The Review of Modern Logic
Volume 9 Numbers 1 & 2 (November 2001–November 2003) [Issue 29], pp. 29–52.

THE FORMAL THEORY OF SYLLOGISMS

MARCEL CRABBÉ

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1. Introduction

By traditional logic is generally meant a whole body of theories that formed the realm of logic before contemporary logic was discovered by Frege. This logic is not strictly speaking Aristotle's theory, though it stemmed from Aristotle. Indeed it took also advantage of contributions by the Stoics and developed to a great extent during the Middle Ages and in modern times. It is certainly not limited to the theory of syllogisms, although this formed its core, at least in education.

There are essentially three ways to consider the relations between the old and the present logics. The first one (exemplified by the few still existing old-fashioned logicians) is to see the two logics as incomparable in the sense that the first one is viewed as essentially non-mathematical, but inseparable from philosophical conceptions about the natures of knowledge, thought, argumentation and language. The second one (instantiated by Russell) is to consider the traditional logic as a trivial part of the larger system constituted by predicate logic or higher order logic. The third one (initiated by Łukasiewicz [4]) is to consider this logic as a system deserving study on its own, like other systems such as modal logic, propositional logic, Boolean algebra, lattice theory, *etc.*

In this paper we adopt this last perspective and we concentrate on the contemporary approach to traditional logic, more specifically to the traditional theory of syllogisms. So we won't refer to this stuff as traditional logic, or worse Aristotelian logic, but we will consider it as an ordinary theory independent of its various traditional motivations. Although this will make our investigation quite disconnected from the traditional concerns, it will however be interesting to keep it loosely connected with the traditional techniques, and, especially, with traditional terminology. In particular, we will not insist on the interpretation of the Aristotelian theory nor worry about its accuracy, as is done in Łukasiewicz [4], Corcoran [2] or Smiley [7].

The traditional field contains a proof theoretic approach and a semantic one. The first one is mainly devoted to the reduction of the validity of the syllogisms to the validity of some of them treated as axioms: Barbara, Celarent, Darii and Ferio. The semantics on the other hand consists in a set of rules motivated in various ways (among which the famous middle term rule) that select the correct syllogisms. The completeness can be established almost empirically, because there are no more than 24 valid forms of syllogisms among the 256 possible forms. This feeling of triviality or at least of "much ado about nothing" that one has when faced with this traditional stuff — combined, we must add, with the general impression that "though this be madness, yet there is a method in it" — will be slightly attenuated here by not imposing exactly two premises in the definition of the notion of syllogism, so that we will have an infinity of valid forms.

Another semantic approach was investigated in a more recent tradition, namely the class or set interpretation, as can be found in the diagram method and of course in the theory of Boolean algebras. This treatment is however not completely adequate, because it proves valid some forms whose traditional validity seems to be questionable. These forms can be divided into two categories: those that derive immediately from a valid reasoning by structural rules, like “No a is b, Every b is c. Therefore No b is a”; and those that don’t result from a valid reasoning in that way, like the logical truth “Every a is a”.

After a proof-theoretic study of the sequents of traditional sentences, the paper consists mainly in two parts. In the first one, we prove the analogue of traditional completeness relative to a semantics formulated in the old terminology. In the second one, we show the completeness relative to a semantics in the present sense. This latter semantics is twofold. Our first treatment uses abstract structures, called Aristotelian algebras, which enable more flexibility than the natural class interpretation and will exclude such by-products as the logical theses. Our second treatment uses class algebras, called Aristotelian families, which will be characterized as special Aristotelian algebras.

2. The language

We suppose that we have a set of constants denoted by the letters a, b, c,.... The sentences of the language are of the form: $\mathsf{A}ab$, $\mathsf{I}ab$, $\mathsf{E}ab$, $\mathsf{O}ab$, which are usually read “Every a is b”, “Some a is b”, “No a is b” and “Some a is not b”.

A *universal sentence* is one of the form $\mathsf{A}ab$ or $\mathsf{E}ab$; a *particular sentence*, one of the form $\mathsf{I}ab$ or $\mathsf{O}ab$; an *affirmative sentence*, one of the form $\mathsf{A}ab$ or $\mathsf{I}ab$; a *negative sentence*, one of the form $\mathsf{E}ab$ or $\mathsf{O}ab$.

Definition of negation (contradiction): $\overline{\mathsf{A}ab} \equiv_{\text{def}} \mathsf{O}ab$, $\overline{\mathsf{I}ab} \equiv_{\text{def}} \mathsf{E}ab$, $\overline{\mathsf{O}ab} \equiv_{\text{def}} \mathsf{A}ab$, $\overline{\mathsf{E}ab} \equiv_{\text{def}} \mathsf{I}ab$. We of course have $\overline{\overline{\varphi}} \equiv \varphi$.

A *multiset* (intuitively “a set allowing repetitions”) Γ is a function from a finite set of sentences to the positive natural numbers. The set of the sentences belonging to a multiset is its domain. Γ, Δ is the multiset defined for the sentences in the domain of Γ or Δ, by $\Gamma, \Delta(\varphi) = \Gamma(\varphi) + \Delta(\varphi)$. φ may be identified with the multiset defined by $\varphi(\varphi) = 1$. Thus $\Gamma, \varphi\,(\varphi) = \Gamma(\varphi) + 1$. It follows, for example, that if φ and ψ are distinct, then φ, ψ, φ, denotes the multiset Γ with domain $\{\varphi, \psi\}$ such that $\Gamma(\varphi) = 2$ and $\Gamma(\psi) = 1$. The multiset MX associated with the set X is the multiset of domain X, defined by $MX(\varphi) = 1$, for $\varphi \in X$. We will, for convenience, identify a set with the associated multiset. If Γ is the multiset $\varphi_1, \ldots, \varphi_n$, then $\overline{\Gamma}$ denotes $\overline{\varphi_1}, \ldots, \overline{\varphi_n}$.

A *sequent* is an ordered pair $\langle \Gamma, \Delta \rangle$ of multisets denoted $\Gamma \vdash \Delta$.

By *inclusion* of multisets we mean inclusion of the respective domains. A sequent $\Gamma' \vdash \Delta'$ is included in $\Gamma \vdash \Delta$ iff Γ' is included in Γ, and Δ' is included in Δ.

3. Proof theory

3.1. The derivation rules.

Initial sequents.

$$\varphi \vdash \varphi$$

Syllogistic rules.

Subalternation

$$\frac{\Gamma \vdash \mathsf{A}ab, \Delta}{\Gamma \vdash \mathsf{I}ab, \Delta}\text{sub}$$

Conversion

$$\frac{\Gamma \vdash \mathsf{I}ab, \Delta}{\Gamma \vdash \mathsf{I}ba, \Delta}\text{conv}$$

Perfect syllogisms

$$\frac{\Gamma_1 \vdash \mathsf{P}ab, \Delta_1 \qquad \Gamma_2 \vdash \mathsf{A}bc, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \mathsf{P}ac, \Delta_1, \Delta_2}$$

where P is A or I. This rule encompasses both Barbara and Darii[1]:

$$\frac{\Gamma_1 \vdash \mathsf{A}ab, \Delta_1 \qquad \Gamma_2 \vdash \mathsf{A}bc, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \mathsf{A}ac, \Delta_1, \Delta_2}\text{Barbara}$$

$$\frac{\Gamma_1 \vdash \mathsf{I}ab, \Delta_1 \qquad \Gamma_2 \vdash \mathsf{A}bc, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \mathsf{I}ac, \Delta_1, \Delta_2}\text{Darii.}$$

[1]Let us record that in a medieval name of a syllogism, like Barbara, Darii, Celarent, Ferio..., the three vowels indicate, in their order, whether the major premiss (the premiss containing the predicate of the conclusion), the minor premiss (the premiss containing the subject of the conclusion), and the conclusion is an A, E, I or O. In order to determine the form from a mood (*EIO* of Ferio, say) one needs to know the figure as well (in this case the first: *i.e.*, the terms of the conclusion are in the same position in their respective premisses).

