Modern Logic
Volume 8, Number 1/2 (January 1998–April 2000), pp. 90–104.

Review of

W. D. HART (ED.), *THE PHILOSOPHY OF MATHEMATICS*

New York: Oxford University Press, 1996
vi + 316 pp. ISBN 0-19-875119-2 and ISBN 0-19-875120-6 (Pbk)

IGNACIO ANGELELLI

This is a collection of thirteen previously published papers on the philosophy of mathematics. Here is the list of authors and titles, in their order within the volume: P. Benacerraf: *Mathematical truth*, W. V. Quine: *Two dogmas of empiricism*, W. D. Hart: *Access and inference*, M. Dummett: *The philosophical basis of intuitionistic logic*, Ch. Parsons: *Mathematical intuition*, P. Maddy: *Perception and mathematical intuition*, W. W. Tait: *Truth and proof: the platonism of mathematics*, H. Putnam: *Mathematics without foundations*, G. Boolos: *The consistency of Frege's "Foundations of arithmetic"*, D. Isaacson: *Arithmetical truth and hidden higher-order concepts*, St. Shapiro: *Conservativeness and incompleteness*, H. Field: *Is mathematical knowledge just logical knowledge?*, Ch. Parsons: *The structuralist view of mathematical objects.* Aside from these thirteen papers, there is an *Introduction* by the editor, *Notes on the contributors*, and *Suggestions for further reading.*

I will describe the main points of each contribution, interjecting, here and there, my comments on some of the numerous themes discussed (in my references, I will just give the page number, omitting "p." or "page").

Benacerraf's paper, as well as much of Hart's anthology (as it appears from Hart's *Introduction* and from the fact that Benacerraf's paper is number one in the selected sequence)[1] hinges on "Benacerraf's dilemma", which is that the requirements of *truth* and *knowledge* cannot be both satisfied in the philosophy of mathematics. On the one

[1]In the *Index of Names* "Benacerraf" boasts 19 references, second only to Quine (22) and ahead of Hilbert and Russell (17), Gödel, Frege and Putnam (16). These figures, however, may have to be revised because the index of names is not accurate in the sense of not listing all pages in which a name occurs (*e.g.* "Cantor", "Husserl" are on 302 but this page is not mentioned in the index).

hand, truth for mathematics requires bringing in abstract objects. Benacerraf rightly rejects the identification of truth with derivability from axioms, which might avoid the introduction of abstract objects (25); he rightly wants *truth*, and not a truth *ad hoc*, just for mathematics, but a general theory of truth (this is the commendable "first condition" stated in section II of the paper). On the other hand, *knowledge* of abstract objects appears to Benacerraf as very hard to understand because he defends a "causal account of knowledge" ("for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S", 22). Benacerraf also maintains a causal theory of reference, "thus making the link to my saying knowingly that S *doubly* [*i.e.* both epistemologically and semantically or referentially] causal" (*ibid.*). From such a doubly causal standpoint, Benacerraf observes that "if, for example, numbers are the kinds of entities that they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have mathematical knowledge cannot be made out" (24). Thus, a way out is not in sight.

Benacerraf finds in Gödel a good fellow-traveller in the issue of truth (first horn) — but, alas, not in the epistemological side of the coin (second horn). Gödel's proclamation that the axioms of set theory "force themselves upon us as being true" (25) is not satisfactory for Benacerraf, who misses in Gödel an account of the key phrase "forcing themselves" (26). Benacerraf says that "instead of tinkering with the logical form of mathematical propositions or with the nature of the objects known, [Gödel] postulates a special faculty through which we 'interact' with these objects. We seem to agree on the analysis of the fundamental problem, but clearly disagree about the epistemological issue — about what avenues are open to us through which we may come to know things."(26). Uncomfortable with the platonists, Benacerraf turns to the other side — the "combinatorial" people — and focuses on Quine, where Benacerraf finds truth ... but truth without reference, and this is unacceptable for Benacerraf, for whom truth without reference is not truth: "truth and reference go hand in hand" (29).

Curiously, if the friends of Benacerraf's dilemma explored the classical sense of "abstract(ion)" rather than taking for granted the sense that has become standard in the mainstream of the logico-analytic-foundational tradition (to be abstract is to be intangible, neither in space nor in time), they might find, perhaps, some help. In the original, traditional meaning, abstraction is an operation that in any case involves, as Locke puts it, a "retaining" and a "leaving out". The abstract object, or *abstractum*, is largely the effect of that operation, so that there is plenty of causal activity — at least in one way, from the knower to the known.

