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THE ORIGINS OF THE DEFINITION OF ABSTRACT RINGS

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Contents

1. Introduction

The theory of rings had deep historical roots in several of the mainstream, classical disciplines of nineteenth-century mathematics, such as the theory of algebraic number fields, the theory of algebraic functions, the theory of polynomial forms, the theory of quaternions and hypercomplex numbers, and others[1]. The unification of all these trends reached its peak in the works of Emmy Noether (1882-1935) and Wolfgang Krull (1899-1970). Using the abstract concept of ring, these two mathematicians published systematic and comprehensive studies of the theorems-especially factorization theorems-which are common to all these domains.

The first attempt to analyze the concept of abstract ring as an autonomous mathematical entity was published by Abraham Halevy Fraenkel (1891-1965) in 1914. However, as many historical accounts justly care to stress, Fraenkel's work in this direction had a rather

The issues discussed in the present article are treated in greater detail in Chapter 4 of [4]. I thank Volker Peckhaus and an anonymous referee for helpful comments.

[1] For a recent account of the historical overview of the development of the theory, see [22].

marginal influence on the development of the theory. Fraenkel's work was not organically connected to any of the main streams that played a major role in the development of the theory of rings and it did not directly contribute to change their courses.

What was, then, the historical framework within which Fraenkel's work arose? The present article sketches the answer to this question, by tracing the roots of Fraenkel's ideas back to three main sources: the works of the American school of postulational analysis, Kurt Hensel's theory of p-adic numbers, and Ernst Steinitz's work on abstract fields. By doing this, the actual historical place of Fraenkel's work and its actual significance for the development of the theory will become clearer. At the same time an additional historical point will be stressed, namely, that the idea of an abstract ring and that of an ideal — two ideas so closely connected in the modern conception of algebra — arose in two separate paths that were only connected in the work of Emmy Noether.

2. Postulational Analysis in the USA

During the first two decades of the twentieth century, several American mathematicians, particularly in the circle associated with Eliakim Hastings Moore (1862-1932) at the University of Chicago, dedicated much of their time and efforts to analyze systems of postulates that lie at the basis of different mathematical disciplines. This activity, which came to be known as "postulational analysis"[2], was triggered by their attentive study of David Hilbert's recently published *Grundlagen der Geometrie*.

In the fall of 1901 Moore conducted a seminar in Chicago entirely devoted to Hilbert's influential book. Moore discussed with his students the possibility of revising the proofs of independence between the axioms of the five groups that Hilbert had introduced in his study of the foundations of geometry. Moore had read a recent paper by Friedrich Schur (1856-1932), who had pointed out a logical redundancy not mentioned in Hilbert's book, concerning the axioms of connection and of order taken together [36]. Moore considered Schur's criticism to be essentially correct, but noticed that Schur had not correctly identified the actual redundancy. Moore proved that Hilbert's system involved, in fact, a redundancy between one of the axioms of connection and one of order [31].

The kind of axiomatic analysis pursued by Schur and Moore was in appearance very similar to that of Hilbert. However, there are also some significant differences between them that must be stressed here.

[2]On the American school of postulational analysis, see [4, pp. 173–183] and [35].

Hilbert's work on the foundations of geometry originated in the last decade of the nineteenth century, as he joined the current efforts developed by several German and Italian mathematicians to clarify certain technical aspects pertaining to the study of the basic theorems of projective geometry. A particularly important issue in this respect was the elucidation of the actual place of continuity considerations in the proof of the basic theorems of this discipline. In 1898, Schur had been able to prove Pappus's theorem (or Pascal's theorem for two lines, as Hilbert called it) without recourse to continuity, and this important result definitively led Hilbert to concentrate all his efforts on the study of the foundations of geometry[3] . He thus undertook to elucidate in detail the fine structure of the logical interdependence of the various fundamental theorems of projective and Euclidean geometry and, more generally, of the structure of the various kinds of geometries that can be produced under various sets of assumptions.

The *Grundlagen der Geometrie* contained the results of Hilbert's work in this direction. The axiomatic structure of geometry described here was based on a long list of axioms, divided into five separate groups, each of which was meant to isolate and allow a clear understanding of the basic building blocks of our immediate spatial intuition. The *Grundlagen* was a book about geometry, and the axiomatic analysis conducted in it was conceived as an ancillary, yet very effective, tool meant to enhance our understanding of this specific mathematical discipline. Accordingly, in the *Grundlagen* Hilbert focused mainly on the interrelations among the various groups of axioms, rather than on the individual axioms. Thus, when Schur analyzed in his 1901 article the logical dependence among axioms of *separate* groups, he was actually going one small step beyond Hilbert's own analysis of the structure of geometry, in a direction and to an extent not formerly contemplated by Hilbert himself.

Moore's article of 1902 departed from the original spirit of Hilbert's axiomatic analysis in an even more significant, though still subtle, sense. When analyzing in the *Grundlagen* the systems of axioms that define geometry (paying special attention to their independence, more than to any other property, at this stage) Hilbert was implicitly bestowing upon them the status of objects with inherent mathematical interest. Still, this interest remained secondary to his main current concern, namely, the logical structure of the theorems of geometry. For

[3]For the events around the publication of Schur's proof and its effect on Hilbert, see [39, pp. 114-122].