In a name like Fesapo, denoting syllogisms in the fourth figure (no term of the conclusion is in the same position as in its premiss), the consonant F means that its validity can be reduced to that of the syllogism (Barbara, Celarent, Darii or Ferio) in the first figure beginning with the same letter, *i.e.*, in this case Ferio. The consonant 's' means that one can do this reduction by performing a simple conversion (*Simpliciter*) on the premiss E, and the consonant 'p' an accidental (*Per accidens*) conversion on the premiss A. The consonants 'm' and 'c' refer to a perMutation of the order of the premisses and a reduction by Contradiction.

Let us finally note that in the second and third figure, it is only the predicate and the subject of the conclusion, respectively, that don't occupy the same position in their premiss.

Thus the syllogistic rules have the general form:

$$\frac{\Gamma_1 \vdash \varphi, \Delta_1 \qquad [\Gamma_2 \vdash \psi, \Delta_2]}{\Gamma_1[, \Gamma_2] \vdash \chi, \Delta_1[, \Delta_2]}$$

Contradiction (negation) rules.

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \overline{\varphi} \vdash \Delta}\,\mathrm{neg}_L \qquad\qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \overline{\varphi}, \Delta}\,\mathrm{neg}_R$$

Structural rules.

Weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\mathrm{weak}_L \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\,\mathrm{weak}_R$$

Contraction

$$\frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,\mathrm{contr}_L \qquad\qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\,\mathrm{contr}_R$$

Cut

$$\frac{\Gamma_1 \vdash \varphi, \Delta_1 \qquad \Gamma_2, \varphi \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\mathrm{cut}$$

A derivation not using the rule of subalternation is termed an *F-derivation* (“F” for “free”).

An *affirmative sequent* is a sequent containing affirmative sentences only.

Let Γ_+, Δ_+ be a multiset of affirmative sentences, and Γ_-, Δ_- be a multiset of negative sentences. Then the *affirmative transform* of the sequent $\Gamma_+, \Gamma_- \vdash \Delta_+, \Delta_-$ is the sequent $\Gamma_+, \overline{\Delta_-} \vdash \Delta_+, \overline{\Gamma_-}$.

A *derivation* is called *affirmative* if it is made up of affirmative initial sequents and syllogistic rules only. One sees that every sequent in an affirmative derivation is affirmative, and has only one sentence on the right side.

A *derivation* is called *strict* if it consists in an affirmative derivation followed by a sequence of contradiction rules.

3.2. **Normalization of derivations.**

We show now that the derivations can be brought into a very strong canonical form. First, let us note that the **weakening** rules can be put at the bottom of the derivations:

Lemma 1. *Every derivation [or F-derivation] of a sequent $\Gamma \vdash \Delta$ can be transformed into a derivation [or F-derivation] of a sequent $\Gamma' \vdash \Delta'$, not using the weakening rules, such that $\Gamma \vdash \Delta$ results from $\Gamma' \vdash \Delta'$ by using weakenings only.*

PROOF
This almost trivial fact is proved by induction on the derivation.

Suppose for example that the derivation ends in a syllogistic rule:

$$\frac{\Gamma_1 \vdash \varphi_1, \Delta_1 \qquad [\Gamma_2 \vdash \varphi_2, \Delta_2]}{\Gamma_1[, \Gamma_2] \vdash \varphi, \Delta_1[, \Delta_2]}$$

The result is clear, if the inductive hypothesis provides a required derivation of $\Gamma_1' \vdash \Delta_1'$ or of $\Gamma_2' \vdash \Delta_2'$.

If there are required derivations of $\Gamma_1' \vdash \varphi_1, \Delta_1'$ [and of $\Gamma_2' \vdash \varphi_2, \Delta_2'$], then by the same rule there is a required derivation of $\Gamma_1'[, \Gamma_2'] \vdash \varphi, \Delta_1'[, \Delta_2']$.

The other cases are handled similarly, and the weakening rules are of course skipped. ∎

The next lemma will entail the analogue result for **contradiction** rules:

Lemma 2. *A derivation remains a derivation if each sequent in it is replaced by its affirmative transform, and if the contradiction rules are skipped.*

PROOF
Observing that the affirmative transform of the premiss of a contradiction rule is the same sequent as the affirmative transform of its conclusion, the inductive proof follows by simple inspection of the rules.

∎

Now we show how, under some conditions, to push the **contractions** downwards:

Lemma 3. *Every derivation [or F-derivation] without weakenings or contradictions can be transformed into a derivation [or F-derivation] of a sequent $\Gamma' \vdash \varphi$, not using weakenings, contradictions nor contractions, such that the initial conclusion results from $\Gamma' \vdash \varphi$ by applying left contractions only.*

PROOF
The only non trivial case of the inductive proof is the case where the derivation ends with a cut rule:

$$\frac{\Gamma_1 \vdash \psi \qquad \Gamma_2, \psi \vdash \varphi}{\Gamma_1, \Gamma_2 \vdash \varphi}$$

The inductive hypothesis provides derivations of $\Gamma_1^* \vdash \psi$ and $\Gamma_2^*, \psi, \ldots, \psi \vdash \varphi$. Applying repeatedly the cut rule, we obtain a derivation of $\Gamma_2^*, \Gamma_1^*, \ldots, \Gamma_1^* \vdash \varphi$, that can be (left) contracted in $\Gamma_2, \Gamma_1 \vdash \varphi$. ∎

A preliminary normal form is provided by

Lemma 4. *Every derivation [or F-derivation] can be transformed in a derivation [or F-derivation] of the same sequent, consisting of a strict derivation [or strict F-derivation], followed by a sequence of contractions and a sequence of weakenings.*

Proof

We can suppose by lemmas 1, 2 and 3, that the derivation is made up of a part produced by initial sequents, syllogistic rules and cut rules, followed by contraction, contradiction and weakening rules.

It will thus suffice to show that one can commute the contractions and the contradictions. The most complex case is illustrated as follows:

$$\cfrac{\cfrac{\vdots}{\varphi_1, \ldots, \varphi_1, \ldots, \varphi_k, \ldots, \varphi_k, \ldots, \varphi_n, \ldots, \varphi_n \vdash \varphi}}{\cfrac{\varphi_1, \ldots, \varphi_k, \ldots, \varphi_n \vdash \varphi}{\varphi_1, \ldots, \varphi_k, \overline{\varphi} \vdash \overline{\varphi_{k+1}}, \ldots, \overline{\varphi_n}}\ \text{contradictions}}\ \text{left contractions}$$

becomes

$$\cfrac{\cfrac{\vdots}{\varphi_1, \ldots, \varphi_1, \ldots, \varphi_k, \ldots, \varphi_k, \ldots, \varphi_n, \ldots, \varphi_n \vdash \varphi}}{\cfrac{\varphi_1, \ldots, \varphi_1, \ldots, \varphi_k, \ldots, \varphi_k, \overline{\varphi} \vdash \overline{\varphi_{k+1}}, \ldots, \overline{\varphi_{k+1}}, \ldots, \overline{\varphi_n}, \ldots, \overline{\varphi_n}}{\varphi_1, \ldots, \varphi_k, \overline{\varphi} \vdash \overline{\varphi_{k+1}}, \ldots, \overline{\varphi_n}}\ \text{contractions}}\ \text{contradictions}$$

∎

We end this sequence of lemmas with a **cut elimination** result:

Lemma 5. *Every affirmative derivation [or F-derivation] can be transformed into a cut-free affirmative derivation [or F-derivation] of the same sequent.*

Proof

The main induction is on the number of cuts. We select an uppermost cut:

$$\cfrac{\begin{matrix}\Sigma \\ \\ \Gamma_1 \vdash \varphi\end{matrix} \qquad \begin{matrix}\Delta \\ \\ \Gamma_2, \varphi \vdash \psi\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \psi}\ \text{cut}$$

i.e., Σ and Δ are cut-free. We show how to remove this cut by a secondary induction on Δ.

