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Review of JON BARWISE AND LAWRENCE MOSS, *VICIOUS CIRCLES. ON THE MATHEMATICS OF NON-WELLFOUNDED PHENOMENA.*

Stanford: CSLI Publications, 1996
Lecture Notes Number 60
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In 1988, Peter Aczel published his now famous *Non-Well-Founded Sets*, where he introduced his Anti-Foundation Axiom AFA. Twelve years later, Jon Barwise and Lawrence Moss publish, in the same series, their *Vicious Circles*, which is also devoted to anti-foundation and non-well-founded sets.

This very pleasant book has been written for a wider readership than *Non-Well-Founded Sets*: students or researchers in logic, computer science and philosophy. Besides the technicalities of anti-foundation, it describes a wide range of applications of non-well-founded sets. Its style is clear and lively; mathematics are introduced and motivated by means of well chosen examples. The book can be used as a classroom textbook or for individual study; it contains many (solved) exercises. It is of a particular interest for researchers who want to start working on anti-foundation: it presents recent results, sometimes not yet published, and introduces some research projects.

Vicious Circles is divided into six parts. Parts I and II introduce the background set theory and circularity in various domains. Then parts III and IV present the basic theory and elementary applications. Finally, advanced theory and applications are described in parts V and VI.

Let us review each part in turn. Part I is a very short introduction to set theory (the reader is expected to have read an introduction to ZFC elsewhere). It gives the axioms of the underlying set theory: ZFC^-, that is ZFC with atoms (*Urelemente*), but without foundation, of course. This explicit use of atoms is much clearer than in Aczel's book.

Part II gives a naive introduction to circularity in computer science (streams, transition systems, etc.) and in philosophy of language (common knowledge). It also introduces vicious circles underlying some famous paradoxes (the liar, Russell's paradox and the hypergame paradox of Zwicker).

The technicalities begin in Part III. There, Barwise and Moss introduce systems of equations in set theory as for example

$$\begin{aligned} x &= \{a, y\} \\ y &= \{x\}. \end{aligned}$$

The Axiom of Anti-Foundation (AFA) states that such a system has a unique solution (here, a set x such that $x = \{a, \{x\}\}$ and a set y such that $y = \{\{a, y\}\}$); this corresponds to Aczel's Solution Lemma. Part III also includes a chapter on bisimulations and strong extensionality and a chapter describing a model of ZFC^- + AFA.

Part IV is devoted to elementary applications of AFA. The first chapter discusses the original formulation of AFA by Aczel: each graph has a unique decoration (with atoms, this is no more precisely equivalent to AFA). The second chapter presents an alternative semantics for modal logic, where Kripke structures are replaced with labeled graphs. It is shown how to characterize some sets by the formulæ they satisfy when these sets are considered as graphs and thus as modal structures. In the next chapter, the notion of bisimulation is reformulated in terms of games, and the hypergame paradox is given a more satisfying solution than simply stating that the hypergame does not exist. Then Barwise and Moss show how to use AFA to solve the liar paradox. To that end, they introduce a notion of structure whose domain is a set of structures, possibly including the structure itself, for a language with a truth predicate. Part IV ends with a chapter devoted to streams, a classical application of AFA.

Parts V and VI present further theory and applications. Mainly they prove the existence of least and greatest fixed points of suitable set theoretic operators. The coalgebras are introduced to generalize the systems of equations of Part III. The presentation is more mature than in Aczel's book. Aczel's artificial use of categories has now disappeared. The definition of uniform operator becomes more complicated (it should be worked on again), but it is more interesting. Part V ends with the Corecursion Theorem, corresponding to Aczel's Special Final Coalgebra Theorem. Part VI begins with some classical constructions of greatest fixed points, mainly related to semantics of transition systems. Anyway, the reviewer regrets that Part VI does not contain Aczel's presentation of Milner's SCCS; it would have been nice to correct Aczel's mistakes about this and to complete his presentation. Besides this, another chapter of Part VI generalizes methods of Part IV concerning modal structures, showing how each element of the greatest fixed points of some operators can be characterized by a set of formulæ of some modal logic. This is still a research subject; at first sight, it seems interesting. But Barwise and Moss do not present any application of such characterizations (it is probably too early to do it). The books ends with two chapters listing open problems related to non-well-founded sets, including the project of constructing a strongly extensional theory of classes.

Vicious Circles is a valuable book. As we have seen, anti-foundation and non-well-founded sets are covered much more widely than in Aczel's book. Nevertheless, the whole book of Aczel is not covered: Aczel's discussion of several axioms of anti-foundation is missing in *Vicious Circles*.

Barwise and Moss state the underlying axiomatic set theory with more care than Aczel, in particular with respect to atoms. Anyway, in both books, formal difficulties concerning classes are sometimes hidden: some classes are defined informally and it is not always easy (especially for a beginner) to see that these classes are indeed formally definable collections of sets. Also some mistakes remain: the most important mistake found by the reviewer is a flaw in the proof of the important Theorem 8.1. This mistake has been indicated to the authors. It will be probably listed some day with the other known mistakes (and recent developments) on the World Wide Web at the URL

`http://www.phil.indiana.edu/~barwise/kjbbooks.html`

To summarize, the reviewer highly recommends *Vicious Circles* as a lively, valuable reference book about anti-foundation and non-well-founded sets. He is sure that the publisher will soon ask the authors to write a new edition of this book and ... that this new edition will be still more mature than the first one.

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