One virtue of Hart's *Introduction* (which includes an in-depth analysis of the dilemma) is that Hart appears to feel not entirely comfortable with the standard, quick notion of abstractness as intangibility. He incisively, and insistingly, asks: "What is worrisome about abstract objects?", "Could there be perception of the abstract objects required for mathematical truth? What is it to be abstract anyway?" (3). Hart's puzzlement over the nature of abstractness is nicely revealed by the following question, asked in connection with his comments on Maddy's paper: "can the wall between the concrete and the abstract be broken down?" (7). Hart admits that he does not have any positive answers to these queries, and in fact, he remains within the standard conception of abstraction, where only a few "general, negative things" are said about abstract objects, for instance "they do not reflect light, nor do they bump into anything" (4). Indeed, most, if not all, the contributors to this volume share this understanding of abstract(ion).

The two senses of "abstract(ion)" are inter-connected in a very complex history. The original sense prevailed throughout the classical philosophical tradition (scholasticism in its three stages: first or medieval, second or post-medieval, third or "neo"; modern philosophy, Husserl's phenomenology, *etc.*), and was borrowed by some of the early workers in logic and the foundations of mathematics (*e.g.* Cantor, Dedekind, Peano). For partially good reasons, abstraction was rejected by Frege (who rightly complained about Cantor's "impossible abstractions") and, with Russell's endorsement, finally banned from the central 20th century philosophical tradition proceeding from these two authors (the rare exceptions being Peano, Weyl, Lorenzen). To complicate things, the void left by abstraction was filled with many pseudo-uses of the term, such as: 1) the so-called "definitions by abstraction" accomplished in moving from given elements to their equivalence classes (relative to some equivalence relation), 2) the merely symbolic operations of "abstraction" so much mentioned in Quine's and many others' works, and last but not least 3) the above mentioned standard sense of abstractness as intangibility. All these *spurious* senses of "abstract(ion)" (spurious because of neglecting the operation of *abstracting from*), prevailing in the 20th century logico-analytical-foundational studies mainstream, are to be distinguished from the *genuine* classical sense (genuine because the operation of *abstracting from* plays the central rôle). One way of appreciating the contrast between the two meanings is to observe that in the spurious sense of abstract = intangible, a religious person should say that God is abstract, whereas in the genuine sense God appears to that person as the least abstract, or the most concrete of all entities[2].

In the famous paper that occurs second in this collection, Quine denounces two "dogmas" in modern empiricism: the belief in a sharp separation between analytic and synthetic, and the belief that each meaningful sentence can be referred, ultimately, to immediate experience. "At root" the two dogmas are identical (47). Quine devotes a final section to "empiricism without the dogmas", and here it is where the reader learns that, in Quine's empiricism, the place left by the second dogma is filled by holism ("No particular experiences are linked with any particular statements in the interior of the field [of the totality of knowledge], except indirectly through considerations of equilibrium affecting the field as a whole" (48)), while the place left by the first dogma is occupied by pragmatism (51).

Hart defends a platonism, with regard to the existence of abstract objects such as numbers, that I would call "contextual", or "that-platonism", or "inference-platonism". This means that it is enough to know *that* there are abstract objects, or more specifically, to *infer that* such objects exist, without demanding any "more intimate" grasping of them (61). Thus, in his paper, Hart reiterates, in different words, his *negative* answer to the question (*Introduction*) concerning what it is to be abstract. Hart shares today's spurious sense of "abstract" but it must be granted that those who understand abstraction in the genuine way — the "great masters of abstraction" to which Berkeley refers — do not fare much better in offering *positive* answers to the crucial question: What exactly are we left with upon performing abstraction?

