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Review of

SOLOMON FEFERMAN, *IN THE LIGHT OF LOGIC*

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Published in commemoration of its author's seventieth birthday, this anthology comprises survey and expository essays dealing with historical, logical and philosophical issues in the foundations of mathematics. With the exception of the first chapter, the text of a previously unpublished lecture to a general audience at Stanford, all the items have appeared in print before. They are drawn from a wide variety of sources, including some that are less familiar and accessible. Their collection into one volume is thus both a useful and welcome undertaking, especially since the texts have been annotated to take account of subsequent publications and developments.

The volume as a whole provides an overview of the principal areas of research to which Professor Feferman has contributed since the late 1970s. The individual chapters are grouped thematically into five sections, entitled "Foundational Problems", "Foundational Ways", "Gödel", "Proof Theory", and "Countably Reducible Mathematics". There is considerable overlap among their contents, in part because the essays were composed at different times during the course of active foundational investigations, and in part because they are addressed to readers of differing degrees of technical sophistication. That is no defect, however: It serves rather to unify the volume and make it accessible to a wider audience.

In general, the essays within each section are arranged in order of increasing technical prerequisites, though none are addressed to specialists. The emphasis throughout is on the philosophical significance of metamathematical results, rather than their proofs. Where proofs are discussed they are presented in outline rather than detail, with appropriate citations to the research literature.

Several themes recur throughout the volume: the reinterpetation of Hilbert's program in the light of Gödel's incompleteness theorems; the extent to which mathematical theories can be predicatively justified;

the rôle of transfinite concepts and methods in applicable mathematics; and the cross-relevance between logical investigations and mathematical practice. In the preface, the author describes himself as "a convinced anti-platonist". Accordingly, a principal thrust of these essays is to advocate a predicative foundation for mathematics as an alternative to mathematical platonism.

Chapter 1, "Deciding the undecidable", recalls the problems posed by Hilbert in his turn-of-the-century address to the International Congress of Mathematicians — particularly the first (to determine the cardinal number of the continuum), the second (to prove the consistency of arithmetic), and the tenth (to devise an algorithm for determining the solvability of Diophantine equations) — and Hilbert's proposal for justifying mathematics through finitistic analyses of formal proofs (proof theory). Gödel's incompleteness theorems and consistency results in set theory, as well as the classical undecidability theorems and more recent results in computational complexity theory, are surveyed as background to the question to what extent Hilbert's program remains viable.

Chapters 2 and 12 consider the role of infinitistic methods in mathematical practice, in the wake of Cantor's creation of transfinite number theory and Zermelo's axiomatization of set theory. Problematic foundational issues arising from those theories are discussed, together with the responses to them by Brouwer (intuitionism), Hilbert (formalism) and Weyl (who proposed an arithmetical approach to analysis within a predicative framework). Among the questions discussed are: Is there a need for new axioms to settle such questions as the truth of the Continuum Hypothesis? Is the notion of truth for such statements meaningful at all for a non-platonist? Do countable foundations suffice for present-day applications of mathematics? Various proposed candidates for "natural" undecidable statements are considered, and their significance as evidence for the necessity of transfinite considerations is challenged.

Chapter 3 is a trenchant, balanced critique of Imre Lakatos' book *Proofs and Refutations* ([2]). After summarizing Lakatos' position, the author declares (p. 78) that he finds "much to agree with both in [Lakatos'] general approach and in his detailed analysis". Yet he also raises a number of serious objections to Lakatos' claims for the "method of proofs and refutations" — among them, that it fails to account for the development of mathematics prior to 1847; that it ignores those "internal organizational foundational moves" which "do not arise as responses to critical examination of fallacious proofs" (p. 86); and that all of the conjectures Lakatos gives as examples have the same underlying logical structure (namely, $\forall x\,[A(x) \to B(x)]$), which casts doubt on the universality of the method. Feferman concludes that although the logical analysis of mathematics may seem "irrelevant to everyday experience ..., it alone throws light on what is distinctive about mathematics, its concepts and methods" (p. 93). While granting that "Lakatos' successes should inspire us to seek a more realistic theory of mathematics", he suggests that there is not just one "logic of mathematical discovery" and that Pólya's heuristic approach may "provide one bridge from theory to practice" (ibid.).

