminology to Kuhn's, it would not be too inaccurate to describe the theme of this book as the claim that the rise of structure can best be understood as a paradigm shift. Corry makes only one brief mention of Kuhn in a footnote; he has written elsewhere ([5]) on the connection between Kuhn's ideas and Elkana's.

Since the importance of the formal axiomatization of theories to the development of the idea of structure has often been remarked upon, it might be of interest to readers of this journal to see how Corry treats David Hilbert's particular contribution to this development, the most significant aspect of which was undoubtedly the impetus he gave to the axiomatization of theories. A persistent theme running throughout the book is that a structural point of view does not automatically grow out of application of the axiomatic method. Hilbert, to begin with, did not see axiomatization as a way of defining new mathematical entities or generalizing already existing traditional objects of research. Corry shows how Hilbert's 1905 axiomatization of vector addition, for example, was unquestionably rooted in traditional intuitions about vectors in ordinary three-dimensional space; in his work along these lines, "no connection is made between [his axioms] and the axioms for abstract groups, which were by then already well-established," nor with "the field properties of the real numbers, not to speak of the idea of an abstract field" (p. 167). The importance of axiomatics for Hilbert lay solely in legitimizing already existing mathematical concepts by establishing mathematical truth through logical consistency. "It was not part of Hilbert's axiomatic conception — at least in its early stages — to encourage a new conception of algebra, in which concepts defined by abstract systems of postulates would assume the central rôle, and to which the systems of real and complex numbers would be conceptually subordinate" (p. 168).

Hilbert, from the beginning of his career, also pioneered in finding connections between different areas of mathematics, drawing upon ideas from separate and not obviously related disciplines to prove results such as his generalized finite basis theorems. These connections sometimes involved what we would now call structural properties. But here again, an indirect use of structural parallels did not amount to a new vision of the aims of mathematical research. In his research on invariant theory and algebraic number theory he still followed the traditional approach which characterized the work of Dedekind and Weber. Fields and groups, for example, were merely tools for solving problems in other areas, and were not of much interest in their own right; Hilbert's 'image of algebra' did not see them as instances of a more general unifying idea, that is, as abstract sets on which operations were defined according to stated axioms. In fact, fields and groups were for Hilbert two completely different kinds of objects, the one being essentially numerical while the other was non-numerical in nature, and it would have been foreign to his point of view to have considered them as part of a single hierarchy of abstract algebraic structures. Thus, in Corry's view, Hilbert's work, important as it was in the evolution of the structural approach to algebra, was not yet truly 'modern', that is, not yet structural in the way that, for example, Emmy Noether's was.

As far as the history of logic is concerned in this part of the book, aside from the discussion of Hilbert's contribution to the axiomatic approach there is a short but interesting section devoted to postulational analysis in the US at around the turn of the century, and a few remarks on how Steinitz and Noether dealt with the use of the axiom of choice in their proofs. On both of these topics, Corry does not add much to already existing accounts, but these matters are somewhat peripheral to his concerns anyway. (On postulational analysis, see [10]. For Steinitz's attitude towards the axiom of choice, see [9]).

The remaining third of the book is devoted to an assessment of three early attempts at defining the notion of structure formally within mathematics: Oystein Ore's research program of the late 1930s and early 1940s, Bourbaki's theory of 'structures' as found in the first volume of his *Eléments*, and category theory as initially formulated by Samuel Eilenberg and Saunders Mac Lane.

Of these three, Oystein Ore's lattice-theoretic approach to this particular problem has received the least attention from historians, although there are, of course, accounts of his contribution to the development and application of lattice theory in general. (See, for example, [8].) Ore's pioneering work on lattice theory in the 1930s seems actually to have been motivated to a great extent by a concern for foundational issues, specifically the problem of characterizing formally the notion of structure. (Ore's term for lattice was in fact 'structure'.) The program he and his followers pursued for almost a decade starting in 1935 led to the discovery of many important connections between algebraic domains and their lattices of subdomains, particularly in the area of chain conditions and direct product decompositions. Interest in Ore's approach began to fade when it became increasingly evident that the lattice concept was too narrow to serve the broad generalizing rôle Ore originally had in mind for it.

Ore's goal had been to show that all the important features of an algebraic domain could be studied through its lattice of subdomains. Clearly the success of such a program depended in part on what one considered to be the important features. For Ore, decomposition properties "represented the main (and probably the only) target of algebraic research" (p. 278), so that, according to Corry, Ore could in good conscience ignore results outside of this area which appear to us now to have been clear threats to his program. Prominent among these were results in group theory showing that a lattice-theoretic generalization such as Ore's could not treat the commutative and non-commutative cases simultaneously; as early as 1928, it was known that the existence of an inclusion- and conjugation-preserving isomorphism between the lattices of subgroups of two groups only insures that the groups themselves are isomorphic if at least one of the groups is Abelian.