Dummett's question is: Why intuitionistic rather than classical logic? He says to be "solely concerned with the logical constants", not with any other aspect of intuitionism (63). However, he wants to consider the logical constants not *per se* but specifically within mathematical reasoning. This is in my judgment a first drawback in Dummett's approach to intuitionistic logic. Mathematical reasoning, because of the infinite domains it considers (or reasoning about future contingent human events), may *motivate* certain special considerations having an impact on the logical constants, but the theory of the logical constants — *i.e.* logic — should be autonomous. At any rate, not even the mathematically viewed logical constants show up to the extent expected by

[2]For these and further remarks on abstraction I must refer to some of my publications: [2], [3], [4], [5], [6], [7], [8], and [9].

the reader in this long essay. In fact, from approximately p. 64 through p. 79, one finds a lengthy discussion of the only "two lines" available "for repudiating classical reasoning in mathematics in favour of intuitionistic reasoning" (64): 1) the thesis that meaning is exhaustively determined by use[3], and 2) "the celebrated thesis that mathematical statements do not relate to an objective mathematical reality existing independently of us" (75). Then there are more pages discussing truth intuitionistically understood. At the end of the tunnel (84) the logical constants begin to emerge. Still, they emerge involved in a disturbing ambiguity: there are logical constants for classical logic, and logical constants for intuitionistic logic. Wisely, Dummett tries to remove this ambiguity. This he attempts by focusing on the notion of proof, but then the analysis of the notion of proof leads nowhere: it requires a special notion of "canonical proof", which is "obscure" (88). Dummett tries to overcome the obscurity by introducing a distinction between "canonical proofs and demonstrations" (89), but, alas, by now the essay has almost reached its end — and the reader still ignores what to do with the logical constants, intuitionistically, mathematically, or otherwise, and must content herself with the following generic advice: if, in repudiating classical logic, one does not wish to follow the above mentioned "meaning" line, one should be "hard-headed" enough to follow the other line, the truth line, acknowledging "that there is no notion of truth applicable even to numerical equations save that in which a statement is true when we have actually performed a computation (or effected a proof) which justifies that statement" (94).

The disappointment with regard to Dummett's answer to his own question concerning the meaning of the logical constants, is experienced again by the reader of Dummett's [1977], published two years after the paper reproduced in this collection, commended by the editor in his *Introduction* (fn. 11) as well as included in the *Suggestions for further reading*. In that book the logical constants undergo the following four, or five, stages treatment: 1) sections 1.2: *The meaning of the logical constants* and 1.3: *Examples of logical principles*. 2) Chapter 4: *The formalization of intuitionistic logic*. 3) Chapter 5: *The semantics*

[3] "If to know the meaning of a mathematical statement is to grasp its use; if we learn the meaning by learning its use, and our knowledge of its meaning is a knowledge which we must be capable of manifesting by the use we make of it: then the notion of *truth*, considered as a feature which each mathematical statement either determinately possesses or determinately lacks, independently of our means of recognizing its truth-value, cannot be the central notion for a theory of the meanings of mathematical statements [...] We must, therefore, replace the notion of truth [...] by the notion of proof." (73).

of intuitionistic logic. 4) There is, further, section 7.3 in the concluding chapter 7: *Are the intended meanings of the logical constants faithfully represented on Beth trees?*. The unprepared reader, who thinks that "meaning" means meaning, and wants to know what the logical constants mean, believes that section 1.2 answers the question. Such a belief seems further confirmed by the fact that what is said in that section appears to be good enough to establish quite serious, far reaching results, such as, for instance, that the move from $\neg p \rightarrow (q \vee r)$ to $(\neg p \rightarrow q) \vee (\neg p \rightarrow r)$ is invalid (28), a result used in the construction of the calculi (*cf.* 136: "which we saw to be invalid in section 1.3"), and that intuitionistic logic, contrary to classical logic, is not "smooth" but "rough" (169). However, at a certain point the reader discovers that "meaning" in the title of section 1.2 of Dummett's book is meant as something curiously hybrid: strong enough to support logical claims as momentous as the just indicated ones; vague enough to prevent any attempt to construct a completeness proof of the calculi proposed in ch. 4 (214-5). Similarly to what happens in the paper reproduced in Hart's collection, Dummett disappoints those who are anxious to be enlightened about the logical constants, and engages in pessimistic considerations about an ever growing "theory of constructions", which has "not yet attained a satisfactory state" and therefore forces us to give up hopes of obtaining completeness proofs, and to content ourselves, for the time being, with such things as Beth trees, which may provide, at best, only a "specific" but not an "absolute" semantics (*ibid.*).

It is difficult to exactly describe Parsons' attitude toward the notion expressed by the title of the first of his two papers in this collection: mathematical intuition in the sense of intuition of mathematical objects — it combines both distrust of that notion and willingness to rescue it as far as possible.