$\begin{array}{c}\Delta\\ \Gamma_2,\varphi\vdash\psi\end{array}$ is the initial sequent $\varphi \vdash \varphi$. $\begin{array}{c}\Sigma\\ \Gamma_1\vdash\varphi\end{array}$ is a required derivation.

$\begin{array}{c}\Delta\\ \Gamma_2,\varphi\vdash\psi\end{array}$ ends in a one premiss rule: $\dfrac{\begin{array}{c}\Delta'\\ \Gamma_2,\varphi\vdash\psi'\end{array}}{\Gamma_2,\varphi\vdash\psi}$. By the inductive hypothesis, we have a cut-free affirmative derivation of $\Gamma_1,\Gamma_2\vdash\psi'$, to which the same rule can be applied.

$\begin{array}{c}\Delta\\ \Gamma_2,\varphi\vdash\psi\end{array}$ ends in a Barbara or Darii rule: $\dfrac{\begin{array}{c}\Delta'\\ \Gamma_2',\varphi\vdash\psi'\end{array}\quad\begin{array}{c}\Delta''\\ \Gamma_2''\vdash\psi''\end{array}}{\Gamma_2,\varphi\vdash\psi}$. By the inductive hypothesis, we have a cut-free affirmative derivation of $\Gamma_1,\Gamma_2'\vdash\psi'$, to which the same rule applies with the same second premiss. And similarly if φ is in the second premiss.

∎

Putting lemmas 4 and 5 together, we obtain our normal form result:

Proposition 6. *Every derivation [or F-derivation] can be transformed into a cut-free derivation of the same sequent, composed of a strict derivation [or strict F-derivation] followed by a sequence of contractions and a sequence of weakenings.*

Remarks.

(1) This normal form divides a derivation into two parts: a strict one, representing a traditional proof; and a structural one composed of contractions and weakenings, usually not allowed in the traditional frame.
(2) It is not difficult to see that a sequent $\Gamma'\vdash\Delta'$ is included in $\Gamma\vdash\Delta$ iff $\Gamma\vdash\Delta$ can be obtained from $\Gamma'\vdash\Delta'$ by using a sequence of contraction rules, followed by a sequence of weakening rules.

4. Traditional semantics: the notion of correct syllogism

Apart from the informal mode of speech, we will avoid the traditional usual notions of form and of instantiation, by distinguishing constants and terms. A *term* is a set of occurrences of a constant. An element of a term will be called an occurrence of this term. Normally no confusion will arise when a term is denoted by the same letter as the constant of which it contains the occurrences.

Definition of quantity.

An occurrence of a term in a sentence is *universal* iff the sentence is universal and this occurrence is the first one (the subject position), or the sentence is negative and the occurrence is the second one (the predicate position) *i.e.*, a is universal in $\mathsf{A}ab$, $\mathsf{E}ab$, $\mathsf{E}ba$, and $\mathsf{O}ba$, but not in $\mathsf{A}ba$, $\mathsf{I}ab$, $\mathsf{I}ba$, or $\mathsf{O}ab$. This corresponds to the fact that the term would appear in a negative part of a formula, when the sentence is canonically translated in the predicate calculus.

An occurrence is *particular* iff it is not universal.

Observe that this *quantity* of the indicated occurrence of a and b in $\varphi(a,b)$ is universal iff it is particular in $\overline{\varphi(a,b)}$.

A term is *taken universally in* Γ iff one of its occurrences is universal in a sentence of Γ. A term is *taken universally in* $\Gamma \vdash \Delta$ iff it is taken universally in $\Gamma, \overline{\Delta}$, *i.e.*, iff it is taken universally in Γ, or particularly in Δ. A term is *taken particularly* iff it is not taken universally.

Definition of syllogism.

A *cycle* of the constants $c_1, \ldots, c_n$ $(n > 1)$ is a multiset of the form $\varphi_1(c_1, c_2)$, …, $\varphi_i(c_i, c_{i+1})$, …, $\varphi_n(c_n, c_1)$. Thus a cycle contains at least two sentences. The terms of the cycle are the indicated occurrences of the constants c_i in $\varphi_{i-1}(c_{i-1}, c_i)$ and $\varphi_i(c_i, c_{i+1})$, in case $1 < i \leq n$, and the two indicated occurrences of c_1 in $\varphi_1(c_1, c_2)$ and $\varphi_n(c_n, c_1)$. Thus a cycle of the constants $c_1, \ldots, c_n$ has n terms, each having exactly two occurrences in the cycle.

A *syllogism* is a sequent $\Gamma \vdash \Delta$ such that Γ, Δ is a cycle.

A syllogism $\Gamma \vdash \Delta$ is *correct* [or *freely correct*] iff each term is taken universally at least [or exactly] once in $\Gamma \vdash \Delta$, and exactly one term in predicate position is taken universally in $\Gamma \vdash \Delta$.

Clearly, a syllogism $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is correct [or freely correct] iff $\Gamma, \overline{\Delta}' \vdash \Delta, \overline{\Gamma}'$ is correct [or freely correct].

4.1. **Special cases: antilogism and reasoning.**

1. An *antilogism*[2] is a syllogism of the form $\Gamma \vdash$. It is clear that an antilogism $\Gamma \vdash$ in the constants $c_1, \dots, c_n$ is correct [or freely correct] iff Γ contains exactly one negative sentence and at least one [or exactly one] occurrence of each c_i $(1 \leq i \leq n)$ is universal.

We also have that a syllogism $\Gamma \vdash \Delta$ is correct [or freely correct] iff the antilogism $\Gamma, \overline{\Delta} \vdash$ is correct [or freely correct].

2. A *syllogistic reasoning* is a syllogism of the form $\Gamma \vdash \varphi$. The sentences in Γ are the *premisses* and φ is the *conclusion*. This is more or less the traditional polysyllogism; the traditional syllogism is a syllogistic reasoning with two premisses; more exactly it is one of the form $\psi, \chi \vdash \varphi$.

4.2. **The correct affirmative syllogisms.**

We will here determine the correct syllogisms, by determining the correct affirmative syllogisms, to which all other correct syllogisms reduce by affirmative transformation.

The former definition of the notion of syllogistic reasoning can be illuminated by introducing the notion of a chain.

A *chain* of the constants $c_1, \dots, c_n$ is a multiset of sentences of the form $\varphi_1(c_1, c_2), \dots, \varphi_i(c_i, c_{i+1}), \dots, \varphi_{n-1}(c_{n-1}, c_n)$. Such a chain is said to connect the extremes c_1 and c_n through the *middle terms* c_i $(1 < i < n)$. A chain is non-empty.

Thus we obtain a chain from a cycle by removing one sentence; and so, a syllogistic reasoning is a sequent $\Gamma \vdash \varphi ab$ such that Γ is a chain connecting a and b.