Moore, *the system of postulates as such became the main subject of interest*, independently of the mathematical issue that gave rise to them in the first place. Moore asked how these systems can be formulated in the most convenient and succinct way from the deductive point of view, without caring very much whether these axioms actually convey any intuitive, geometrical meaning. His questions could equally be applied to a system of postulates defining Euclidean geometry, groups or fields, or, in fact, to any arbitrarily defined system. No doubt, the point of view and the techniques introduced by Hilbert in the *Grundlagen* could in principle have been applied in this more general context as well, but Hilbert's actual motivations never did contemplate the realization of that possibility. Neither he nor his students in Göttingen published any work in that direction. For Hilbert, axiomatic analysis was never a starting point, but always a very late stage in the development of any scientific discipline. From the beginning he was convinced of the importance of applying it not only to geometry, but also to other elaborate, "concrete", mathematical and physical theories such as arithmetic or mechanics[4] . Thus, Moore's perspective implied a slight shift away from Hilbert's initial interest in geometry and the other "concrete" entities of classical, nineteenth century mathematics, and towards the study of a new kind of autonomous mathematical domain: the analysis of systems of postulates.

This new perspective was followed and intensively developed over the next years by Moore himself and by other mathematicians in the USA. In 1902, Edward Huntington (1847-1952) published an analysis of two different sets of postulates defining abstract groups [20]. The first one he took from a classical article published in 1893 by Heinrich Weber (1842-1913) [42]. The second system had been proposed by William Burnside (1852-1927) in his classical book on finite groups [3]. As Huntington explicitly remarked, all the then existing systems of postulates defining groups contained many redundancies; his analysis was meant to clear these redundancies away. Moreover, Huntington proposed two new, presumably irredundant, systems defining the same concept. Moore soon published a new article in which he further discussed the four systems of Huntington [32].

This was only the beginning of a new trend. Moore's first doctoral student, Leonard Eugene Dickson (1874-1954), published his own contributions to the analysis of the postulates defining fields, linear associative algebras, and groups [7, 8, 9]. Oswald Veblen (1880-1960) was foremost among Moore's students to pursue this trend of research

[4]For more details on this , see [5]

wholeheartedly. His dissertation discussed a new system of axioms for geometry, using as basic notions point and order, rather than point and line as was traditional [40]. Veblen was joined in the same direction by another distinguished doctoral student of E.H. Moore, Robert Lee Moore (1882-1974).

Works on postulational analysis continued to appear in the leading American journals, mostly in the *Transactions of the American Mathematical Society* but also in the *Bulletin of the American Mathematical Society*. According to the statistics of E.T. Bell [2, p. 6], 8.46% of the mathematical papers written in the USA between 1888 and 1938 were devoted to postulational analysis. In the index to the first ten volumes of the *Transactions* (1900-1909), works on postulational analysis appear under the classificatory heading of "Logical Analysis of Mathematical Disciplines." The disciplines whose systems of postulates were analyzed include "real and complex algebra", groups, fields, algebra of logic, and geometry (especially projective geometry). The same heading appears in the index to the next ten volumes of the Transactions, but the number of works included under it is considerably lower. In those years (1910-1919) the main issues of research were the postulate systems for Boolean algebra and for geometry (which included projective geometry and "analysis situs"). This change of emphasis in the disciplines investigated indicates that standard postulate systems for algebraic disciplines had been adopted meanwhile in the algebraic domains investigated in the first decade of the century as part of this trend.

The analysis of these systems is similar in all the articles on postulational analysis. All postulate systems are required to comply with the guidelines established by Hilbert in the *Grundlagen*. These requirements are explicitly stated again in each new paper. Thus, it is required that the postulates be independent and consistent. If there exists more than one possible system for a given mathematical branch, equivalence among different systems should then be proven. Some articles also introduced new concepts and ideas into postulational analysis, that appeared neither in the *Grundlagen* nor in previous works of the same kind. Some of these ideas were to become significant for future developments of foundational research, while others led nowhere. To the first kind belongs the concept of "categoricity", first introduced by Veblen in 1904. Veblen called a system of postulates categorical if the addition of some new axiom necessarily renders the system redundant; otherwise he called the system disjunctive. An additional requirement which was absent from the *Grundlagen* and was first introduced in this trend — but which contrary to categoricity was not to remain central to the study of postulational systems — is that the postulates be "simple." This requirement, however, was seen as somewhat problematic from the outset, since "the idea of a simple statement is a very elusive one which has not been satisfactorily defined, much less attained" [21, p. 290].

The accumulated experience of research on postulational analysis brought about an increased understanding of the essence of postulational systems as an object of intrinsic mathematical interest. In the long run, it had a great influence on the development of mathematical logic in the USA, since it led to the creation of model theory, by helping incorporate into a broader and more organic picture concepts that had emerged, incipiently and isolated, in separate mathematical contexts. This is particularly the case with early work on the algebra of logic, to whose systems of postulates Huntington dedicated considerable attention[5]. Associated with the same train of ideas are the early works of Richard Dedekind on lattices, which arise in the context of his number-theoretical investigations[6]. This context is the same where the earliest roots of the definition of rings can be found, as will be mentioned in what follows.

But as a more direct by-product of its activity, postulational analysis also provided a collection of standard axiomatic systems that were to become universally adopted in each of the disciplines considered. In this way, by affording standards for axiomatic definitions of various mathematical branches, it provided the natural framework within which abstract, structural research on those issues was later to proceed. This was indeed the case, as will be seen below, for Fraenkel's definition of abstract rings.