On the one hand, Parsons calls "outrageous" Gödel's view that mathematical intuition is "something like a perception" (98). This is not just based, uninterestingly, on a short-sighted empiricism or nominalism. Parsons' doubts about mathematical intuition have more profound reasons. One of these seems to be what he himself describes as the "incompleteness" and lack of individuality of mathematical objects (99-100). Incisively, Parsons asks: "Now the question is, how can mathematical intuition place objects 'before our minds' when these objects are not identifiable individually at all?" (100). Commenting on this question, Parsons writes: "One could press the matter further and urge the possibility of an interpretation of mathematics which dispenses with distinctively mathematical objects" (*ibid.*) and "What is really essential to mathematical objects is the relations constituting the structure to which they belong" (101).

On the other hand, Parsons tries to salvage, as far as possible, the intuition of objects in mathematics. He distinguishes intuition *that* and intuition *of*; the former refers to propositions, the second refers to objects (96), and plans to show that "there is at least a limited application of the notion of mathematical intuition *of* which is able to meet these objections" (101). This "limited application" has to do with the grasp of types as opposed to tokens. While pointing, as an example thereof, to the familiar stroke-introduction of numbers (|, ||, |||, ...), Parsons rightly emphasizes that such type-grasping is not something that "comes into play only in doing pure mathematics" but is "perfectly ordinary" (104), and types are "no more mysterious than other objects, in spite of their 'abstractness' (103). Parsons is so inclined to favor the intuitive grasping of types that he is reluctant to agree with Husserl in viewing "intuition of a type as founded on perception of a token" (105).

After displaying his interest in the intuitability of types, Parsons turns to the question of how can we know such "acausal" or "incomplete" entities (109). Here Parsons makes a point of minimizing his claim that there is an intuition of abstract strokes. His minimalization is twofold: 1) stroke types are minimally abstract (*ibid.*), 2) intuition is not (in my, not Parsons', words) passive but active: "what is intuited depends on the concept brought to the situation by the subject" (110).

I believe that Parsons concludes his paper more pessimistically than he ought to. While granting that he has reached a "significant positive result" he also adds that "the result is of very limited scope" (111). But this self-pessimism is wrong and due, in the first place, to Parsons' prejudice against stroke-types as good for the foundation of arithmetic. In fact, he thinks of them, regrettably, as rather geometrical (103) and foreign to arithmetic, so that he ends up viewing his result as more geometrical than arithmetical. As the paper reaches its end, Parsons appears to become less pessimistic, and even says that "the natural numbers, considered as 'numbering' objects of intuition, are objects of intuition' (112).

Parsons' paper shows the need for the restoration of a theory of abstraction in the genuine, classical sense. It must be noted that it is abstraction *theory* that disappeared from the *philosophy* of mathematics, and that needs to be restored — genuine abstraction practice never disappeared (for instance, in such a standard introduction to set theory as Kamke's, abstraction is used, and correctly).

One point in Parsons' paper where the potential abstraction connection is obvious (to those who acknowledge genuine abstraction) is where mathematical intuition is described as an intuition of an entity *as* falling under a concept (111). This reduplicative[4] little word calls for a striking comparison of Parsons' paper with Lear's [12]. Where Lear says "qua", Parsons uses the more familiar "as", but both particles are equally used as a gateway to the realm of genuinely abstract entities, as rightly observed by Lear[5].

Parsons' efforts to make sense of the intuition of mathematical objects can be read, at least in part, by the friends of abstraction, as efforts to understand the nature of (genuine, not spurious) abstracta. But perhaps in such a reading of Parsons, the word "intuition", used by him, has to be dropped, if it turns out that abstracta are not intuitable, and that we *should not* expect them to be intuitable. Exercise: What intuitions can we have of Peano's abstractum obtained by leaving out, in our consideration of fractions, anything not invariant[6] with respect to the relation of having equal cross-products? The answer to this query is discouragingly difficult. Such a difficulty in having intuitions of abstracta did not help, obviously, the cause of abstraction, and must have pushed Peano's students, Russell and others toward entities that are, seemingly at least, more intuitable, *e.g.* the equivalence class, or for that matter anything at all that is compatible with the given equivalence relation. But perhaps the right approach was, and is, not to demand or to expect intuitability where the latter should not be expected — there was, after all, a tradition opposing abstractive and intuitive knowledge.