Corry begins his treatment of Ore by reviewing Ore's classically oriented work on polynomial theory in the early 1930s, which he carried out in spite of his awareness that Noether's more modern approach was already by then generally considered to be the approved methodology. Corry goes on to describe the inexplicably sudden change in orientation towards the structural viewpoint Ore underwent in 1935, and then summarizes the contents of several of Ore's subsequent papers as these bear specifically on Ore's project to find an abstract foundation for all of algebra. Corry's account necessarily gives a condensed view of only one aspect of Ore's work, but it has the merit of bringing back into the light an interesting episode in the history of metamathematics which seemed to have become all but forgotten. There is no question in Corry's mind that Ore's research program "was a significant stage in the rise and development of the structural approach to algebra" (p. 292), if only because of the influence it had on the thinking of mathematicians such as Saunders Mac Lane and on the initial growth of category theory.

The evolution of lattice theory in the 1930s and 1940s was closely connected with the development of universal algebra and with the early stages of model theory. Corry has little to say about either subject in relation to structure, mainly because he feels they are best treated in a history of logic, which is not what he is writing. He therefore refers the interested reader to works such as [11], [3], and [8].

The second approach to defining structure which Corry considers is Bourbaki's, an approach that has sometimes been characterized as model-theoretic (for example, in [2] and [1]) because of its emphasis on axiomatization applied to particular kinds of sets. Bourbaki's formal axiomatic theory of *structures* is found in *Theory of Sets*, the first volume of his encyclopedic magnum opus *Eléments*. In his book Corry uses italics to make a distinction between the structures Bourbaki defines formally and the usual informal notion of structure. This distinction turns out to be an important one, to say the least.

Corry's treatment of Bourbaki is based on a previous paper ([4]) and is highly critical. A major theme is that Bourbaki's attempt to formalize the meaning of structure, motivated by his desire to establish rigorously what mathematics in essence should be for the working mathematician, was undermined from the start by the popular but inconsistent philosophical position, perhaps shared by most working mathematicians, commonly described as "Platonism on weekdays and formalism on Sundays." For it seems to be the case that the results presented formally in *Theory of Sets* not only are rarely used in the subsequent volumes of the *Eléments*, but in effect are abandoned explicitly in the *Fascicule de résultats* (summary of results) concluding the first volume, where the reader is actually encouraged to rely on *ad hoc*, informal definitions of basic concepts such as 'set' and 'structure' which are quite independent of the formal definitions given earlier. More refined notions such as 'poor' and 'rich' *structures*, 'fine' and 'coarse' *structures*, and 'deduction' of *structures*, carefully worked out in the main body of *Theory of Sets*, are never applied at all in the other volumes. Corry claims that in the *Eléments*, for the most part, "no new theorem is obtained through the *structural* approach and standard theorems are treated in the standard way" (p. 327). He finally concludes that Bourbaki's treatment of *structure* is "forced and unnatural" and can "safely be skipped by any potential reader" of the remainder of Bourbaki's work (p. 334). In his mind it is therefore not surprising that *Theory of Sets* has not had nearly as great an influence on mathematics as the other volumes in the series.

If Bourbaki's theory of *structures* has proved to be of so little relevance to mathematics as a whole, and even to Bourbaki's own enterprise, why is Bourbaki so often associated uniquely, in the minds of mathematicians and non-mathematicians alike, with the idea of structure? Corry suggests that one reason for this might lie in the influence of Bourbaki's historical writings, especially those collected in the *Eléments d'histoire des mathématiques* (1964), in which the rise of structure often plays a central rôle. It is ironic, however, that Bourbaki's historiography is apparently vulnerable to criticism especially where it makes assertions about the rise of structure. Corry cites Bourbakian statements which are innocent enough, perhaps even trivial, when interpreted as statements about the informal notion of structure, but are patently false if they are meant to refer, as Bourbaki evidently intended (judging from the evidence), to Bourbaki's own *structures*.

Perhaps the identity of Bourbaki with structure is due to little more than his vigorous campaign to present his work as what mathematics really is and should be, which has elevated its undeniably structural character (in the informal sense) to something approaching a paradigm for mathematical research. In any case, Bourbaki's use of the term 'structure' is by no means accidental, for it reflects what was already, before he began writing, a widespread informal usage. His formal notion of *structure*, on the other hand, seems to be an accidental feature of his work as a whole, judging from Corry's fairly detailed account. (It should be pointed out, however, as Corry does in a footnote, that "a renewed interest in Bourbaki's concept of *structure* has arisen lately in the framework of current research in model theory" (p. 334), where some of its limitations are being examined with a view to making the concept a little more serviceable. See, for example, [6].)