3. Theory of p-adic Numbers

The second main source of ideas from which Fraenkel's definition of rings arose is found in the work of Kurt Hensel (1861-1941) on p-adic numbers. Hensel published his results on this topic in an article of 1899, and in two textbooks published in 1908 and 1913 [16, 17, 18]. In this section we discuss them briefly.

In a classical article on algebraic functions, published in 1882, Richard Dedekind (1831-1916) and Heinrich Weber had applied to this field ideas originally developed in the framework of algebraic number theory [6]. In developing his own theory of p-adic numbers, Hensel also combined ideas from these two closely connected disciplines but he

[5]See [1].

[6]See [30].

went in a direction contrary to that followed by Dedekind and Weber: he took ideas developed by Karl Weierstrass (1815-1897) in his lectures on the representation of complex functions as infinite power series, and applied them to study fields of algebraic numbers. In particular, Hensel focused on the fact that, in the surroundings of a given point a, every algebraic or rational complex function can be represented as an infinite series of integer and rational powers of linear factors $(z - a)$. Hensel thought that many of the limitations encountered in the study of specific domains of numbers were due to the fact that numbers had traditionally been represented in a unique way, namely, using the decimal representation. Any function, on the other hand, can be variously represented as a power series by choosing different points around which to develop it. If functions could only be represented only either around zero, or around an infinitely distant point — claimed Hensel — then one would find in the theory of functions the same limitations theretofore encountered in the study of fields of algebraic numbers. Thus Hensel intended to enlarge the scope of the existing theory, by providing various alternative representations of algebraic numbers, through suitable "changes of basis."

A *rational g-adic number* was defined by Hensel as a series

$$A = a_r + a_{r+1}g^{r+1} + a_{r+2}g^{r+2} + \cdots ,$$

where r is any integer and g is a positive integer, and where the a_i's are rational numbers whose denominator (after eliminating common factors with the numerator) has no common factors with g. If the a_i's are restricted to be only positive integers between 0 and $g - 1$, then A is called a *reduced* g-adic number. It can be shown that, for any given g, every rational number can be uniquely represented as a reduced g-adic number. This was the starting point for Hensel's theory, since it afforded — as he wished — the possibility to represent a rational number in several different ways by taking different bases.

Hensel showed how the usual arithmetical operations can be defined on these numbers. Nevertheless, a division exists only if one takes a prime number p as basis, instead of an arbitrary rational g. In his 1908 textbook, Hensel began by considering that case. He noticed that the product of two p-adic numbers is zero if and only if one of the factors is zero. By contrast, this is not the case if one chooses a non-prime basis g. Since Hensel saw his theory as an extension of previous works on algebraic numbers, it was totally unnatural for him to consider a system containing zero-divisors. In his 1913 textbook, however, he changed this perspective and opened with the more general definition, using a general basis g. Only in later sections of that book did he introduce the prime basis p, as a particular case.

There is a second, significant change of perspective between Hensel's two books that must be noticed here. One of the main issues which Hensel had in mind when devising his theory was that of unique factorization in algebraic fields. In the last third of the nineteenth century this theory had totally been transformed by the contributions of Dedekind and Leopold Kronecker (1823-1891). Though the works of these two mathematicians have much in common, they also essentially differ in their basic approach; it is usual to characterize Dedekind's approach as more "conceptually oriented" and Kronecker's as more "algorithmic"[7]. As it happened, it was Dedekind's "conceptual" approach to the issue that became dominant over the following decades. Among the main reasons for this dominance, one must count the fact that Hilbert essentially followed Dedekind's approach in his influential report on the state of the art in the discipline, the famous *Zahlbericht* [19].

Hensel had studied with Kronecker in Berlin, but he obviously was acquainted with Dedekind's work as well, and in particular with his theory of ideals. Still, this latter concept is totally absent from Hensel's treatment in his first textbook, and so are all other concepts that later came to constitute the main core of "modern" algebra, such as groups and fields. This situation changed slightly in Hensels book of 1913, in which one notices several changes of emphasis and overall approach manifest in the exposition of the theory. In particular, unlike the 1908 textbook, this one introduces from the beginning, albeit in a somewhat ad-hoc fashion, many of the new concepts that were then being adopted as part of the new standard approach to the theory of algebraic fields. Rings, for instance, are defined in the closing section of the first chapter of Hensel's 1913 textbook. The term is reportedly taken from Hilbert, who had used it in the *Zahlbericht*, and it denotes a field of numbers (not an abstract one) in which division is not always warranted. Some basic facts about rings are also proven, and the procedure for building a field of quotients out of two rings is also discussed. These rings defined in Hensel's textbook are, in fact, integral domains, namely, domains in which non-zero elements cannot have zero product.

The slight differences between Hensel's two books reflect overall, basic changes that were affecting algebra and number-theory at that time. However, it seems unlikely that Hensel would have adopted them as part of the presentation of his theory were it not for the fact that in

[7]See [10].

writing his second textbook he received considerable assistance from his young student Fraenkel — an assistance that Hensel explicitly acknowledged in the introduction. Fraenkel was then involved in a study of Hensel's system of p-adic numbers, along the lines of the American school of postulational analysis. This study, which he published in 1912 and which will be discussed below, led him to see the theory from a rather abstract perspective that was alien to Hensel's much more classical view. Thus, Fraenkel's participation in the book markedly influenced the approach followed in it, and brought the theory to a closer relation with the new trends in algebraic research and somewhat away from the original framework within which Hensel had originally conceived it.

