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Review of
SAHARON SHELAH, *CARDINAL ARITHMETIC*

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ROBERT MIGNONE

Cardinal arithmetic, as Shelah observes, was the impetus for two major currents of modern set theory: generic extensions of the set theoretic universe, and inner model theory. Understanding the actual bounds or potential boundlessness of cardinal exponentiation — in particular for singular cardinals — and the underlying reasons for these phenomena are recurring themes of *Cardinal Arithmetic* by Saharon Shelah. However it is the main approach to cardinal arithmetic which is most notable in Shelah's book. *Cardinal Arithmetic* is a source book for Shelah's theory of possible cofinalities (pcf theory). It mines the mother lode of pcf theory down to what seems like the last mathematical speck, while pursuing applications which include spectacular results in cardinal arithmetic.

Pcf theory, which was first introduced by Shelah in a 1978 paper [5] to build Jónsson algebras on $\aleph_{\omega+1}$, is arguably a natural approach, or more accurately an effective alternative to cardinal exponentiation. Consider λ^κ, with $\kappa < \lambda$ and κ regular. Thinking of this cardinal exponentiation first as a set of functions from κ into λ leads to equating λ^κ with $|S_{\leq\kappa}(\lambda)|$, where $S_{\leq\kappa}(\lambda) = \{\, X \subset \lambda : |X| \leq \kappa \,\}$. The point here is that to understand the bounds or boundlessness of λ^κ, first effective ways of measuring the size of $S_{\leq\kappa}(\lambda)$ need to be studied. The study begins with the cofinalities of products $\prod \lambda_i$ modulo ideals J on κ, where $\lambda_i < \lambda$ are regular cardinals extending to λ. This results in a set of regular cardinals — the set of possible cofinalities of $\{\, \lambda_i : i < \kappa \,\}$ — which can be viewed as a set of candidates for the cofinality of $S_{\leq\kappa}(\lambda)$. Here lies the first step toward measuring the size of $S_{\leq\kappa}(\lambda)$.

Cardinal Arithmetic is, in part, a compilation of Shelah's work on pcf theory from work included in the 1978 paper [5], through Fall 1989 when he gave a series of lectures on pcf theory as part of the Logic Year at the Mathematical Sciences Research Institute in Berkeley, California. The book includes work offered in print for the first time, as well as previously published work.

The reader is given perspective on the history and motivation for the major themes of the book. A general introduction lays out Shelah's arguments for using pcf theory to study cardinal arithmetic, and other problems related to cardinal numbers. It also sets the book apart from a main thrust of contemporary set theory — amply contributed to by Shelah — which is concerned with the relative consistency of statements relating to properties of cardinal numbers. *Cardinal Arithmetic*, as Shelah emphasizes, is concerned with proving theorems true in ZFC. He playfully repeats the contemporary folklore that if Cantor were to rise from the grave he would not only fail to understand relative consistency proofs, he would fail to understand the statements of the theorems. Although Shelah disclaims this as criticism of relative consistency proofs, he readily professes that Cantor would not have this difficulty with *Cardinal Arithmetic*.

Shelah highlights the contrast between cardinal-arithmetic-like statements being provable in ZFC versus being altered via forcing — the technique used to build generic extensions of the set theoretic universe. Cardinal exponentiation, in particular for regular cardinals, can be made of varying "size" via forcing. However, the bounds which pcf theory gives for the size of sets of cardinals such as $S_{\leq\kappa}(\lambda)$ for specific λ and κ are provable in ZFC, or at the very least changeable via forcing only with strong consistency-strength assumptions.

The general introduction to the book is followed by a historical background of cardinal arithmetic. It traces central currents from the classical results of Cantor and König; through the advent of forcing; to results of Galvin, Hajnal, Jensen, Magidor, Shelah, Silver, Solovay, and others in the 1970's (see [3]).

In addition to the general introduction and historical background, each chapter begins with an introduction which gives history, motivation and layout to the material in that chapter. Even many of the sections within chapters begin in the spirit of background and strategy. This helps keep the reader somewhat triangulated in space, time and thought in this otherwise dense and disorienting material.