Corry's final section on category theory begins with a short review of its basic concepts and early history. Category theory developed partly out of an idea that was central to Ore's program, namely that the structural nature of algebra can best be understood by entirely ignoring the existence of the elements of a given algebraic domain and concentrating on the relationships between the domain and its subdomains. While it is not merely an extension of Ore's approach, category theory expands on it by including within its scope relationships between all possible domains of a given kind, and even domains of different kinds. Eilenberg and Mac Lane were not, however, initially concerned with the general problem of defining the meaning of 'structure'. Instead, as is well known, they developed category theory in order to answer certain questions they were investigating in homology theory, and it was slightly later that they, and other mathematicians, saw its potential for much greater generalization. In this respect, the genesis of category theory was quite different from that of Ore's program.

This section of the book also does not fail to note some important historical connections between category theory and Bourbaki's *structures*. Corry describes in some detail the debate within the Bourbaki group over whether categories should play an explicit rôle in the *Eléments*, drawing upon documents circulated within the group, to the extent these are available, as well as public statements made by individual members. One reason for the long gap between the publication of the main part of *Theory of Sets* in the 1950s and the much earlier appearance in 1939 of the concluding *Fascicule des résultats*, seems to have been technical complications introduced by results from work going on in category theory outside of the group during that interval of time, which had to be dealt with before the finished volume could be published. Perhaps it would not be too much of an exaggeration to say that Bourbaki's attempt to overcome these difficulties without making explicit use of category theory was to a certain degree a consequence of *hubris*, in that Bourbaki overreached himself in trying to include so much of mathematics within a single comprehensive theory of *structure*. While correctly recognizing that category theory had its limitations in some areas of mathematics, Bourbaki evidently failed to see the even greater limitations of his own theory of *structures* in almost all of the areas covered by the *Eléments*. Perhaps most serious of all, he never really acknowledged the greater power of category theory in formalizing one of the most important aspects of the structural approach, namely its exploitation of mappings between structures to reveal their properties.

In a short space, Corry has succeeded in at least sketching a more complete picture than we have had up to now of the historical connections between these three major attempts at defining structure formally within mathematics. In particular, category theory was a direct outgrowth of neither Ore's program nor Bourbaki's, each approach having its own unique origin and emphasis. Most mathematicians interested in these matters would probably agree that category theory has a decided advantage over the other two, but as noted earlier in this review, Corry does not seem to feel that *any* formal definition of structure could do justice to the use of the concept in actual mathematical practice, however rewarding such a formal exercise might prove to be for other reasons. Given the "reflexive" nature of mathematics — to use a favorite term of Corry's to denote the peculiar ease with which mathematics can be turned to the study of mathematics itself — attempts at formalization such as those Corry describes in the last third of his book were an inevitable outgrowth of mid-twentieth century mathematics. Nevertheless, Corry's view could be summarized as the belief that 'structure' refers essentially to a way of *doing* mathematics, and is therefore a concept probably just as far from being precisely definable as the cultural artifact of mathematics itself.

References

[1] Awodey, S., "Structure in mathematics and logic: a categorical perspective", *Philosophia Mathematica* **4** (1996), 209-237.

[2] Bell, J. L., "Category theory and the foundations of mathematics", *British Journal for the Philosophy of Science* **32** (1981), 349-358.

[3] Chang, C. C., "Model theory 1945-1971", in: L. Henkin et. al. (eds.), *Proceedings of the Tarski Symposium*, Providence: AMS, 1974, 173-186.

[4] Corry, L., "Nicolas Bourbaki and the concept of mathematical structure", *Synthese* **92** (1992), 315-348.

[5] ———, "Kuhnian issues, scientific revolutions and the history of mathematics", *Studies in History and Philosophy of Science* **24** (1993), 95-117.

[6] Da Costa, N. C. A. and R. Chuaqui, "On Suppes' set theoretical predicates", *Erkenntnis* **29** (1988), 95-112.

[7] Elkana, Y., "A programmatic attempt at an anthropology of knowledge", in : E. Mendelsohn and Y. Elkana (eds.), *Sciences and Cultures*, Sociology of the Sciences, Vol. 5, Dordrecht: Reidel, 1981, 1-70.

[8] Mehrtens, H., *Die Entstehung der Verbandstheorie*, Hildesheim: Gerstenberg, 1979.

[9] Moore, G. H., *Zermelo's Axiom of Choice: Its Origins, Development, and Influence*, New York: Springer, 1982.

[10] Scanlan, M., "Who were the American postulate theorists?", *Journal of Symbolic Logic* **50** (1991), 981-1002.

[11] Vaught, R. L., "Model theory before 1945", in: L. Henkin et. al. (eds.), *Proceedings of the Tarski Symposium*, Providence: AMS, 1974, 152-172.

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