4. Abstract Theory of Fields

A major milestone on the way to the rise of the structural approach to algebra was the publication in 1910 of Ernst Steinitz's work on the theory of abstract fields: *Algebraische Theorie der Körper*. In this work, Steinitz (1871-1928) presented an exhaustive account of the results of the theory to that date and, at the same time, opened new avenues for research on any abstractly formulated algebraic concept. Fraenkel's definition of abstract rings provides a classical example of the works that were published under the direct influence of Steinitz's theory.

Steinitz's article articulated a new program of research which he fully applied for the first time to a specific subject-matter (abstract fields), but which could also be extended, and in fact was later adopted, for algebra at large. Steinitz opened his article by explicitly stating his methodological outlook: he announced what might be called today a "structural research program", albeit only for a particular algebraic domain, *i.e.*, for the study of fields. His subject matter would be abstract fields as they were defined by Weber in his article of 1893. However, his research would diverge from Weber's in an important sense. In Steinitz's own words:

> Whereas Weber's aim was a general treatment of Galois theory, independent of the numerical meaning of the elements, for us it is the concept of field which represents the focus of interest... The aim of the present work is to advance an overview of all the possible types of fields and to establish the basic elements of their interrelations. [38, p. 5]

Steinitz also explained in detail the steps to be followed in order to attain this aim. First, it is necessary to consider the simplest possible fields. Then, one must study the methods through which from a given field, new ones can be obtained by extension. One must then find out which properties are preserved when passing from the simpler fields to their extensions.

Incidentally, the main direct source of inspiration for Steinitz's interest in the study of abstract fields was Hensel's work on the theory of numbers. In the introduction to his 1910 article Steinitz wrote:

> I was led into this general research especially by Hensel's *Theory of Algebraic Numbers*, whose starting point is the field of p-adic numbers, a field which counts neither as the field of functions nor as the field of numbers in the usual sense of the word. [*ibid.*]

This testimony stresses the significant fact that, in spite of the existence of abstract formulations of the concept, the only fields considered by algebraists before Steinitz were particular fields of numbers or fields of functions. These were studied as part of the theories from which they arose, rather than as new mathematical entities worthy of independent research. Steinitz was certainly well-acquainted with Dedekind's and Hilbert's works, but he did not learn from them that there is a special mathematical interest in the abstract treatments of fields; this he learnt only form Hensel's work. Steinitz was a personal friend of Hensel, and the two had close professional contacts. Steinitz had the opportunity to discuss at length Hensel's theory of p-adic numbers directly with its creator, and this work confronted him with a completely new instance of field, one which was neither the typical field of numbers nor the typical field of functions.

In particular, a central concept, whose importance Steinitz claimed to have realized while studying Hensel's theory, was the characteristic of a field. In his 1893 definition of an abstract field, for instance, Weber had not envisaged the possibility of considering fields of characteristic other than zero. Steinitz showed that any given field contains a "prime field", which, according to the characteristic of the original field, is isomorphic either to the field of rational numbers or to the quotient field of the integers modulo p (p prime). Then, after thoroughly studying the properties of these prime fields, Steinitz proceeded to classify all possible extensions of a given field and to analyze which properties are transferred from any field to its various possible extensions. Since every field contains a prime field, by studying prime fields, and the way properties are passed over to extensions, Steinitz would attain a full picture of the structure of all possible fields.

5. Alfred Loewy and Postulational Analysis in Germany

A last source of influence on Fraenkel that must be mentioned in relation to his work on rings is found in the works of his uncle, the Jewish mathematician Alfred Loewy (1873-1935). Loewy exerted a decisive influence in shaping Fraenkels early academic career. For one, it was Loewy who induced Fraenkel to travel to Marburg to study under Hensel. Moreover, Fraenkel derived his early interest in the study of axiomatic systems from his uncle. Very much as he helped Hensel in writing the 1913 textbook, Fraenkel also assisted his uncle in the preparation of the latter's book on the foundations of arithmetic, at the time when he was involved in his own study of the postulates for p-adic numbers. Loewy's *Lehrbuch der Algebra*, published in 1915, did not reach a wide audience, but it was among the first to introduce in Germany the methodology, the terminology and the achievements of postulational analysis as practiced in the USA. Loewy — who had published several articles in the *Transactions* during the first decade of the century — was well-acquainted with the aims and methods of this trend.

The combination of Fraenkel's interest in postulational analysis (derived from Loewy's influence), the influence of Steinitz's work, and the direct contact with Hensel and his theory of p-adic numbers, all these provide the raw materials from which his early works on abstract ring theory were derived.

6. Fraenkel's Axioms for p-adic Systems

Several months after his arrival in Marburg in 1912, Abraham Fraenkel wrote the first article which brought him some recognition: an axiomatic foundation for Hensel's system of p-adic numbers [11]. Following the conception behind Hilbert's *Grundlagen der Geometrie* and behind those works of the American postulationalists which were intended to provide minimal systems of independent postulates for the known, concrete mathematical entities — such as the real and natural numbers, the continuum, etc. — Fraenkel took another known entity which had not been considered thus far — the system of p-adic numbers — and provided a suitable system of independent postulates for it. The influence of this article on the further development of the ideas initially introduced by Hensel was rather marginal. Later in his life, Fraenkel himself claimed that the importance then accorded to this work was far greater than it actually deserved [15, p. 111]. But on the other hand, it was this axiomatic study of Hensel's systems of g-adic numbers that led directly to Fraenkel's definition of abstract rings.