Chapter I introduces the basic definitions and properties of pcf theory. If $\mathbf{a}$ is a set of cardinals the smallest of which is greater than the cardinality of $\mathbf{a}$, and J is an ideal on $\mathbf{a}$, then $\text{tcf}\,(\prod \mathbf{a}/J)$ — the true cofinality of $\prod \mathbf{a}/J$ — is the minimal cardinality of a set of functions in $\prod \mathbf{a}$, unbounded and well ordered by $\leq_J$. In this case, define the set of possible cofinalities of $\mathbf{a}$, denoted $\mathrm{pcf}(\mathbf{a})$, by

$$\mathrm{pcf}(\mathbf{a}) = \{\, \mathrm{tcf}\left(\prod \mathbf{a}/J\right) : J \text{ is a maximal ideal on } \mathbf{a} \,\}.$$

Another basic definition is of $J_{<\lambda}[\mathbf{a}]$, defined by

$$J_{<\lambda}[\mathbf{a}] = \{\, \mathbf{b} \subseteq \mathbf{a} : \text{ if } I \text{ is a maximal ideal on } \mathbf{a} \text{ such that } \mathbf{b} \notin I, \text{ then } \mathrm{tcf}\left(\prod \mathbf{a}/I\right) < \lambda \,\}.$$

Some properties of $J_{<\lambda}[\mathbf{a}]$ proved here include: $J_{<\lambda}[\mathbf{a}]$ is an ideal on $\mathbf{a}$; if $B \subseteq \prod \mathbf{a}/J_{<\lambda}[\mathbf{a}]$ and $|B| < \lambda$, then B is bounded in $J_{<\lambda}[\mathbf{a}]$. Also, $\mu < \lambda$ implies $J_{<\mu}[\mathbf{a}] < J_{<\lambda}[\mathbf{a}]$, and $J_{<\lambda}[\mathbf{a}] = \bigcup_{\mu \leq \lambda} J_{<\mu}[\mathbf{a}]$.

The ideal $J_{<\lambda}[\mathbf{a}]$ is used to prove that $\mathrm{pcf}(\mathbf{a})$ includes a maximal cardinal, $\max(\mathrm{pcf}(\mathbf{a}))$, and if $\mathbf{b} \subseteq \mathrm{pcf}(\mathbf{a}), |\mathbf{b}| < \min \mathbf{b}$, then $\mathrm{pcf}(\mathbf{b}) \subseteq \mathrm{pcf}(\mathbf{a})$.

Next, a notion of normality for a cardinal in $\mathrm{pcf}(\mathbf{a})$ is defined. A cardinal $\lambda \in \mathrm{pcf}(\mathbf{a})$ is normal if there exists $\mathbf{b} \subseteq \mathbf{a}$ such that $J_{\leq\lambda}[\mathbf{a}] = J_{<\lambda}[\mathbf{a}] + \mathbf{b}$. If it happens that $\mathbf{a}$ is normal, that is, every λ in $\mathrm{pcf}(\mathbf{a})$ is normal, then $J_{<\lambda}[\mathbf{a}] = \{\, \mathbf{b}_\mu : \mu \in \lambda \cap \mathrm{pcf}(\mathbf{a}) \,\}$. A sequence of all such $\mathbf{b}_\mu$ gives what is defined as a *generating sequence* for $\mathbf{a}$. The last section of the chapter studies generating sequences and related ideas.

Chapter II begins with further advancements of pcf theory in preparation for several varied applications. These advancements answer basic questions about pcf theory and are building blocks within its structure. They also support Shelah's effort to promote pcf theory as an effective means for studying the cardinal arthmetic of singular cardinals.

A keystone of these building blocks says that if J is a maximal ideal on $\mathrm{pcf}(\mathbf{a})$, and the limit of $\mathbf{a}$ modulo J is λ (that is, for every $\alpha < \lambda$, $\{\, \beta \in \mathbf{a} : \beta < \alpha \,\} \in J$), then for every regular cardinal μ between λ and $\mathrm{tcf}\,(\prod \mathbf{a}/J)$, there is an $\mathbf{a}' \subseteq \mathbf{a}$ and a J' on $\mathbf{a}'$, such that the limit of $\mathbf{a}'$ modulo J' is λ, and $\mathrm{tcf}\,(\prod \mathbf{a}'/J')$ is μ. One nice consequence of this fact is that if $\mathbf{a}$ is an interval of regular cardinals with limit λ, then $\mathrm{pcf}(\mathbf{a})$ is an interval of regular cardinals immediately above λ.

