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Review of
G. SAMBIN AND J. M. SMITH, EDS., *TWENTY-FIVE YEARS OF CONSTRUCTIVE TYPE THEORY*

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This is a fine collection of essays about and around Per Martin-Löf's intuitionistic type theory. The theory was introduced in the 1970s, and Martin-Löf's seminal paper of 1972 is published here as chapter 8 (pp. 127-172) — one should consult also [4]. We begin our discussion with that central article.

Martin-Löf's motivation is constructive mathematics, and the type theory he discusses is distinct from, and at the same time more general than, the classical Russellian original brand. While Russell introduced types to build a hierarchy for arguments of propositional functions over the non-negative integers (simple theory of types) and supplemented the type hierarchy with orders for bound variables (the ramified theory of types), modern type theory is designed as a categorization or cataloguing of mathematical objects of all sorts, formulas and proofs: it is built essentially upon the Curry-Howard isomorphism between formulas and types (see [3]). The formulas-as-types motto allows for a combinatorial type theory in an intuitionistic logical framework. This is the intent of Martin-Löf's work. The author admits that his notion of type is quite similar to Bishop's notion of set: "The totality of all mathematical objects constructed in accord with certain requirements is called a set" ([1, p. 13]). For Brouwer, a set was a law, and a closer reading of Brouwer's reconstruction of set theory would have avoided the inconsistency of a first formulation of type theory with the type of all types. Martin-Löf credits Girard with the construction of the paradox, but although he mentions Brouwer's theory of ordinals of the second number class, the author fails to notice that Cantor's second number class does not exist for Brouwer. Brouwer calls it rather an unfinished domain (*Bereich*), since it has room for what Brouwer calls indeterminate ordinals, an idiom inspired by Kronecker's theory of forms (homogeneous polynomials) with indeterminates (*Unbestimmte*). It is in such a domain that one can locate the ordering without infinite descending chains that Girard evokes to construct his paradox. Martin-Löf opts finally for an open universe of types or a multiplicity of universes which Grothendieck resorted to in his foundation of algebraic geometry — where the totality of U-topoi constitutes a U-topia, as one could say. Category theory has also followed suit with the notion of category of small sets, and Martin-Löf adopts a universe with the type of small types together with a reflection principle borrowed from the cumulative rank structure of Zermelo-Fraenkel set theory. There the reflection principle is equivalent to the replacement axiom + infinity, here it looks like a *detypification* of the universe of types. The elaboration of intuitionistic type theory proceeds along proof-theoretical lines in a Gentzenian style. Is type theory, like set theory and category theory, a candidate for *the* foundational system of mathematics? Martin-Löf and others seem to think so.

In their contribution on "The Hahn-Banach theorem in type theory" (pp. 57-72), J. Cederquist, T. Coquand and S. Negri offer a type-theoretical treatment of "formal" topology similar to the category-theoretical analysis of a point-free functional analysis which is distinct from Bishop's constructive reformulation of the Hahn-Banach theorem (see [1], pp. 262-263) based on standard point-set topology.

G. Sambin and S. Valentini propose a type-theoretical account of subsets in their "Building up a toolbox for Martin-Löf's type theory: subset theory"(pp. 221-244). Here one finds sound philosophical reasons for a simple constructivist notion of subset. A. Setzer offers a not so simple characterization of well-ordering proofs in Martin-Löf's type theory (pp. 245-263), although the proof-theoretical resources of his paper are well known, from Cantor's normal form theorem for the ordinals of the second number class — which does not exist for Brouwer! — to the Veblen hierarchy for ordinal notations. According to the author, no system of such ordinal notations *from below*, that is by infinite induction, is sufficient to denote the proof-theoretic ordinal of a theory. As we know since Ackermann and Gentzen, one needs some form of transfinite induction beyond ε_0. This is set theory in proof-theoretical clothing with big ordinals and a large type W.

In "On universes in type theory" (pp. 191-204), E. Palmgren studies the universes of type theory leading to the construction of superuniverses and higher-order universe operators in the freer spirit of model theory where the constructive content slips away in the impredicative realm.

M. Hofmann and T. Streicher come back to a more down-to-earth interpretation of type theory in their paper "The groupoid interpretation of type theory" (pp. 83-111). Their syntactic approach to the model construction of groupoids (categories with invertible arrows) enables them to define extensions of Martin-Löf's type theory, reformulate category theory and show that uniqueness of identity is not derivable in pure type theory.