Fraenkel's 1912 system of postulates was meant to characterize a "concrete" mathematical entity, rather than any abstract concept. In order to account for all the basic properties of that entity it was necessary to introduce three separate sub-systems of postulates. The first of them defines the order-type (*Ordnungstypus*) of the p-adic numbers. The second sub-system establishes that, from the point of view of the arithmetic defined on them, the p-adic numbers constitute a field. The third system of postulates is needed for rendering the system of p-adic numbers a categorical one.

Consider a system S of abstract elements, each of which is contained in one out of a collection of pairwise disjoint classes C_i. An order-type is defined on this system by means of two binary relations: for any two elements of the system a,b one can say either that a is "smaller" than b (denoted $a < b$) or that a is "lower" than b (denoted $a \ll b$). These two relations are defined through the postulate systems Γ and Λ respectively.

The system Λ, of seven independent axioms, postulates that the collection of classes C_i is totally ordered, and that it contains a single smallest class C_∞. Moreover, every class C in the collection (except for C_∞) has an immediate predecessor, denoted by $/C$, and also an immediate successor, $C/$. Further, the following generalized principle of induction is introduced:

> $\mathbf{\Gamma_\gamma}$. If a system S of classes defined as above satisfies the following two conditions:
> **I :** It contains an additional class beyond C_∞,
> **II:** If it contains the class C then it also contains the classes $/C$ and $C/$,
> then the system in question contains all the classes of S.

Fraenkel also indicated how this postulate enables the definition of the operations of addition and product on S.

Fraenkel summarized the discussion on the system Γ by claiming that, if one takes "$>$" as the basic order relation, the system defines an order of the type $^*\omega + \omega + 1$, which is equivalent to the order-type of the rational integers, with an additional element placed at the end. This justifies denoting the classes with indexes C_i, and asserting that $C_i > C_j$, whenever $i > j$ with respect to the usual order of the rational integers.

The second binary relation, "$\ll$", defines an internal order within each of the classes of S, based on the nine independent postulates of the system Λ. This system establishes that each class C is totally ordered by $\ll$, while the class C_∞ contains a single element. Further, within each class C there exists a special subset N, which is countably infinite and everywhere dense in C with respect to $\ll$, and each of whose elements (except probably the lowest one) has an immediate predecessor belonging to $C - N$. A further postulate establishes that each class C is continuous in the sense of Dedekind (*i.e.*, using cuts). Finally, the system postulates the existence of both a (unique) lowest and a (unique) highest element in each class of the system (except for C_∞).

Thus the relation "$<$" orders the classes of S with respect to each other, while the relation "$\ll$" orders the elements within each given class. In order to complete the definition of the order-type of S it is also necessary to consider two elements α and β belonging to two different classes C_m and C_n. In such case one says that $\alpha < \beta$ if and only if $C_m < C_n$. If α and β belong to the same class C, one says that α is equivalent to β ($\alpha \sim \beta$); this is clearly an equivalence relation.

Among several examples of systems satisfying the above two collections of postulates, Fraenkel discussed in some detail the domain of (reduced) rational p-adic numbers, with p a fixed prime integer. Consider the system of expressions

$$\sum_{n}^{\infty} a_i\, p^i \quad (a_i = 0, 1, \ldots, p-1)\ .$$

Call the index of the first non-zero coefficient of an expression its "order"; then, each class C_i is formed by all the numbers having the same order i. Two numbers belonging to the same class are considered as equivalent, since they behave similarly with respect to division by p. Given two non-equivalent numbers α, β one says that $\alpha < \beta$, whenever the order of the former is lower (in the usual sense) than that of the latter. Clearly the classes C_i are ordered according to their respective indexes. C_∞ contains only the zero element, whose order is ∞. Likewise, C_0 is the class of the units, whose order is zero. Within a given class C_i, two numbers are ordered as follows. Given any two elements,

$$\alpha = \sum_{n}^{\infty} a_i\, p^i \quad \text{and} \quad \beta = \sum_{n}^{\infty} b_i\, p^i$$

then $\alpha \ll \beta$ if $a_n = b_n, a_{n+1} = b_{n+1}, \ldots, a_{n+m} = b_{n+m}$, but $a_{n+m+1} < b_{n+m+1}$ (in the usual sense).

The set N stipulated by the system Λ is the infinite set of those p-adic numbers whose expression breaks off after a finite number of factors. This set is clearly an ordered subset of the natural numbers, and it is therefore countable. Moreover, given a number in that set, $a_n p^n + a_{n+1} p^{n+1} + \ldots + a_r p^r$, then its immediate predecessor (with respect to $\ll$) is

$$a_n p^n + a_{n+1} p^{n+1} + \ldots + a_{r-1} p^{r-1} + (a_r - 1) p^r + (p-1) p^{r+1} + (p-1) p^{r+2} + \ldots$$

which is in itself not a member of N. Also, the lowest element of the class C_n is p^n, while the highest is $(p-1)p^n + (p-1)p^{n+1} + \ldots$ Fraenkel thus concluded that each class of equivalent p-adic numbers constitutes a perfect, nowhere dense set with an initial and final element, which therefore has the power of the continuum.

So much for the order-type of the system S. Fraenkel defined now two operations by postulating a system Π of twelve independent axioms. The first nine of them involve the standard requirements for fields: Fraenkel took them from Dickson's definition of field, with some slight changes [7]. Given the aims of postulational analysis as practiced by Dickson, this system is neither the earliest, nor the most illuminating, nor the clearest one for fields; it is just the most logically-non-redundant one.