If $\Gamma \vdash \varphi$ is a correct syllogistic reasoning, then there is exactly one negative sentence in $\Gamma, \overline{\varphi}$; that is, either $\Gamma \vdash \varphi$ is affirmative, or φ is negative and there is exactly one negative sentence in Γ. Therefore, unfolding the above definitions of correctness, a syllogistic reasoning is correct [or freely correct] iff the following traditional conditions are satisfied:

− middle term rule: each middle term is taken universally at least [or exactly] once (see [3] for a connection with the interpolation theorem),

− latius hos: the quantity of a term in the conclusion is not greater[3]

[2]This notion was due to Ladd-Franklin, and sometimes used to characterize the valid syllogisms in an abstract way (*e.g.*, in [7]). We will use the notion of affirmative syllogism for the same purpose.

[3]The universal quantity is greater than the particular one.

than [or is the same as] in its premiss, *i.e.*, if a term is universal in φ it is universal in its premiss [or a term is universal in φ iff it is universal in its premiss],
– there is at most one negative premiss,
– if one of the premisses is negative, then the conclusion is negative,
– if all the premisses are affirmative, then the conclusion is affirmative.

It is by now clear that an affirmative correct [or freely correct] syllogism is an affirmative correct [or freely correct] syllogistic reasoning. Therefore a syllogism is correct [or freely correct] iff its affirmative transform is a correct [or freely correct] syllogistic reasoning.

The next definition captures the notion of a set of affirmative premisses allowing a syllogistic conclusion. An affirmative chain is *correct* [or *freely correct*] iff each middle term is universal at least [or exactly] once, that is iff each middle term occurs at least [or exactly] once as subject of an affirmative universal sentence. Thus the affirmative syllogism $\varphi_1(a, c_1), \ldots, \varphi_{n-1}(c_{n-1}, b) \vdash \mathsf{P}ab$ is correct [or freely correct] iff $\varphi_1(a, c_1), \ldots, \varphi_{n-1}(c_{n-1}, b)$ is a correct [or freely correct] chain and $\varphi_1(a, c_1)$ is $\mathsf{A}ac_1$ if [or iff] $\mathsf{P}ab$ is $\mathsf{A}ab$.

In order to determine the correct affirmative syllogisms, we begin by determining the correct chains of affirmative sentences (the "affirmative chains").

A *Barbara-chain* is a chain of the form $\mathsf{A}ac_2, \mathsf{A}c_2c_3, \ldots, \mathsf{A}c_{n-1}b$. It may be denoted by $\mathsf{A}a\_b$. It will be useful to employ the notation $\mathsf{A}a{=}b$ to denote a chain $\mathsf{A}a\_b$ if $a \neq b$, and either a chain $\mathsf{A}a\_b$ or the empty (multi)set, if $a = b$.

Let $\Gamma = \varphi_1(a, c_2), \ldots, \varphi_{n-1}(c_{n-1}, b)$ be a correct [or freely correct] affirmative chain.

If a is universal in $\varphi_1(a, c_2)$ or b universal in $\varphi_{n-1}(c_{n-1}, b)$, then Γ is a Barbara-chain

$$\mathsf{A}ac_2, \ldots, \mathsf{A}c_{n-1}b \quad \text{or} \quad \mathsf{A}c_2a, \ldots, \mathsf{A}bc_{n-1}.$$

If a is particular in $\varphi_1(a, c_2)$, and b particular in $\varphi_{n-1}(c_{n-1}, b)$, then we have two subcases.

First, if Γ contains a particular sentence, $\mathsf{I}c_{i-1}c_i$ say, then Γ is

$$\mathsf{A}c_2a, \ldots, \mathsf{A}c_{i-1}c_{i-2}, \mathsf{I}c_{i-1}c_i, \mathsf{A}c_ic_{i+1}, \ldots, \mathsf{A}c_{n-1}b,$$

which we call a *Darii-chain*, *i.e.*, one of the general form

$$\mathsf{A}c{=}a, \mathsf{I}cd, \mathsf{A}d{=}b.$$

Second, if Γ contains no particular sentence, then the chain has the form:

$$\mathsf{A}c_2a, \ldots, \mathsf{A}c_ic_{i-1}, \mathsf{A}c_ic_{i+1}, \ldots, \mathsf{A}c_{n-1}b,$$

which we call a *Darapti-chain*, *i.e.*, one of the general form

$$\mathsf{A}c\_a, \mathsf{A}c\_b,$$

and it is not freely correct.

Having determined the correct affirmative chains (those multisets of affirmative sentences enabling a syllogistic conclusion), we are in a position to describe the correct and freely correct affirmative syllogisms:

Theorem 7. *An affirmative syllogism is freely correct iff it has one of the following forms:*

$\mathsf{A}a\_b \vdash \mathsf{A}ab$ *(general Barbara);*
$\mathsf{A}c{=}a, \mathsf{I}cd, \mathsf{A}d{=}b \vdash \mathsf{I}ab$ *or* $\mathsf{A}c{=}b, \mathsf{I}cd, \mathsf{A}d{=}a \vdash \mathsf{I}ab$ *(general Darii).*

An affirmative syllogism is correct iff it is freely correct, or has one of the following forms:

$\mathsf{A}a\_b \vdash \mathsf{I}ab$ *or* $\mathsf{A}b\_a \vdash \mathsf{I}ab$ *(general Barbari);*
$\mathsf{A}c\_a, \mathsf{A}c\_b \vdash \mathsf{I}ab$ *(general Darapti).*

We recover the eight correct, and the five freely correct, traditional affirmative syllogisms as particular cases:

Barbara: $\mathsf{A}ac, \mathsf{A}cb \vdash \mathsf{A}ab$;
Barbari and Bramantip: $\mathsf{A}ac, \mathsf{A}cb \vdash \mathsf{I}ab$ and $\mathsf{A}bc, \mathsf{A}ca \vdash \mathsf{I}ab$;
Darapti: $\mathsf{A}ca, \mathsf{A}cb \vdash \mathsf{I}ab$;
Darii, Dimaris, Disamis, and Datisi: $\mathsf{I}ac, \mathsf{A}cb \vdash \mathsf{I}ab$; $\mathsf{I}bc, \mathsf{A}ca \vdash \mathsf{I}ab$; $\mathsf{A}ca, \mathsf{I}cb \vdash \mathsf{I}ab$ and $\mathsf{A}cb, \mathsf{I}ca \vdash \mathsf{I}ab$.

As further particular cases, we obtain the correct affirmative syllogisms without middle terms, known as immediate inferences: $\mathsf{A}ab \vdash \mathsf{A}ab$ (general Barbara); $\mathsf{A}ab \vdash \mathsf{I}ab$ and $\mathsf{A}ba \vdash \mathsf{I}ab$ (general Barbari); $\mathsf{I}ab \vdash \mathsf{I}ab$ and $\mathsf{I}ba \vdash \mathsf{I}ab$ (general Darii), among which the syllogistic forms $\mathsf{A}ab \vdash \mathsf{I}ab$ and $\mathsf{A}ba \vdash \mathsf{I}ab$ are not freely correct.

We end this section by indicating how to make explicit the existential import:

Proposition 8.
1. If $\Gamma \vdash \Delta$ *is a correct syllogism, then, for some* c*,* $\Gamma, \mathsf{I}cc \vdash \Delta$ *is a freely correct syllogism.*
2. If $\Gamma, \mathsf{I}cc \vdash \Delta$ *is a freely correct syllogism and* $\mathsf{O}cc$ *is not in* Γ *and* $\mathsf{I}cc$ *is not in* Δ*, then* $\Gamma \vdash \Delta$ *is a correct syllogism.*

Proof

1. If $\Gamma \vdash \Delta$ is correct but not freely correct, then the affirmative transform of $\Gamma \vdash \Delta$ is a general Darapti $\mathsf{A}c\_a, \mathsf{A}c\_b \vdash \mathsf{I}ab$ in which only the term c is taken twice universally. The conclusion follows from the fact that $\mathsf{A}c\_a, \mathsf{A}c\_b, \mathsf{I}cc \vdash \mathsf{I}ab$ is a general Darii, which is freely correct.