Chapter 4, "Foundational ways", aims to restore the logical approach to the foundations of mathematics while avoiding preoccupation with grand schemes. In it the author categorizes foundational activity as falling into several "characteristic modes", including conceptual clarification, the method of interpretations, syntactic transformations, ways of dealing with problematic methods, organization and axiomatization, and reflective expansion of concepts and principles. Examples from the history of mathematics are given for each mode. Chapter 5 ("Working foundations") then presents more detailed and technical treatments of the same topics.

Gödel's work, its genesis and impact are the subject of four of the fourteen chapters in the volume. Chapter 6, reprinted from Gödel's *Collected Works* ([1]), is a sketch of Gödel's life and work — the best essay-length discussion thereof. Chapter 8, from the same source, is a commentary on Gödel's 1933 lecture to the Mathematical Association of America ("The present situation in the foundations of mathematics"), in which, in stark contrast to his later pronouncements, Gödel proclaimed that to interpret the axioms of set theory or the simple theory of types "as meaningful statements, necessarily presupposes a kind of Platonism, which cannot satisfy any critical mind" (p. 168). Feferman notes that the lecture, left unpublished during Gödel's lifetime, "could hardly have been bettered for its time" and has held up remark- ably well "after more than half a century's subsequent intensive development of those areas of mathematical logic that most bear on Gödel's foundational concerns" (p. 171).

Chapter 7 ("Conviction and caution") is the text of a lecture delivered to a special session on Gödel at the 1983 Salzburg Congress on Logic, Methodology and Philosophy of Science. A speculative essay, it seeks to explain why, prior to 1940, Gödel was so reluctant to reveal the deep platonistic convictions to which he later attributed much of his success in achieving his sensational results of the 1930s. In particular, why were some incisive philosophical remarks from his dissertation omitted from its published version? Why did he hold back from announcing the undefinability of truth, which he had recognized early on, independent of Tarski? And what accounts for his conspicuous absence from the further development of recursion theory after he defined the class of (general) recursive functions in 1934?

On the basis of various materials in Gödel's *Nachlass*, Feferman suggests (p. 160) that Gödel feared the reaction of "the foundational establishment, dominated as it was by Hilbert's ideas", to any objective conception of mathematical truth, and that, given such a conception, he may not have felt the need (as Tarski did) to give a set-theoretic definition of the notions of satisfaction and truth. In addition, as we know from Gödel's remarks to the Princeton Bicentennial Conference of 1946 (vol. II of his *Collected Works* ([1]), p. 150), he did not expect it to be possible to give "an absolute definition" of such "an interesting epistemological notion" as that of effective computability, and he only acceded to Church's thesis (that the notion of recursive function provides such a characterization) after Turing's work. Feferman believes Gödel's strategy of avoiding controversy by eschewing "new concepts as objects of study" while freely employing them as "tool[s] for obtaining results" may have been "exactly right for the times" (p. 163); but at the same time he wonders "how logic might have been different" had Gödel been bolder in expressing his philosophical convictions (ibid.).

Chapter 11, in the section on proof theory, is the most technical of the essays devoted to Gödel's work. It presents a historical overview of Gödel's functional interpretation of intuitionistic number theory — the so-called *Dialectica* interpretation — and later extensions of that method to second-order analysis. The works surveyed include three lectures by Gödel first published posthumously in volume III of his *Collected Works* ([1]): that of December 1933 to the Mathematical Association of America, an untitled one delivered in January 1938 to an informal seminar in Vienna organized by Edgar Zilsel, and one given at Yale University in April 1941 ("In what sense is intuitionistic logic constructive?"). Gödel's aim in all three was to present a constructive consistency proof for arithmetic by admitting, in addition to the strictly finitary methods that the incompleteness theorems had shown to be inadequate for the purpose, the abstract notion of computable functional of finite type. Feferman traces the development of Gödel's ideas along these lines, culminating in their belated publication in 1958, and then briefly outlines the later work of Kreisel and Spector. In a final section he discusses the rôle of the *Dialectica* interpretation as a general proof-theoretical tool. He notes (p. 223) that the goal of "establishing [the] consistency of formal systems" has since been "replaced by a more general reductive program" of proof theory. But "at the same time, proof-theoretical tools have been applied to obtain results of a more mathematical[ly] ... extractive" character, in which "explicit ... computational information [is drawn] from proofs of statements of existential or universal-existential form" (ibid.); and for the latter purpose, the *Dialectica* interpretation "has proved to be a rather powerful and versatile" method (p. 224).