The last three axioms of the system Π refer to properties involving both the two operations and the above defined order-properties of S. Thus, axiom Π_{10} may be formulated as follows:

> Let ε be a unity with respect to multiplication, and assume ε belongs to the class C_n. If $k\varepsilon$ denotes the sum $\varepsilon + \varepsilon + \ldots + \varepsilon$ (k times), then, in the sequence $2\varepsilon, 3\varepsilon, 4\varepsilon, \ldots$, there exits a first multiple of ε, $p\varepsilon$, such that $p\varepsilon$ belongs to C_{n-1}. p is called the ground number (*Grundzahl*) of the system, and it can be proven to be unique, since in fact ε can be proven to be the only unity of the system S.

Axiom Π_{11} expresses a similar property:

> Let α be any non-zero element of S and let ε be another element of S, belonging to the class C_n, and satisfying the property that no number of the series $\varepsilon, 2\varepsilon, 3\varepsilon, \ldots, (p-1)\varepsilon$ belongs to C_{n-1}. Then from $\beta < \varepsilon$ it follows that $\alpha.\beta < \alpha.\varepsilon$.

Finally axiom Π_{12} states that if α and β are any two elements of S such that $\beta < \alpha$, then $\alpha + \beta \sim \alpha$.

Fraenkel derived some immediate consequences of these axioms. Thus, for any non-zero element α of S,

$$p\alpha = \alpha+\alpha+\ldots+\alpha = \alpha.\varepsilon+\alpha.\varepsilon+\ldots+\alpha.\varepsilon = \alpha.(\varepsilon+\varepsilon+\ldots+\varepsilon) = \alpha.p\varepsilon$$

but $p\varepsilon < \varepsilon$, hence by Π_{11}: $p\alpha < \alpha$.

Now let m be the smallest integer for which $m\alpha < \alpha$. Suppose p is not a multiple of m, i.e., $p = mr + n$, with $0 < n < m$. In that case $n\alpha \gtrsim \alpha > m\alpha$. Hence, by Π_{12}, $(n + m)\alpha = n\alpha + m\alpha \gtrsim \alpha$, $(n+2m)\alpha \gtrsim \alpha$, $\ldots$, $(n+rm)\alpha = p\alpha \gtrsim \alpha$, which contradicts the above result. It follows that m is necessarily a divisor of p.

Notice that in the systems Γ and Λ there is no mention whatsoever of the ground number p. This is also the case for the first nine axioms of the system Π. The ground number appears for the first time in axiom Π_{10}. Thus it is only at this point that a difference can be established between systems with prime ground number p, or p-adic systems, and those with composite ground number g, or g-adic systems. In other words, it is only by means of the last three axioms of system Π that one can establish the difference between systems containing divisors of zero and systems which do not contain such divisors. An important basic result based on that differentiation is the following:

> If α is a non-null element of a p-adic system, then $p\alpha < \alpha$. In particular, $p\alpha$ is the first element of the sequence of multiples $2\alpha, 3\alpha, \ldots$ which is smaller than α. In a g-adic system, if $m\alpha$ is the first multiple of the sequence which is smaller than α, then α is a divisor of g (probably g itself).

Fraenkel thus considered that the differences between the p-adic and the g-adic systems could be reduced to this result. This result, moreover, makes manifest the much simpler structure of the former systems when compared to the latter, but at the same time it suggests that there is room for separate research of g-adic systems. Fraenkel's definition of abstract rings arises when this line of research is effectively pursued.

It is thus interesting to notice how the subtle differences in approach between Hensel and Fraenkel opened a new line of research. Hensel's research had been directly motivated by number-theoretic concerns; therefore, the mere existence of zero-divisors in g-adic systems was a limitation that discouraged research of such systems. Fraenkel's postulational analysis of Hensel's system, on the other hand, suggested to him the convenience of pursuing what for his teacher was a limitation.

The model followed by Fraenkel in his exploration of g-adic systems was the one put forward by Steinitz in his research of abstract fields.

This became the main issue of his doctoral dissertation which appeared in print in 1914; it introduced for the first time the axiomatic definition of a ring and discussed systematically the basic properties of this mathematical entity. In his 1912 paper, Fraenkel had taken a "concrete" mathematical entity — the system of p-adic numbers — and sought to characterize it in minimal axiomatic terms. The system of numbers was here the focus of interest, while the axioms were just the means to improve his understanding of the former. In this sense, Fraenkel had pursued a task very close to Hilbert's own axiomatic concerns, although in a domain originally not envisaged by the latter. The opening pages of Fraenkel's 1914 paper would seem to bring him closer to postulational works of the kind that were published in the USA during the early years of the century. The definition of an "abstract" concept by a system of postulates appears on first sight as the main concern of his paper; the study of the postulates themselves, rather than that of the entity which they define, would seem to attract all of the attention. But in fact, after introducing the system of postulates that define abstract rings, and after applying to this system the standard techniques of postulational analysis, Fraenkel immediately proceeded to study the rings themselves, following the model put forward by Steinitz in his study of abstract fields. In Steinitz's own work, it must be added, there was no "postulational analysis" of the axiom system defining fields.