Certainly, the above hinted at restoration of genuine, classical abstraction in our contemporary philosophy should also be a *revision*, meeting first of all the objections raised by Frege, and putting things in a logico-linguistic framework rather than in the psychological language of the philosophical tradition.

---

[4]Reduplicative expressions are "qua", "as", "insofar as", *etc.* Scholastic logic usually devoted a section to the analysis, in the sense of "explaining away", of such phrases. There is now one substantial monograph on reduplication: Bäck's [10].

[5]Abstraction and reduplication, prima donnas in the classical tradition, are the cinderellas of modern logic, from whose textbooks they have vanished almost entirely. The reasons for their disappearance are known in the case of abstraction, but in the case of reduplication remain, to me, as a mystery. Regardless of these historical accidents, they go, theoretically, hand in hand: to talk reduplicatively is to talk abstractively.

[6]The word "invariant" seems to occur the same sense in Parsons' second paper, p. 279, second paragraph: one more cryptic occurrence, perhaps, of genuine abstraction in this volume.

Maddy begins with an excellent terminological point: "The term 'platonism' is often applied to views of this sort [sets existing independently of human thought], but I will avoid it. To me 'realism' seems more appropriate, since sets, on the view I am concerned with, are taken to be individuals or particulars, not universals." (114, fn. 1). In fact, claiming that sets exist independently of human thought is more similar to claiming the existence of angels than the existence of universals.

Maddy would like to baptize sets: "We imagine our baptist in his study saying things like: 'All the books on this shelf, taken together, regardless of order, form a set'"(117), but she notes that supporters of a causal theory of knowledge would not be satisfied, since, as she puts it: "the set-dubber causally interacts only with the members of some samples", and "sets seem unable to enter into causal relations" (118).

To overcome this difficulty, Maddy attempts to show that "people often perceive sets of physical objects" (120). This develops, she says, "much the same way as that in which our ability to perceive physical objects develops" (120-126). Maddy gives an example: a set of three eggs. A person P who "reaches into the refrigerator for the egg carton, opens it, and sees three eggs there" (126) is a person who "acquires the perceptual beliefs that there is a set of eggs [...], that it is three-membered, and that is has various two-membered subsets" (127). Maddy anticipates the objection that sets do not have location, but quickly dismisses it, claiming that sets of physical objects do have location: the set of eggs is located in the egg carton. Quite surprisingly, Maddy wants to continue to uphold, at the same time, the standard view that sets are abstract objects. So, entities can both be abstract and exist in space and time. This is a momentous episode in the history of the term "abstract(ion)", genuine or spuriously taken. Maddy's claim deserves being quoted in full: "I have now denied that abstract objects cannot exist in space and time, and suggested that sets of physical objects do so exist" (127, fn. 39). Surprising at it is, Maddy's thesis should not be rejected before a careful analysis of it is made. In the spurious sense of "abstract(ion)", Maddy's claim would be like assigning a space and time location to an angel; in the genuine sense, the claim would require that, in performing abstraction on objects that are in space and time, these two features be not "abstracted from", *i.e.* not "left out" but "retained". Perhaps theologians or logicians tell us that there is nothing absurd about either proposal. At any rate, for friends of abstraction Maddy's essay is one of the most stimulating readings available in the recent literature.

Anticipating objections from the friends of sets as abstract entities, Maddy quite rightly says: "I find it hard to be sure what the difference is between believing that 'three' applies to a particular physical aggregate under the property 'egg' and believing that a particular set of eggs is three-membered. What is the set over and above the physical aggregate individuated in a certain way?" (127-8). The view of sets as aggregates considered "under a property" is very interesting, and has occurred in the history of these notions (*cf.* my [1], 8.31).

Another important move — perhaps the most difficult one — in Maddy's paper is the claim that "at least some of the basic axioms of set theory" can be seen as fruits of the intuitive evidence we have of sets. Maddy claims that "from the writings of Zermelo on his original axioms and the axiom of choice, the work of Fraenkel on replacement and choice, and the remarks of numerous authors on the iterative conception, historical evidence could be adduced for my claim that intuitions form a basis for (but do not exhaust) the scientific theory of sets, that they can be confirmed or disconfirmed like any theory, and that their status as intuitions is evidence in their favour" (134).