For readers who are less well versed in mathematical logic, Chapter 9 ("What does logic have to tell us about mathematical proofs?") reviews some basic logical concepts and symbolism before discussing proof trees and the so-called *Hauptsatz*, or principal theorem, concerning them (that every proof in either the classical or first-order predicate calculus can be reduced to a unique normal form), proved by Gerhard Gentzen for the calculus of sequents and later by Dag Prawitz for natural deduction systems. This leads to a discussion of an important open problem in proof theory: that of characterizing the informal notion of identity of proofs. It is a proposal of Prawitz and Per Martin-Löf that two informal proofs should be regarded as the same if and only if their formalizations as tree proofs both reduce to the same normal form, but their proposal has not yet gained widespread acceptance. The essay concludes with a discussion of various conservation results.

Rounding out the section on proof theory is the more technical chapter 10, "What rests on what? The proof-theoretical analysis of mathematics". Here Feferman distinguishes several senses in which the question "What rests on what?" might be understood: One might say that an informal body of mathematics rests on a formal axiomatic system if it can be formalized therein; that an axiomatic system rests on a general foundational framework if that framework justifies it; or that one axiomatic system rests on another if the first is reducible to the second according to some precise criterion of reducibility.

One such criterion is that of the *interpretability* of a theory (such as Peano arithmetic) within a more comprehensive one (Zermelo-Fraenkel set theory, for example). Feferman notes that although such an interpretation provides a conceptual reduction of numbers to sets, it does not yield a *foundational* reduction of number theory to set theory, because the latter, but not the former, requires an uncountable infinitary framework for its justification.

In the other direction, it is sometimes possible to show that a theory T_1 is *proof-theoretically reducible* to a theory T_2 that is a proper part of it. The definition of proof-theoretic reducibility presumes that both theories contain primitive recursive arithmetic, so that binary predicates can be constructed that represent the provability relations in T_1 and in T_2; then T_1 is said to be *proof-theoretically reducible to* T_2*, conservatively over a primitive recursive class* Φ *of formulas*, if there is a partial recursive function f for which one can prove in T_2 that whenever y is a code for a proof in T_1 of a formula with code x, then $f(y)$ is defined and is a code for a proof in T_2 of that same formula. If T_1 is proof-theoretically reducible to T_2 then any formula of Φ that is provable in T_1 must also be provable in T_2, so the consistency of T_2 implies that of T_1.

A proof-theoretic reduction of a theory T_1, justified by a foundational framework F_1, to a theory T_2, justified by a foundational framework F_2, is said to provide a *partial foundational reduction of* F_1 *to* F_2, and it is such reductions that are the subject of the rest of the chapter. Examples are given there of reductions of uncountable infinitary theories to finitary theories, of uncountable infinitary theories to countable infinitary theories, of impredicative to predicative theories, and of non-constructive to constructive theories.

No "foundationally informative" proof-theoretic reduction of (full) analysis has yet been achieved, despite much effort expended toward that end. One may ask, however, whether "concrete analysis" — that based on finite-dimensional real and complex spaces — admits informative reductions. In that regard, the author mentions the "reverse mathematics" program of Friedman and Simpson, together with modern extensions of Weyl's approach, including his own theory W of variable finite types. He concludes (p. 207) that the scientific "indispensability arguments" advanced by Quine and Putnam in support of mathematical realism "are considerably weakened" by the sorts of results surveyed in this chapter.