Fraenkel's definition is somewhat more cumbersome and less general than the one used nowadays for rings; the differences between the two may in most cases be traced back to their roots in Fraenkel's work on g-adic systems. Fraenkel defined rings as systems R on which two abstract operations are postulated: addition and multiplication. The first operation is assumed to satisfy the axioms of a group, and the second one is assumed to be associative and distributive with respect to the addition. Further, R is assumed to contain at least an identity relative to the second operation. Under these assumptions it is possible that R contains divisors of zero; an element which is not a divisor of zero is called a regular element of the ring. Fraenkel added two axioms which do not appear in the standard, modern definition of rings. These are:

- R_8**:** Every regular element must be invertible with respect to multiplication in the ring.
- R_9**:** For any two elements a, b of the ring there exists a regular element $\alpha_{a,b}$ such that $a.b = \alpha_{a,b}.b.a$ and a second regular element $\beta_{a,b}$ such that $a.b = b.a.\beta_{a,b}$.

Notice that axiom R_8 implies that the set of regular elements of the ring constitutes a field. Obviously, then, even the immediate case of the "ring of integers" is not covered by Fraenkel's definition. In fact, axiom R_8 describes a situation typical of the system of g-adic numbers, in which the units constitute a group regarding multiplication. Moreover, a unity in a g-adic system is not divisible by g, and therefore it cannot be a zero-divisor.

Fraenkel's 1914 paper deals mainly with the factorization properties of divisors of zero and with additive decompositions of the elements of the abstract rings in terms of some elementary divisors of zero. A ring is called separable (*zerlegbar*) if whenever three elements a, b, c are given, such that c divides ab, then it is always possible to write c as a product $c = c_1c_2$, where c_1 divides a and c_2 divides b. Fraenkel observed that an exact analog of this condition holds for the rational integers, but it does not always hold for integers in an arbitrary field of algebraic numbers. This divergence had constituted the point of departure for Dedekind's theory of ideals in 1871. Dedekind had attempted to overcome the failure of unique factorization in certain domains of algebraic integers by imbedding the theory of integers in a more general one. Fraenkel, on the contrary, restricted his treatment of factorization problems to separable rings. One should notice that in Fraenkel's treatment of factorization in abstract rings there is not even a clue to the connections between rings, on the one hand, and concepts such as ideals or modules, on the other hand. From the point of view of "modern algebra", modules and ideals are intimately linked, and in fact, both are subordinate to abstract rings. This was neither the case in Dedekind's conception in the late nineteenth century, nor in Fraenkel's pioneering work on abstract rings, as late as 1914. Also, in Hilbert's *Zahlbericht*, ideals appear only in the more restricted context of rings of algebraic integers. It was only after 1920, with the work of Wolfgang Krull and Emmy Noether, that the theory of ideals became organically integrated into the theory of abstract rings.

Rather than providing a framework for studying decomposition of ideals, Fraenkel introduced a special kind of decomposition property, which was not subsequently developed in later research on rings, but which is a direct extension of Steinitz's line of thought into the domain of rings. Given two elements a, c of R, if a divides c and c divides a, then they are called equivalent (*unwesentlich verschieden oder äquivalent*); otherwise they are called essentially different (*wesentlich verschieden*). A zero-divisor is called a prime divisor, whenever it contains no proper divisor, except for regular elements. A ring is called simple (*einfach*) if all of its prime divisors are equivalent to each other. With this terminology, Fraenkel proved a main decomposition theorem for rings which may be formulated as follows:

> If a separable ring R contains n essentially different prime divisors $p_1, p_2, \ldots, p_n$, then there exist exactly n uniquely-determined, simple rings $R_1, R_2, \ldots, R_n$, satisfying the following conditions:
>
> **I:** The simple rings $R_1, \ldots, R_n$ contain only elements of R.
>
> **II:** The intersection of any two of the above n simple rings contains only the zero element.
>
> **III:** The product of two elements of the ring, belonging to two different rings R_i, is always zero.
>
> **IV:** Every element of R may be written in a unique way as a sum of n elements of R, each belonging to one of the n rings R_i.

This theorem implies the possibility of reducing any separable ring into simple rings. Thus, simple rings play in Fraenkel's theory of rings a rôle similar to that played by prime fields in Steinitz's theory. Of course there are important differences between the two concepts (e.g., fields contain only one prime sub-field), but there is also a basic functional similarity between them, namely, both play the rôle of building stones of their respective theories. In fact, as in the case of prime fields in Steinitz's theory, a full structural knowledge of separable rings is attained by establishing the properties of simple rings, and by inquiring how these properties are transmitted through the different kinds of extensions. In this article Fraenkel did not pursue this point further, but he did so in his next two published works [13, 14].

For Fraenkel, the central achievement of his dissertation had been the proof that the algebraic properties of any separable ring may be reduced to the consideration of a finite or an infinite number of "simpler rings", *i.e.*, rings that in essence contain only one prime divisor of zero. In his following works he undertook the task of extending, in the framework of abstract rings, the whole range of questions that Steinitz had worked out for fields. In particular, Fraenkel addressed the task of characterizing all the possible algebraic and transcendental extensions of a given ring. Although Steinitz had dealt with this question for arbitrary fields, Fraenkel — in order to avoid unnecessary complications — considered here only finite and infinite rings of "finite degree" (*i.e.*, separable rings containing only a finite number of essentially different zero-divisors). The study of these rings, Fraenkel thought, was necessary in order to cover "all the arithmetical and algebraic applications of the theory". In fact, he had two specific such applications in mind: the formulation of a Galois theory for separable rings and the determination of all possible types of finite rings.