Tait claims that he makes of platonism (or realism: propositions are true regardless of our knowledge of them, 142) a "truism" (143). This he achieves by identifying truth with existence of a computation, and by further distinguishing between the computation and the "presentation" of the computation or "proof" (160). Tait acknowledges that his use of the term "computation" may raise objections, because computing is normally associated with a human activity (161). To this he replies: "but the term is also used in my sense, for instance in the mathematical theory of computation" (161). It seems difficult to understand Tait's point: "computations in themselves", disconnected from their human presentations or proofs, appear to play the same old role of the "truths in themselves" that exist independently of our knowledge of them.

Not quite aside from its main theme, Tait's paper offers a jewel to friends of abstraction (if any). Tait coins the phrase "Dedekind abstraction" to describe certain abstractions performed by Dedekind (165, fn. 12). What is important here is to find at least and at last one reference — perhaps unique in the volume — to *genuine* abstraction.

Putnam distrusts the famous "isms" in the philosophy of mathematics (170) and rejects the common idea of a "foundational crisis" (168). He highlights two "equivalent descriptions" of mathematical facts: "mathematics as set theory" and "mathematics as modal logic" (171). Putnam says that while the notion of set has been used to clarify the modalities, he wishes "to go in the reverse direction" (181).

He refers to this as "modalism". One philosophical significance Putnam sees in his modalism is that it allows us to give a clear sense to statements about all sets without assuming any maximal model: "in metaphysical language, it is not necessary to think of sets as one system of objects in some one possible world in order to follow assertions about all sets" (183).

Boolos begins by saying that the plausible view that Frege's *Die Grundlagen der Arithmetik* (*Foundations of arithmetic*, or GRL as I will briefly write) is, once suitably formalized, inconsistent, is mistaken (186). Boolos presents a formal theory, FA (Frege Arithmetic) "that captures the whole content of these central sections [of GRL] and for which a simple consistency proof can be given, one that shows *why* FA is consistent" (186).

Boolos' FA has as its underlying logic the standard axiomatic second-order logic with Peano-Russell notation. There are three sorts of variables: object variables a, b,...; unary predicate variables F, G, H,...; binary predicate variables ϕ, ψ, ... Just one non-logical symbol: η, read as "is in the extension of" (reminiscent of membership). Aside from usual axioms and rules for second-order systems, there are two comprehension axioms: (i) $\exists F \forall x\,(Fx \longleftrightarrow A(x))$ and (ii) similar for binary predicates. Only one non-logical axiom is introduced, called "Numbers": there is, for every concept F, exactly one number, which is the extension of "equinumerous with F".

Next, Boolos moves to the consistency proof of FA. He finds it helpful to begin by showing the obvious satisfiability of Hume's principle. The latter is stated by Frege as follows: the number of a concept $F =$ the number of a concept G iff F and G are equinumerous. In a universe of discourse consisting of all natural numbers and $\aleph_0$, and interpreting NF as the cardinal of F, Hume's principle is satisfiable, and the satisfiability of Numbers is equally obvious: every subset of U has a cardinality.

Interesting as it is, Boolos' contribution too sweepingly ignores the main objective pursued by Frege in GRL: an analysis (hopefully in logical terms) of the notion of natural number. All that remains of this central endeavour of Frege's in Boolos' paper is a seven line, or so, sentence: "In the course of replicating in FA Frege's treatment of arithmetic, we shall of course make definitional extensions of FA. For example, as Frege defined the number belonging to the concept F as the extension of the concept 'equinumerous to F'', so we introduce a function symbol N, taking a concept variable and making a term of the type of the object variables, and then define $NF = x$ to mean $\forall G\,(G\eta x \longleftrightarrow F\,\mathrm{eq}\,G)$; the introduction of the symbol N together with this definition is of course licensed by Numbers" (193). Contrary to Frege's plan, in Boolos' interpretation a number, which is always the number of a concept F (namely, NF), is dissolved into sheer equinumerosity.

How much of the essence of Frege is lost in Boolos may be appreciated by his simplistic reading of GRL §73 as a proof of Hume's principle without even wondering what sense does it make to prove in §73 the proposition that in §68 was used by Frege to show that the famous definition of number was licit. Properly, in Frege *first* comes Hume's principle, *then* the observation that the singular terms of the form NF *can* receive as denotation, *thanks to Hume's principle*, the extension of the concept "being equinumerous to F", and *finally* the famous explicit definition: $NF =$ the extension of the concept "being equinumerous to F". With his axiom Numbers, Boolos bypasses all that is essential for Frege in the core of GRL: §§62-69.