Chapter II is also where Shelah introduces an instrument for measuring the size of $S_{\leq\kappa}(\lambda)$. The instrument is $\mathrm{pp}(\lambda)$, which represents the supremum of all the true cofinalities of all possible products of cardinals modulo all possible ideals over these sets of cardinals, whose limits modulo these ideals is λ. $\mathrm{pp}(\lambda)$ is offered as the proper substitute for cardinal exponentiation with respect to singular cardinals.

Another interesting fact of pcf theory proved in Chapter II is that $\mathrm{cf}(\prod \mathbf{a}, <) = \max(\mathrm{pcf}(\mathbf{a}))$, where $\min \mathbf{a} > |\mathbf{a}|^+$.

The applications begin with the use for which Shelah first introduced pcf theory in 1978, namely a Jónsson algebra is constructed on $\aleph_{\omega+1}$ (A Jónsson algebra is an algebra with countably many finitary operations, and no proper subalgebra of the same cardinality, see [3].). This result is generalized to λ^+ for many singular cardinals λ.

The applications in chapter II touch several topics: Boolean algebras; entangled linear orders; negative partition relations; covering numbers — which are closely related to $\mathrm{pp}(\lambda)$ and help measure $\mathrm{cf}(\,S_{\leq\kappa}(\lambda))$ under inclusion; λ-freeness — which touches on combinatorial set theory, group theory and topology; and model theory, in particular the existence of $L_{\infty,\lambda}$-equivalent, non-isomorphic models of singular cardinality λ.

Chapters III and IV depart from pcf theory and singular cardinals. Jónsson cardinals (cardinals which lack Jónsson algebras) and strong coloring theorems, primarily on inaccessible cardinals, are investigated. The technique introduced here is called "guessing clubs". One of the main results shows that if λ is an inaccessible which has a stationary subset that does not reflect in any inaccessible cardinals, then λ has a Jónsson algebra. Related to this is the result that the first regular cardinal λ, which does not have a Jónsson algebra, must be λ-Mahlo (that is, the set of all regular cardinals below λ must be stationary; see [3] for more on large cardinals).

Chapters V and VI return to cardinal exponentiation. Chapter V begins with some history and a lengthy discussion on the general problem to be addressed. Bounds are achieved for $\aleph_\delta^{\mathrm{cf}(\delta)}$, where $\mathrm{cf}(\delta) = \aleph_1$, $\aleph_\delta > 2^{\aleph_1}$, and $\aleph_\delta = \delta$; that is, at fixed points. Since pcf theory fails at fixed points, here Shelah develops and employs a notion of the rank of a function $f \in {}^{\aleph_1}Ord$, using normal filters on ω_1. This extends work of Silver, Galvin and Hajnal. The proofs employ some forcing and inner model theory, which the author promises can be cirmumvented.

Chapter VII is a rewriting of [6] Chapter VIII. The chapter begins with a discussion of covering lemmas in the spirit of Jensen's covering lemma, which shows that if $O^\#$ is not in the model V of ZFC, then every set of cardinals in V is a subset of (covered by) a similarly sized set of cardinals in the constructible universe L (see [3]). Several generalizations are made by replacing L with a transitive class W of V, where W is a model of ZFC and has the same ordinals as V. The generalizations lead to definitions of strong covering lemmas for pairs (W, V), which then are seen to exist for pairs (L, V), under the same condition which yields Jensen's covering lemma, namely "$O^\#$ is not in the model V".

Next, there is a review of results and questions related to what conditions must be present in order to preserve the continuum hypothesis when adding reals to a model of ZFC satisfying the generalized continuum hypothesis. The last section of chapter VII uses, among other things, the notion of strong covering lemmas to prove variants of questions concerning the preservation of the continuum hypothesis when reals are added. It also considers related consistency strength results. The chapter concludes by showing that if r is a real added to V, a model of ZFC satisfying the continuum hypothesis, such that $\aleph_1^V = \aleph_1^{V[r]}$, then a necessary consequence of the continuum hypothesis failing in V is that $\aleph_2^V$ is inaccessible in L.