2. If $\Gamma, \mathsf{I}cc \vdash \Delta$ is a freely correct syllogism, then its affirmative transform is a general Darii $\mathsf{A}c{=}a, \mathsf{I}cc, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, *i.e.*, it has one of the forms $\mathsf{A}c\_a, \mathsf{I}cc, \mathsf{A}c\_b \vdash \mathsf{I}ab$, or $\mathsf{A}c\_a, \mathsf{I}cc \vdash \mathsf{I}ac$, or $\mathsf{I}cc, \mathsf{A}c\_b \vdash \mathsf{I}cb$, or $\mathsf{I}cc \vdash \mathsf{I}cc$. The last possibility is excluded by hypothesis. Therefore, the affirmative transform of $\Gamma \vdash \Delta$ is the general Darapti $\mathsf{A}c\_a, \mathsf{A}c\_b \vdash \mathsf{I}ab$, or the general Barbari $\mathsf{A}c\_a \vdash \mathsf{I}ac$ or $\mathsf{A}c\_b \vdash \mathsf{I}bc$. Whence $\Gamma \vdash \Delta$ is correct. ∎

4.3. **Traditional completeness.**

Proposition 9.
1. Every affirmative correct [or freely correct] syllogism is affirmatively derivable [or F-derivable] without cuts.
2. Every correct [or freely correct] syllogism is strictly derivable [or F-derivable] without cuts.

Proof
2. follows from 1. and the fact that if a syllogism is correct [or freely correct], then so is its affirmative transform. In accordance with theorem 7, the following examples will suffice to indicate how to carry out the proof of 1.

General Barbara:

$$\dfrac{\dfrac{\mathsf{A}ac_2 \vdash \mathsf{A}ac_2 \quad \mathsf{A}c_2c_3 \vdash \mathsf{A}c_2c_3}{\mathsf{A}ac_2, \mathsf{A}c_2c_3 \vdash \mathsf{A}ac_3}\,\text{Barbara} \qquad \mathsf{A}c_3c_4 \vdash \mathsf{A}c_3c_4}{\mathsf{A}ac_2, \mathsf{A}c_2c_3, \mathsf{A}c_3c_4 \vdash \mathsf{A}ac_4}\,\text{Barbara}$$

$$\vdots$$

$$\mathsf{A}ac_2, \ldots, \mathsf{A}c_{n-1}b \vdash \mathsf{A}ab$$

General Darii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{I}cd \vdash \mathsf{I}cd}{\mathsf{I}cd \vdash \mathsf{I}dc}\,\text{conv} \qquad \begin{array}{c}\text{General Barbara}\\ \vdots \\ \mathsf{A}c\_a \vdash \mathsf{A}ca\end{array}}{\mathsf{A}c\_a, \mathsf{I}cd \vdash \mathsf{I}da}\,\text{Darii}}{\mathsf{A}c\_a, \mathsf{I}cd \vdash \mathsf{I}ad}\,\text{conv} \qquad \begin{array}{c}\text{General Barbara}\\ \vdots \\ \mathsf{A}d\_b \vdash \mathsf{A}db\end{array}}{\mathsf{A}c\_a, \mathsf{I}cd, \mathsf{A}d\_b \vdash \mathsf{I}ab}\,\text{Darii}$$

General Darapti:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{General Barbara}\\ \vdots \\ \mathsf{A}c\_a \vdash \mathsf{A}ca\end{array}}{\mathsf{A}c\_a \vdash \mathsf{I}ca}\,\text{sub}}{\mathsf{A}c\_a \vdash \mathsf{I}ac}\,\text{conv} \qquad \begin{array}{c}\text{General Barbara}\\ \vdots \\ \mathsf{A}c\_b \vdash \mathsf{A}cb\end{array}}{\mathsf{A}c\_a, \mathsf{A}c\_b \vdash \mathsf{I}ab}\,\text{Darii}$$

∎

Proposition 10.
1. An affirmatively derivable [or F-derivable] sequent is an affirmative correct [or freely correct] syllogistic reasoning.
2. A strictly derivable [or F-derivable] sequent is a correct [or freely correct] syllogism.

Proof
1. By induction on the length of the derivations. We can use the general definition of correctness, or theorem 7. Here, we mix the two methods.

$\varphi \vdash \varphi$ is a freely correct affirmative syllogism, if φ is affirmative.

Subalternation. If $\Gamma \vdash \mathsf{A}ab$ is a correct syllogism, then it is a general Barbara. Hence $\Gamma \vdash \mathsf{I}ab$ is a general Barbari.

Conversion. If $\Gamma \vdash \mathsf{I}ab$ is a correct [or freely correct] syllogism, then it is a general Darii or Darapti [or a general Darii]; and $\Gamma \vdash \mathsf{I}ba$ is of the same kind.

Perfect syllogisms. If $\Gamma_1 \vdash \mathsf{P}ab$ and $\Gamma_2 \vdash \mathsf{A}bc$ are correct [or freely correct] syllogisms, then so is $\Gamma_1, \Gamma_2 \vdash \mathsf{P}ac$, because the mentioned occurrence of the term b is universal in Γ_2 [and particular in Γ_1]; and those of a, c keep their quantity in $\mathsf{P}ac$.

Cut. a [or b] has not the same quantity in $\Gamma_1^* \vdash \mathsf{P}ab$ and in $\Gamma_2^*, \mathsf{P}ab \vdash \psi$. Thus one can show that if $\Gamma_1^* \vdash \mathsf{P}ab$ and $\Gamma_2^*, \mathsf{P}ab \vdash \psi$ are correct [or freely correct] syllogisms, then so is $\Gamma_2^*, \Gamma_1^* \vdash \psi$. But we can also dispense with this step and use lemma 5 instead.

2. follows from 1. and the fact that a syllogism is correct if its affirmative transform is correct. ∎

Proposition 11.
1. A sequent is strictly derivable [or F-derivable] without cuts iff it is a correct [or freely correct] syllogism.
2. $\varphi_1, \ldots, \varphi_n \vdash \psi$ is a correct [or freely correct] syllogistic reasoning iff it is affirmatively derivable [or F-derivable], or, for some i,

$\varphi_1, \dots, \varphi_{i-1}, \overline{\psi}, \varphi_{i+1}, \dots, \varphi_n \vdash \overline{\varphi}_i$ *is affirmatively derivable [or F-derivable].*

Proof

1. The 'if' part is covered by proposition 9.2; and the 'only if' part is a consequence of proposition 10.2.

2. The 'if' part is a particular case of 1. For the 'only if' part, we assume that $\varphi_1, \dots, \varphi_n \vdash \psi$ is a correct syllogism. Then it is strictly derivable, by 1. Therefore it is either affirmatively derivable, or it follows by the contradiction rules from an affirmative syllogism, which can only be of the form $\varphi_1, \dots, \varphi_{i-1}, \overline{\psi}, \varphi_{i+1}, \dots, \varphi_n \vdash \overline{\varphi}_i$. ∎

Corollary 12 (Barbara, Celarent...)**.**
There are exactly 15 forms of freely correct traditional syllogistic reasoning; and exactly 24 forms of correct traditional syllogistic reasoning.