Nevertheless, Boolos' surgery on Frege's corpus is, aside from its intrinsic theoretical interest, useful as a preliminary cleaning up operation that disconnects Frege from the philosophically infelicitous method[7] employed by him in the definition of number, and opens the way for a reconstruction of Frege's analysis of number in terms of (genuine) abstraction.

Isaacson starts from the recognition of the incompleteness of formal systems of arithmetic as "a fact of mathematical life" (203). Nevertheless, he wants "to raise some issues" about that fact, "particularly on the nature of those true statements in the language of arithmetic which are unprovable in it" (203). While Gödel shows that any particular formal system must be provisional, Peano arithmetic "seems a natural and intrinsically important axiomatization" (203). Isaacson wonders whether we are so impressed by Peano arithmetic just because it was historically arrived at first, "or does it reflect rather some underlying conceptual fact?" (203). Isaacson suggests that any first-order arithmetical truth beyond Peano Arithmetic is such that there is no way by which its truth "can be perceived in purely arithmetical terms" and in this sense "Peano arithmetic may be seen as complete for finite mathematics." (203-4). Known examples of arithmetical true sentences unprovable in Peano arithmetic "could not be taken as axioms in an extension of Peano arithmetic" (222).

Shapiro says that in H. Field's *Science without numbers* it is argued that the addition of mathematics to a nominalistic theory preserves

[7]I have referred in [2] to Frege's method as "looking around method", because of a similar phrase found in Carnap's exposition of it.

nominalism. Shapiro's plan is to undermine this "conservativeness notion", by showing that "for any reasonable physical and mathematical theories in Field's programme, either the mathematical theory is not conservative in the philosophically relevant way or the mathematics is not applicable to the physical theory in the usual way" (226).

Field's essay begins with a strong attack on logicism. Assuming with Kant that "logic [...] can never categorically assert the existence of anything", Field argues that logic cannot entail mathematics, because mathematics — "if taken at face value" (235) — includes existential claims.

Here I would like to insist on understanding, and thereby to some extent defending, logicism in a historical perspective. Arithmetic had been largely confined, in classical philosophy, to a particular region of reality: the physical, material world, and number properly had been understood as emerging from the division of material quantity. Leibniz, Husserl, Frege and others react against such a restriction of arithmetic, and point out that everything is countable, not only material entities. Frege, in particular, draws the conclusion that a discipline whose subject matter applies to everything must be very closely related to logic, where "logic" means ontology, a term banned by Kant. This is the plausible core of logicism[8] that survives, perhaps because of its vagueness, Field's criticism or Russell's contradiction.

However, it turns out that it was not necessary to defend logicism from Field's attack: shortly after his initial anti-logicist statement, Field shocks the reader by affirming that "the idea that mathematical knowledge is just logical knowledge is largely correct" (237). This is not achieved by magic but by revising or reconstructing the notion of mathematical knowledge in a "deflationist" direction. Deflation here means the following: the knowledge that a mathematician has falls entirely into two classes: i) "empirical knowledge (*e.g.*, about what other mathematicians accept and what they use as axioms)" (237), ii) "knowledge of a purely logical sort — even on the Kantian criterion of logic according to which logic can make no existential commitments." (*ibid.*) Obviously, Field's surprising mutation from logicist to non-logicist must be possible because he manages to remove from mathematics the existential assertions that, as he claims at the beginning of the paper, mathematics apparently makes. This prediction is right: deflationism will reconstruct mathematical knowledge as a *modal* knowledge of purely logical possibility (240, fn., where Field remarks that his modalism is, contrary to Putnam's, purely logical). Field devotes thirty long pages

[8]To which I refer as "philosophical logicism" in my [1], 10.6.

to surveying problems related to his de-existentialization of mathematics. Therein Field emphasizes that every statement of possibility is, if true, logically true, and if false, logically false (242). This idea is presented by Field as non-Kripkean, and as having "its roots in chapter 5 of Carnap's *Meaning and Necessity*" (271).