Those same themes are discussed further in the last two chapters of the book. The first of them ("Weyl vindicated: *Das Kontinuum* seventy years later"), the more technical of the two, begins by summarizing Hermann Weyl's career and his foundational contributions, focussing on his 1918 book *Das Kontinuum* ([3]). Shortly after its publication Weyl became a convert to Brouwer's intuitionism and gave up further attempts to develop his own foundational scheme. In later years, however, he regretfully admitted that "in advancing to higher and more general theories" the intuitionistic proscription against use of the Law of Excluded Middle "results in an almost unbearable awkwardness" ([4], p. 54).

The author endeavors to show (p. 268) that the approach of *Das Kontinuum* can in fact be carried through "in terms which meet the modern requirements of formalization while remaining faithful as far as possible to [Weyl's predicativist] ... ideas." After describing the contents of *Das Kontinuum*, he concludes that in reconstructing Weyl's program one is faced with two options: because unrestricted iteration of set-valued functions leads outside the sets of lowest level of definability, one must either restrict the principles of iteration so as to prevent that or else show that unrestricted iteration is predicatively justified. In fact, he describes formalizations of both kinds: a theory $K^{(\alpha)}$ of second order with suitably restricted iteration principles; an extension thereof to a theory W of flexible finite types, which "maintains the arithmetical interpretation" but allows "a much greater body of analysis and other parts of mathematics" to be readily formalized (ibid.); and a third-order theory $K^{(\beta)}$ that can be embedded in a simple extension of W, which admits full functional iteration. $K^{(\alpha)}$ is shown to be a conservative extension of Peano arithmetic, and W is stated to be so as well. The reader is referred to the research literature for a proof of the latter result. The paper concludes with a discussion of how much of scientifically applicable mathematics can be formalized in W.

The much shorter final chapter, "Why a little bit goes a long way: logical foundations of scientifically applicable mathematics", addresses the indispensability arguments for mathematics put forward by Putnam and Quine, according to which (in Penelope Maddy's formulation) we have good reason to believe in mathematical entities because they are indispensable to our best scientific theories, which we have good reason to believe. But precisely *which* mathematical entities are indispensable to scientific practice? Feferman notes (p. 285) that "both Quine and Putnam were led to accept set-theoretical notions" to a significant extent, but neither of them examined how essential such notions really were.

The paper lists (p. 288) various problematic assumptions that are implicit in set-theoretical foundations, among them that "abstract entities are assumed to exist independently of any means of human definition or construction", that "the totality of all subsets of any infinite set are assumed to exist", that "impredicative definitions of sets are routinely admitted", and that "the Axiom of Choice is assumed in order to carry through the Cantorian theory of transfinite cardinals". The historical development of predicative foundations for analysis is then again surveyed, including the efforts of Poincaré, Russell and Weyl, after which the author's system W is once more briefly described and compared to the "reverse mathematics" program and to various constructivist and finitist foundational programs. In light of these developments, Feferman argues that "even if one accepts the indispensability arguments, practically nothing philosophically definitive can be said of the entities which are ... supposed to have the same status ... as the entities of natural science" p. 297). He notes that Maddy too, though for different reasons, concluded that the indispensability arguments "do not provide a satisfactory approach to the ontology or the epistemology of mathematics" (ibid.).

The writing throughout exemplifies the clarity, care and incisiveness for which its author is renowned. In a few places, of necessity, the material is rather heavily notational. No uniformity of notation among the different chapters has been imposed, but a ten-page glossary of symbols is provided at the back of the book for readers' convenience. An extensive list of references and a detailed index complete the volume.

This is a stimulating book that can be read and reread with intellectual pleasure and profit.

References

[1] Kurt Gödel, *Collected Works*, vols. I - III, Solomon Feferman et al. (eds.), New York and Oxford: Oxford University Press, 1986, 1990, 1995.

[2] Imre Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, John Worrall and Elie Zahar (eds.), Cambridge, UK: Cambridge University Press, 1976.

[3] Hermann Weyl, *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Leipzig: Veit, 1918.

[4] Hermann Weyl, *Philosophy of Mathematics and Natural Science*, Princeton: Princeton University Press, 1949.

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