Chapter VIII returns to pcf theory. The chapter begins by refining what was developed in chapter II. It shows that regular cardinals between $\sup \mathbf{a}$ and $\mathrm{pp}(\sup \mathbf{a})$ can be represented in a nice way if $\mathbf{a}$ is normal, and section 2 eliminates the assumption that $\mathbf{a}$ be normal, provided $|\mathbf{a}|^+ < \min \mathbf{a}$. In either case, if $\mathrm{cf}(\sup \mathbf{a})=\kappa$, then for every $\lambda \in \mathrm{pcf}(\mathbf{a})$, there is a closed unbounded subset $C \subset \kappa$ and a sequence of cardinals λ_i in $\mathrm{pcf}(\mathbf{a})$ below λ, such that $\lambda = \mathrm{tcf}\left(\prod_{i \in C} \lambda_i / J_C^{bd}\right)$, where $J_C^{bd} = \{X \subseteq C : |X| < \kappa\}$. This yields a sharper representation for $\mathrm{pp}(\lambda)$. Section 2 shows that when $|\mathbf{a}|^+ < \min \mathbf{a}$, then $J_{\leq\lambda}[\mathbf{a}] = J_{<\lambda}[\mathbf{a}] + \mathbf{b}$ for every $\lambda \in \mathrm{pcf}(\mathbf{a})$, hence $\mathbf{a}$ is normal. Section 3 begins with the unfulfilled wish to refute $|\mathrm{pcf}(\mathbf{a})| > |\mathbf{a}|$. In its absence is a notion of "localization", which can be used to show that if $\lambda \in \mathrm{pcf}(\mathrm{pcf}(\mathbf{a}))$, then for some $\mathbf{b} \subseteq \mathrm{pcf}(\mathbf{a})$, with $|\mathbf{b}| \leq |\mathbf{a}|$, $\lambda \in \mathrm{pcf}(\mathbf{b})$. The remaining sections advance many applications from chapter II including entangled linear orders and λ-freeness.

Chapter IX presents the evidence which helps to establish pcf theory as an effective tool for answering questions about the cardinal arithmetic of singular cardinals. The main result of section 1 shows if $\lambda^{\mathrm{cf}(\lambda)} > \lambda^+ + 2^{\mathrm{cf}(\lambda)}$ (along with restrictions when $\mathrm{cf}(\lambda)$ is $\aleph_0$ or $\aleph_1$), then $\mathrm{pp}(\lambda) > \lambda^+$. This result is then used to help determine the consistency strength of "$\exists\lambda(\lambda^{\mathrm{cf}(\lambda)} > \lambda^+ + 2^{\mathrm{cf}(\lambda)})$", which ultimately turns out to be the same as the consistency strength of "$\exists\kappa\ (o(\kappa) = \kappa^{++})$", where $o(\kappa)$ is the length of a sequence of measures on κ, created through the existence of an iterative process of elementary embeddings of length $o(\kappa)$ (see [2]).

Section 2 presents what most would agree to be the main result of the book: $2^{\aleph_0} < \aleph_\omega$ implies $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$. Sections 3 and 5 advance Shelah's arguments for substituting cardinal exponentiation, $\lambda^{\mathrm{cf}(\lambda)}$, with $\mathrm{pp}(\lambda)$, by showing that they are essentially the same; and are in fact equal for most cardinals. Section 4 improves the main result (of section 2) by proving: $\delta < \aleph_4$ and $\mathrm{cf}(\delta) = \aleph_0$ implies $\mathrm{pp}(\aleph_\delta) < \aleph_{\omega_4}$.

The proofs are unfiltered Shelah, and not for the faint-of-heart. The chapters are not rigidly ordered. This invites the reader to skip around once the basics of pcf theory are mastered in the beginning chapters — allowing for some backtracking to fill in necessary details. However, before *Cardinal Arithmetic* is tackled head-on, it might be a good idea to ease into pcf theory by first reading "Shelah's pcf theory and its applications," by M. Burke and M. Magidor [1]. It is a very readable introduction to pcf theory with a selection of some of its most interesting applications.

Cardinal Arithmetic is not a textbook. It is, as mentioned above, a compilation of Shelah's published articles and unpublished manuscripts on cardinal arithmetic and pcf theory. The book lacks the objective filtering and coherent presentation that is expected of a textbook. It is unlikely that *Cardinal Arithmetic* will become the *Joy of Cooking* [4] for the working set-theorist, as *Set Theory* by Thomas Jech [3] has become. However, despite the rawness of the presentation of its theorems and proofs, it is easy to imagine that mathematicians working in cardinal arithmetic and combinatorial set-theory will need ready access to *Cardinal Arithmetic* by Saharon Shelah.

References

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Department of Mathematics, The College of Charleston, Charleston, South Carolina 29424, USA