In the last paper of the anthology, Parsons rejects general structuralism (all mathematical objects vanish in terms of structures): "some mathematical objects for which structuralism is not the whole truth must still have their place" (273). For one thing, structures themselves appear to be set-theoretically conceived, so that at least those mathematical objects called sets are not going to vanish in the seas of structuralism. In this sense, general structuralism cannot be accepted — at most "set theoretical structuralism" can be accepted. This seems to lead Parsons to the consideration of Dedekind as example of such a set-theoretical structuralism. Thereafter, Parsons removes set theory from Dedekind and puts second-order logic in its place (281). After a long discussion, he attacks the "eliminative structuralist" (defined as he who tries to eliminate reference to mathematical objects in favor of statements about structures, 277) with the following dilemma: if you use second-order logic then you cannot avoid ontological commitments that are worse than the mathematical objects you want to avoid; if not, you are faced with unpersuasive relativistic consequences (300). In spite of this result, Parsons continues to ponder (section 7) whether the problems concomitant to second-order logic are really fatal (for the structuralist, I suppose). In the last section of the paper, Parsons moves to consider "whether sets should be an exception to a general structuralism about pure mathematical objects" (303), and decides in favour of replacing the set-theoretical structuralism by a metalinguistic one (308). Still, this is not presented by Parsons as a view that he enthusiastically endorses but rather as an attempt to produce a "defensible version" of structuralism (308). The outcome remains for Parsons that "structuralism is not the whole truth about mathematical objects" (308). It is in this spirit that Parsons writes the penultimate sentence of his paper, and of the volume: "Thus, if the structuralist view of mathematical objects is taken to mean that all mathematical objects are only structurally determined, it has to rest on legislation about what counts as a mathematical object".

There is in my view something nonsensical about extreme or unqualified structuralism, and I believe that the latter can be understood only as an obscure way of filling the gap left by the disappearance of the theory of genuine abstraction. Such an interpretation of structuralism prompts the following, final remark of this review, where I have several times referred to abstraction. If, as critics of old or modern abstraction theories claim, a decent abstraction theory is really shown to be unfeasible, then certainly the friends of abstraction must give up their endeavours. At the same time, however, it should be forbidden, to the just mentioned critics as well as to philosophers at large, to continue to bring in abstraction in cryptic or disguised ways.

Some editorial remarks are the following. The *Index of names* is defective as pointed out in fn. 1 of this review. The print is too small for a comfortable reading. Misprints were very hard to find: p. 223 (Lorenz, not Lorentz) and p. 282 (Putnam, not Putnom).

References

[1] Angelelli, Ignacio, *Studies on Gottlob Frege and Traditional Philosophy*, Dordrecht: Reidel, 1967.

[2] ______, "Abstraction, Looking-around and Semantics," *Studia Leibnitiana* **8** (1979), 108-123.

[3] ______, "Abstracción moderna y tradicional," *Anuario Filosófico*, Navarra, XIV, **2** (1981), 9-21.

[4] ______, "Frege's Notion of 'Bedeutung'," *Proceedings of the sixth international congress of logic, methodology and philosophy of science, Hannover 1979*, Amsterdam: North-Holland, 1982, 735-75.

[5] ______, "Frege and Abstraction," *Philosophia Naturalis* **21** (1984), 453-47.

[6] ______, "La abstracción en la filosofía contemporanea," in: *El hombre: inmanencia y transcendencia (XXV Reuniones Filosóficas, 1988)*, Universidad de Navarra, vol. **1** (1991), 167-180.

[7] ______, "Abstraction and Number in Michael Dummett's 'Frege. Philosophy of mathematics'," *Modern Logic* **4** (1994), 308-318.

[8] ______, "En los orígenes de las tradiciones analítica y continental: Frege y Husserl," in: J. Echeverría et al. (eds.), *Calculemos ... Matemáticas y libertad. Homenaje a Miguel Sánchez Mazas*, Madrid, 1996, 363-375.

[9] ______, "Adventures of Abstraction," in: R. Poli et al. (eds.): *Abstraction*, Poznan Studies in the Philosophy of the Sciences and the Humanities, Rodopi Verlag, (forthcoming).

[10] Bäck, A., *On Reduplication*, Leiden: Brill, 1996.

[11] Dummett, M., *Elements of Intuitionism*, Oxford: Clarendon Press, 1977.

[12] Lear, J., "Aristotle's Philosophy of Mathematics," *The Philosophical Review* **91** (1982), 161-192.

Philosophy Department, The University of Texas at Austin, Austin, TX, 78712

E-mail address: PLAC565@UTXVMS.CC.UTEXAS.EDU