On the more logical side, C. Coquand contributes a paper on "A realizability interpretation of Martin-Löf's type theory" (pp. 73-82), building on Kleene's realizability interpretation of intuitionistic number theory extended by Tait to Brouwer's theory of species. A simple normalization theorem is presented. W. Tait's own contribution "Variable-free formalization of the Curry-Howard theory" (pp. 265-274) deals with the combinatorial-logical foundations of type theory and some classical properties, e.g. the Church-Rosser property for reduction sequences in the lambda calculus. From a historical point of view, as mentioned above, the combinatory logic of Shönfinkel and Curry was the building-block of modern type theory.

In the essay opening the volume, "Yet another constructivization of classical logic" (pp. 1-20), S. Baratella and S. Berardi develop a constructive interpretation of classical logic in terms of continuous computations close to Kreisel's no counterexample interpretation for continuous functionals which generalizes Herbrand's theorem in classical predicate calculus. The authors state their result as a theorem on the intuitionistic completeness of classical logic; their interpretation rests on the notion of "simulation of a formula", a kind of restricted or local instantiation of a formula (or subformulas of a formula). The finiteness requirement for simulation maps and connectivity of finite paths in well-founded trees provide an easy access to the result. In an appendix, the authors show that Gödel's *Dialectica* interpretation in terms of functionals over all finite types is not intuitionistically complete, since Gödel's functionals are not continuous. Nor is Martin- Löf's type theory intuitionistically complete in the author's interpretation.

The next papers belong to theoretical computer science. In "Extension of Martin-Löf's type theory with record types and subtyping" (pp. 21-40), G. Betarte and A. Tasistro extend Martin-Löf's type theory to accomodate abstract data types with the help of what they call dependent record types. The idea is to be able to form types of tuples and by subtyping to generate substructures of algebraic systems in an extended calculus of substitutions for type theory. In his paper "Analytic program derivation in type theory" (pp. 113-126), P.

Mäenpää uses a method of analysis-synthesis for programming problems and gives an interesting example from Descartes' algebra. In the author's view, Descartes' algebraic analysis can almost be seen as a "Discourse on the Method of Programming". With "On storage operators" (pp. 173-190), K. Nour takes advantage of the notion of "storage operators" introduced by J. L. Krivine to present results for memory storage in various systems of the lambda calculus, among others. A similar question is taken up in S. Valentini's paper "The forget-restore principle: a paradigmatic example" (pp. 275-283), here in a multi-level typed lambda calculus. The intent in this case is less technical or more philosophical, since it addresses the problem of forgotten and restored information in a general setting.

I shall close this review by the paper with the most philosophical overtones: "Type- theoretical checking and philosophy of mathematics" (pp. 41-56) by N. G. de Bruijn. It is not directly related to Martin-Löf's type theory for it covers a wide range of topics in the philosophy and history of mathematics. A more specific concern has to do with verification systems, in particular the Automath system which the author had designed in the late 1960s: it was indeed the first system designed for machine or computer-verified proofs, and it is of historical significance.

In all, this volume is a good representative of the North-South axis in constructive type theory. In which sense precisely is this type theory constructive? Unfortunately, there is no precise sense of the term "constructive" in the wide variety of constructivisms. In the case under study, constructive type theory, constructive means essentially intuitionistic logic due to the fact that the Curry-Howard isomorphism exhibits an internal relationship between the logic of operations (or combinatorial operators) and the logic of constructions (or intuitionistic connectives). The analogy found by Curry has become an isomorphism in the hands of Howard and other logicians of intuitionistic inclination, including Martin-Löf. It is not a radical or strict constructivism in the tradition of Kronecker or even Brouwer. It is, assuredly, a theory of (mental) constructions in the sense of Brouwer, Kolmogorov (the operational interpretation of intuitionistic connectives), Kreisel, Goodman, Friedman (intuitionistic set theory) and others, and it is intuitionistic in spirit, for it tends to restrict logical means to intuitionistically admissible principles of proof, but it is most liberal in its toleration of classical extensions and enrichments. *Twenty-Five Years of Constructive Type Theory* is certainly a testimony to the vitality and fruitfulness of Martin-Löf's ideas in the foundations of mathematics and theoretical computer science.

References

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