Proof

Indeed, each of the affirmative forms of correct traditional syllogisms give rise to two new forms.

Here are the details and the names indexed with the figure.

From Barbara$_1$: $\mathsf{A}ac, \mathsf{A}cb \vdash \mathsf{A}ab$, we obtain $\mathsf{A}ac, \mathsf{O}ab \vdash \mathsf{O}cb$ (Bocardo$_3$) and $\mathsf{O}ab, \mathsf{A}cb \vdash \mathsf{O}ac$ (Baroco$_2$).

From the (not freely correct) Barbari$_1$ and Bramantip$_4$: $\mathsf{A}ac, \mathsf{A}cb \vdash \mathsf{I}ab$, and $\mathsf{A}bc, \mathsf{A}ca \vdash \mathsf{I}ab$, we obtain $\mathsf{A}ac, \mathsf{E}ab \vdash \mathsf{O}cb$ (Felapton$_3$), $\mathsf{E}ab, \mathsf{A}cb \vdash \mathsf{O}ac$ (Camestrop$_2$); $\mathsf{A}bc, \mathsf{E}ab \vdash \mathsf{O}ca$ (Fesapo$_4$) and $\mathsf{E}ab, \mathsf{A}ca \vdash \mathsf{O}bc$ (Camenop$_4$).

From the (not freely correct) Darapti$_3$: $\mathsf{A}ca, \mathsf{A}cb \vdash \mathsf{I}ab$, we obtain $\mathsf{A}ca, \mathsf{E}ab \vdash \mathsf{O}cb$ (Celaront$_1$) and $\mathsf{E}ab, \mathsf{A}cb \vdash \mathsf{O}ca$ (Cesaro$_2$).

Finally, from Darii$_1$, Dimaris$_4$, Disamis$_3$, and Datisi$_3$: $\mathsf{I}ac, \mathsf{A}cb \vdash \mathsf{I}ab$; $\mathsf{I}bc, \mathsf{A}ca \vdash \mathsf{I}ab$; $\mathsf{A}ca, \mathsf{I}cb \vdash \mathsf{I}ab$ and $\mathsf{A}cb, \mathsf{I}ca \vdash \mathsf{I}ab$, we obtain: $\mathsf{I}ac, \mathsf{E}ab \vdash \mathsf{O}cb$ (Ferison$_3$), $\mathsf{E}ab, \mathsf{A}cb \vdash \mathsf{E}ac$ (Camestres$_2$); $\mathsf{I}bc, \mathsf{E}ab \vdash \mathsf{O}ca$ (Fresison$_4$), $\mathsf{E}ab, \mathsf{A}ca \vdash \mathsf{E}bc$ (Camenes$_4$); $\mathsf{A}ca, \mathsf{E}ab \vdash \mathsf{E}cb$ (Celarent$_1$), $\mathsf{E}ab, \mathsf{I}cb \vdash \mathsf{O}ca$ (Festino$_2$); $\mathsf{A}cb$, $\mathsf{E}ab \vdash \mathsf{E}ca$ (Cesare$_2$) and $\mathsf{E}ab, \mathsf{I}ca \vdash \mathsf{O}cb$ (Ferio$_1$). ∎

Theorem 13. *A sequent is derivable [or F-derivable] iff it includes a correct [or freely correct] syllogism.*

Proof

The 'if' part follows from proposition 11.1. The 'only if' part is a consequence of propositions 6 (and its second remark) and 11.2. ∎

Examples of non-derivable syllogistic sequents obtained by weakening: $\mathsf{A}ab, \mathsf{A}ab \vdash \mathsf{A}ab$; $\mathsf{I}ab, \mathsf{A}ab \vdash \mathsf{I}ba$; $\mathsf{E}ab, \mathsf{I}bc \vdash \mathsf{E}ba$.

Examples showing the role of the contraction rules:

– with the existential presupposition, implied by the subalternation rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{A}ca \vdash \mathsf{A}ca}{\mathsf{A}ca \vdash \mathsf{I}ca}\,\text{sub}}{\mathsf{A}ca \vdash \mathsf{I}ac}\,\text{conv} \qquad \mathsf{A}ca \vdash \mathsf{A}ca}{\mathsf{A}ca, \mathsf{A}ca \vdash \mathsf{I}aa}\,\text{Barbara}}{\mathsf{A}ca \vdash \mathsf{I}aa}\,\text{contr}_L$$

– and without existential import:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{I}cc \vdash \mathsf{I}cc \qquad \mathsf{A}ca \vdash \mathsf{A}ca}{\mathsf{I}cc, \mathsf{A}ca \vdash \mathsf{I}ca}\,\text{Darii}}{\mathsf{I}cc, \mathsf{A}ca \vdash \mathsf{I}ac}\,\text{conv} \qquad \mathsf{A}ca \vdash \mathsf{A}ca}{\mathsf{I}cc, \mathsf{A}ca, \mathsf{A}ca \vdash \mathsf{I}aa}\,\text{Darii}}{\mathsf{I}cc, \mathsf{A}ca \vdash \mathsf{I}aa}\,\text{contr}_L$$

5. Modern semantics

5.1. General interpretation.

A *free Aristotelian algebra* is a triple $\langle A, \prec, \frown \rangle$, where A is a non empty set, $\prec$ is a transitive relation (included in $A \times A$), and $\frown$ is a symmetric relation (included in $A \times A$), such that, for all $a, b, c \in A$, if $a \frown b$ and $b \prec c$ then $a \frown c$. An *Aristotelian algebra* is a free Aristotelian algebra in which $\prec$ is included in $\frown$.

$a \prec b$, $a \frown b$ may be read b *extends* a, and a *is compatible with* b, respectively.

Examples.

> Any non-empty collection of non-empty sets with the subset relation and non-empty overlap relation is an Aristotelian algebra; if an empty set is allowed, it is a free Aristotelian algebra.
>
> Any Boolean algebra with the appropriate relations ($a \cdot b = a$, $a \cdot b \neq 0$) is a free Aristotelian algebra.
>
> If $\prec$ is transitive, we may define a free Aristotelian algebra by letting $a \frown b$ iff $c \prec a$ and $c \prec b$, for some c. In particular, a forcing notion, with its usual compatibility relation, is an Aristotelian algebra. ∎

Let $\mathcal{A}$ be an Aristotelian algebra $\langle A, \prec, \frown \rangle$, and let there be a function from the terms of a language to A, which we denote also by $\mathcal{A}$. We define the truth relation in the natural way as follows: $\mathcal{A} \models \mathsf{A}ab$ iff $\mathcal{A}(a) \prec \mathcal{A}(b)$; $\mathcal{A} \models \mathsf{I}ab$ iff $\mathcal{A}(a) \frown \mathcal{A}(b)$; $\mathcal{A} \models \mathsf{E}ab$ iff not $\mathcal{A}(a) \frown \mathcal{A}(b)$; $\mathcal{A} \models \mathsf{O}ab$ iff not $\mathcal{A}(a) \prec \mathcal{A}(b)$. $\mathcal{A} \models \varphi$ is read "φ is true in $\mathcal{A}$" $\Gamma \vdash \Delta$ is *valid* [or *freely valid*] iff one of the sentences in Δ is true in all Aristotelian algebras [or free Aristotelian algebras] in which all the sentences in Γ are true.

5.1.1. *Generating Aristotelian algebras.* Let $\mathcal{A} = \langle A, \alpha, \iota \rangle$ be a structure with the binary relations α and ι.