In his 1916 article — which was an elaboration of the *Habilitationsschrift* he had submitted to the University of Marburg while still mobilized at the war — Fraenkel did not bother anymore to produce a minimal system of independent postulates defining a ring. This time it was more important to provide a workable definition, rather than a logically irredundant one. Thus, the postulates advanced here are very similar to those accepted nowadays, although Fraenkel still demanded commutativity for the product. He also added an axiom according to which R must contain at least one regular element, and establishing that if a is a regular element, and b is any element of R, then there exists at least one element x in R, such that $ax = b$. Fraenkel explained, that postulating the existence of at least one regular element excluded the case of the trivial ring having only zero elements, as well as of other, non-trivial systems, *e.g.*, the system of all classes of congruence modulo g^m, which are divisible by g. As in his earlier version, the ring of integers is not covered by Fraenkel's 1916 definition. In fact, Fraenkel did not even mention the integers in the framework of his theory. On the other hand, this last postulate implies the existence of a neutral element for the product.

This time, Fraenkel formulated more clearly the relations between the various algebraic concepts involved in his theory: the elements of a ring constitute a group with regard to addition, while the regular elements constitute a group with regard to the product. If a ring, as defined by Fraenkel here, contains no zero-divisors, then it is obviously a field. Thus, all the results valid for fields are also valid for rings, except for those depending on the existence of division. Also those results on fields are not valid for rings, which depend on the fact that the product of two factors in a field is zero if and only if one of the factors is zero. Loewy had mentioned in his book the other side of the same coin, namely that in the system of integers, though not in itself a field, the product of any two integers is zero, if and only if one of them is zero [28, p. 26]. Fraenkel was clearly not envisaging the system of integer numbers when he devised his theory of rings. This partially explains why he did not elaborate upon the connection between the theory of rings and the theory of ideals.

The main problem addressed in Fraenkel's publications of 1916 and 1921 is that of the extensions of rings. Fraenkel translated all the concepts that Steinitz had introduced for fields, taking all the precautions necessary for rings. He defined prime rings — similar way to Steinitz's prime fields — and classified all possible kinds of such rings. In order to classify the different kinds of extensions, it is necessary to consider the systems $R(x)$ of rational functions in a single variable x, with coefficients in a ring R. Fraenkel studied those systems, stating that they are, in fact, rings. He then deduced several properties of $R(x)$ that depend on those of R. Typical is the theorem stating that:

> If the ring R is simple, then $R(x)$ is also a simple ring, and the only prime zero-divisor of $R(x)$ is equivalent in $R(x)$ to p, the only prime zero-divisor of R. [13, p. 16]

Likewise, Fraenkel defined an Euclidean algorithm for $R(x)$, which was to be the main tool for studying the ring-extension of a given ring. But also in this context there is a noteworthy gap between the problems addressed by Fraenkel and those that were later to become the main problems of the abstract theory of rings. In fact, the main achievement of Emmy Noether's early work on rings was her unified approach to problems of factorization in algebraic number theory and in the theory of polynomials. Her main source in algebraic number theory was the work of Dedekind, whereas in the theory of polynomials it was Hilberts work, as well as those of Emanuel Lasker (1868-1941) and of Francis Sowerby Macaulay (1862-1937). Lasker proved that any ideal of polynomials may be decomposed into "primary" ideals [27] and, later on, Macaulay proved that this decomposition is essentially unique [29]. He also provided an algorithm for actually performing the decomposition. The kind of problem they dealt with was not even mentioned by Fraenkel.

7. Ideals and Abstract Rings after Fraenkel

The task of realizing the potentialities involved in the idea of ring as a natural framework for dealing with the existing theories of factorization, both in the framework of algebraic number theory and the theory of polynomials, was first undertaken separately by Masazo Sono, Wolfgang Krull and Emmy Noether.

Masazo Sono published several results on rings and ideals of rings, p roving that the theorems on unique factorization of ideals — such as had been proved by Dedekind for ideals of algebraic integers — do not always hold in more general rings [37]. Wolfgang Krull presented a systematic account of the results of the theory of ideals in the framework of the theory of abstract rings [23, 24, 25]. Krull explicitly mentioned Fraenkel's earlier work, and stressed the fact that the latter's definition of ring allowed no more than rings that behave essentially like Z_m.

According to Krull, the limitations inherent in Fraenkel's work were a consequence of not having included ideals in the treatment. Krull also published a very influential monograph containing a systematic exposition of the abstract theory of ideals [26]. This monograph was published after van der Waerden's *Moderne Algebra* (1930) [41], which also dedicated some chapters to the study of ideals in abstract rings. However, the main inspiration for Krull's work was not provided by van der Waerden, but by Steinitz and Emmy Noether. Krull applied the "structural program" of Steinitz to the theory of ideals, which had, on the one hand, a long history going back to Kummer's work and, on the other hand, a recently established re-formulation: the abstract one. This abstract formulation was a contribution of Emmy Noether.

When Noether published in 1921 her first important paper on factorization of ideals in abstract rings [33], the concept was still totally unfamiliar to most contemporary mathematicians. Throughout her paper, the most elementary properties of rings are proved as the need arises. In her second major paper on rings [34], she was able to reconsider the whole issue from a much clearer and mature perspective, in which the central rôle of chain conditions in factorization theorems was rendered plain and clear[8].These issues, that were instrumental in opening the way for the flourishing of modern algebra, did not appear in Fraenkel's work on rings. The latter simply provided one component of the conceptual setting necessary for the standard formulation of the former.

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