We define the relations $\prec$, $\preceq$, $\frown$ and $\frown^A$ as follows[4]. $\prec$ is the transitive closure of α: $a \prec b$ iff $a\alpha b$, or, for some $c_1, \ldots, c_n$ $(n \geq 1)$, $a\alpha c_1, \ldots, c_i \alpha c_{i+1}, \ldots, c_n \alpha b$. $\preceq$ is the transitive and reflexive closure of α: $a \preceq b$ iff $a \prec b$ or $a = b$. $a \frown b$ iff $c \preceq a$ and $d \preceq b$, for some $c, d \in A$ such that $c\iota d$ or $d\iota c$. $a \frown^A b$ iff $a \frown b$, or there is a c in A such that $c \preceq a$ and $c \prec b$, or $c \prec a$ and $c \preceq b$.

$\langle A, \prec, \frown \rangle$ is a free Aristotelian algebra such that $\alpha \subseteq \prec$ and $\iota \subseteq \frown$; and $\langle A, \prec, \frown^A \rangle$ is an Aristotelian algebra such that $\alpha \subseteq \prec$ and $\iota \subseteq \frown^A$. In fact they are the smallest such algebras.

Theorem 14 (Completeness)**.**
1. A sequent is freely valid iff it is F-derivable.
2. A sequent is valid iff it is derivable.

PROOF

Observe first that a sequent is derivable [or F-derivable] iff its affirmative transform is derivable [or F-derivable]; and, that a sequent is valid [or freely valid] iff its affirmative transform is valid [or freely valid]. It will therefore be sufficient to prove the theorem for affirmative sequents.

Since the 'if' parts are straightforward verification, we limit ourselves to the 'only if' directions.

1. Suppose that the affirmative $\Gamma \vdash \Delta$ is not F-derivable. Then, by theorem 13, it doesn't include an affirmative freely correct syllogism.

Let $\mathcal{G}$ be $\langle G, \alpha, \iota \rangle$, where G is the set of constants occurring in Γ, Δ, and $a\alpha b$ iff $\mathsf{A}ab$ is in Γ, $a\iota b$ iff $\mathsf{I}ab$ is in Γ. Let $\mathcal{F}$ be the free Aristotelian algebra $\langle G, \prec, \frown \rangle$ generated by $\mathcal{G}$, and let $\mathcal{F}(a) = a$, for the terms in G.

Clearly, $\mathcal{F} \models \varphi$, for every φ in Γ. It remains to be shown that $\mathcal{F} \not\models \varphi$, for the sentences φ in Δ.

[4]It will not be necessary to use a more pedantic writing of the sort $\prec_{\mathcal{A}}$, $\frown_{\mathcal{A}}$.

If $\mathsf{A}ab$ is in Δ, and $\mathcal{F} \models \mathsf{A}ab$, *i.e.*, $a \prec b$, then, by definition of $\prec$, $\Gamma \vdash \Delta$ would include a freely correct syllogism of the form $\mathsf{A}a\_b \vdash \mathsf{A}ab$. Hence $\mathcal{F} \not\models \mathsf{A}ab$.

If $\mathsf{I}ab$ is in Δ, and $\mathcal{F} \models \mathsf{I}ab$, *i.e.*, $a \frown b$, then, by definition of $\frown$, $\Gamma \vdash \Delta$ would include a freely correct syllogism of the form $\mathsf{A}c{=}a, \mathsf{I}cd, \mathsf{A}d{=}b \vdash \mathsf{I}ab$, or $\mathsf{A}c{=}a, \mathsf{I}dc, \mathsf{A}d{=}b \vdash \mathsf{I}ab$. Hence $\mathcal{F} \not\models \mathsf{I}ab$.

Therefore, as every sentence in Γ is true in $\mathcal{F}$, and every sentence in Δ is false in $\mathcal{F}$, the sequent is not freely valid.

2. Suppose that $\Gamma \vdash \Delta$ is not derivable. Then, by theorem 13, it doesn't include an affirmative correct syllogism. We define $\mathcal{G}$ as in the previous part, and we let $\mathcal{A}$ be the Aristotelian algebra $\langle G, \prec, \frown^A \rangle$ generated by $\mathcal{G}$, with $\mathcal{A}(a) = a$, for the terms in G.

The proof that $\mathcal{A}$ is a model of $\Gamma, \overline{\Delta}$ is the same as in the first part, with the extra case where $a \frown^A b$ because Γ includes a multiset of the form $\mathsf{A}c\_a, \mathsf{A}c{=}b$, or $\mathsf{A}c{=}a, \mathsf{A}c\_b$, entailing that $\Gamma \vdash \Delta$ includes a correct syllogism of the form $\mathsf{A}c\_a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, or $\mathsf{A}c{=}a, \mathsf{A}c\_b \vdash \mathsf{I}ab$. ∎

5.2. **Class interpretations.**

5.2.1. *Aristotelian families.* A *free Aristotelian family* is a non-empty collection of sets with the subset relation and non-empty overlap relation; an *Aristotelian family* is a free Aristotelian family not containing the empty set. As remarked above, Aristotelian families [or free Aristotelian families] are Aristotelian algebras [or free Aristotelian algebras].

$\Gamma \vdash \Delta$ is *valid for Aristotelian families* [or *free Aristotelian families*] iff one of the sentences in Δ is true in all Aristotelian families [or free Aristotelian families] in which all the sentences in Γ are true.

Examples.

> The following invalid sequents are valid for free Aristotelian families: $\vdash \mathsf{A}aa$; $\mathsf{I}ab \vdash \mathsf{I}aa$. $\mathsf{O}ab \vdash \mathsf{I}aa$ is valid for free Aristotelian families, but not for free Aristotelian algebras. $\mathsf{I}aa$ is true in all Aristotelian families, but not in all free Aristotelian families.

5.2.2. *Reflexive algebras.* Let $\mathcal{A} = \langle A, \prec, \frown \rangle$ be a free Aristotelian algebra. Let's call a a *non-empty* element of $\mathcal{A}$ iff, for some b, $a \frown b$ or not $a \prec b$. $\mathcal{A}$ is *reflexive* iff $\prec$ is reflexive and the non-empty elements are $\frown$-reflexive; that is $a \prec a$, for every $a \in A$, and $a \frown a$, if a is a non-empty element of $\mathcal{A}$.

Thus an Aristotelian algebra $\langle A, \prec, \frown \rangle$ is reflexive iff $\prec$ is reflexive.

$\Gamma \vdash \Delta$ is *valid for reflexive Aristotelian algebras* [or *free reflexive Aristotelian algebras*] iff one of the sentences in Δ is true in all reflexive Aristotelian algebras [or free reflexive Aristotelian algebras] in which all the sentences in Γ are true.

We will now state and prove a representation theorem that relates families and reflexive algebras. It will clearly evoke Stone's representation theorem for Boolean algebras. Similar results appear in [1], [2] and [6], and a somewhat different one, but to the same effect, is found in [7].

A *filter* on a free Aristotelian algebra $\langle A, \prec, \frown \rangle$ is a set closed under extension and whose elements are pairwise compatible : X is a filter iff for all $a, b \in A$, such that $a \prec b$ and $a \in X$, we have $b \in X$, and if $a, b \in X$ then $a \frown b$.

$F(a)$ is the set (class) of filters containing a.

Proposition 15 (representation)**.** *If $\langle A, \prec, \frown \rangle$ is a reflexive free Aristotelian algebra, then:*

$$a \prec b \textit{ iff } F(a) \subseteq F(b);$$
$$a \frown b \textit{ iff } F(a) \cap F(b) \neq \emptyset.$$

Proof

If $a \prec b$ and $X \in F(a)$, then $a \in X$, and, because X is a filter, $b \in X$. Therefore $X \in F(b)$.

Let $F(a) \subseteq F(b)$, and suppose $a \not\prec b$. Then a is not empty, and $a \frown a$. Therefore $\{ c \mid a \prec c \}$ is a filter containing a. Hence, $\{ c \mid a \prec c \} \in F(b)$, which implies $a \prec b$. We conclude that $a \prec b$.

If $a \frown b$, then a, b are not empty. Hence, $\{ d \mid a \prec d \text{ or } b \prec d \}$ is a filter belonging to both $F(a)$ and $F(b)$.

Let $F(a) \cap F(b) \neq \emptyset$ and X be a filter in $F(a)$ and $F(b)$. Then, $a, b \in X$ and $a \frown b$. ∎

Though this function F is not necessarily an isomorphism, the proposition 15 suffices to characterize the Aristotelian families as special Aristotelian algebras and situates the class interpretations among the general interpretations. We express this as follows:

Corollary 16. *A sequent is valid for Aristotelian families [or free Aristotelian families] iff it is valid for reflexive Aristotelian algebras [or reflexive free Aristotelian algebras].*

Proof

Every Aristotelian family [or free Aristotelian family] is a reflexive Aristotelian algebra [or reflexive free Aristotelian algebra].

Conversely, if $\langle A, \prec, \frown \rangle$ is a reflexive free Aristotelian algebra, then the set of $F(a)$, for a in A, with the inclusion and non-empty overlap relations is a free Aristotelian family; and if $\langle A, \prec, \frown \rangle$ is a reflexive Aristotelian algebra, then it is an Aristotelian family, because in this case $F(a)$ is non-empty, for every a: $\{ b \mid a \prec b \}$ is a filter containing a. ∎

5.2.3. *Generating reflexive algebras.* Let, as above, $\mathcal{A} = \langle A, \alpha, \iota \rangle$ be a structure with the two binary relations α and ι.

$\frown^{FC}$ and $\frown^{C}$ are defined as follows. $a \frown^{FC} b$ iff $a \frown b$, or $c \preceq a$ and $c \preceq b$, for some $c \in A$ such that, for some $z \in A$, $z\iota c$ or $c\iota z$ or $c \not\preceq z$. $a \frown^{C} b$ iff $a \frown b$ or, for some $c \in A$, $c \preceq a$ and $c \preceq b$.

$\langle A, \preceq, \frown^{C} \rangle$ is a reflexive Aristotelian algebra such that $\alpha \subseteq \preceq$ and $\iota \subseteq \frown^{C}$; and $\langle A, \preceq, \frown^{FC} \rangle$ is a reflexive free Aristotelian algebra such that $\alpha \subseteq \preceq$ and $\iota \subseteq \frown^{FC}$. They are even the smallest such ones.

Theorem 17.
1. A sequent is valid for free Aristotelian families iff it is F-derivable, or its affirmative transform includes a sequent of one of the forms $\vdash \mathsf{A}aa$, *or* $\mathsf{I}cd$ [*or* $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, *or* $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab, \mathsf{A}cd$
2. A sequent is valid for Aristotelian families iff it is derivable, or its affirmative transform includes a sequent of the form $\vdash \mathsf{A}aa$ *or* $\vdash \mathsf{I}aa$.

Proof
As in the proof of theorem 14, we can limit ourselves to the case of affirmative sequents.

The 'if' parts are simple exercises. For the 'only if' parts, we will make use of corollary 16, and thus be confined to reflexive free algebras or reflexive algebras.

1. We suppose that $\Gamma \vdash \Delta$ is not F-derivable and does not include a sequent of the form $\vdash \mathsf{A}aa$, or $\mathsf{I}cd$ [or $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, or $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab, \mathsf{A}cd$. It follows, by theorem 13, that it does not include an affirmative freely correct syllogism either. We show that $\Gamma \vdash \Delta$ is not valid for reflexive free Aristotelian algebras.

Let G be the set of constants appearing in Γ, Δ. Let $a\alpha b$ iff $\mathsf{A}ab \in \Gamma$, or some $\mathsf{A}a{=}c, \mathsf{A}a{=}d \vdash \mathsf{I}cd$ is included in $\Gamma \vdash \Delta$. Let further $a\iota b$ iff $\mathsf{I}ab \in \Gamma$. Finally, let $\mathcal{FC}$ be $\langle G, \preceq, \frown^{FC} \rangle$, the reflexive free Aristotelian algebra generated by $\langle G, \alpha, \iota \rangle$, with $\mathcal{FC}(a) = a$, for $a \in G$.

We note that $\mathcal{FC} \models \varphi$, for φ in Γ. Before we proceed to show that $\mathcal{FC} \not\models \varphi$, for φ in Δ, we observe that:

(∗) if $a \preceq b$, then some $\mathsf{A}a{=}b$ is included in Γ, or some $\mathsf{A}a{=}c$, $\mathsf{A}a{=}d \vdash \mathsf{I}cd$ is included in $\Gamma \vdash \Delta$.

Indeed, if $a \preceq b$, then, $a = b$, or, for some $x_1, \ldots, x_n$, $a\alpha x_1$, $x_1\alpha x_2, \ldots,$ $x_n\alpha b$. Hence, if $a \neq b$ and $\mathsf{A}ax_1, \mathsf{A}x_1x_2, \ldots, \mathsf{A}x_nb$ are not all in Γ, then, for some $y \in \{a, x_1, \ldots, x_n, b\}$, some $\mathsf{A}a{=}y$ is included in Γ, and some $\mathsf{A}y{=}c, \mathsf{A}y{=}d \vdash \mathsf{I}cd$ is included in $\Gamma \vdash \Delta$. Thus, some $\mathsf{A}a{=}c, \mathsf{A}a{=}d \vdash \mathsf{I}cd$ is included in $\Gamma \vdash \Delta$.

Now let us suppose that $\mathsf{A}ab$ is in Δ, and that $\mathcal{FC} \models \mathsf{A}ab$, *i.e.*, $a \preceq b$. Then, by $(*)$, some $\mathsf{A}a{=}b$ is included in Γ, or some $\mathsf{A}a{=}c, \mathsf{A}a{=}d \vdash \mathsf{I}cd$ is included in $\Gamma \vdash \Delta$. The second possibility contradicts one of the hypotheses, because $\mathsf{A}ab$ is in Δ. In the first possibility, since $\mathsf{A}aa \notin \Delta$, some $\mathsf{A}a\_b$ is included in Γ, and hence some freely correct syllogism $\mathsf{A}a\_b \vdash \mathsf{A}ab$ would be included $\Gamma \vdash \Delta$.

Next, let us suppose that $\mathsf{I}ab$ is in Δ, and that $\mathcal{FC} \models \mathsf{I}ab$, *i.e.*, $a \frown^{FC} b$. We consider the three possibilities stemming from the definition of $\frown^{FC}$:

- $a \frown b$, *i.e.*, there are c, d be such that, $\mathsf{I}cd$ [or $\mathsf{I}dc$] is in Γ, and $c \preceq a, d \preceq b$. Then, by $(*)$, some $\mathsf{A}c{=}a, \mathsf{A}d{=}b$ is included in Γ, or some $\mathsf{A}c{=}x, \mathsf{A}c{=}y \vdash \mathsf{I}xy$ or $\mathsf{A}d{=}x, \mathsf{A}d{=}y \vdash \mathsf{I}xy$ is included in $\Gamma \vdash \Delta$. In the first case, $\Gamma \vdash \Delta$ would include the freely correct syllogism $\mathsf{I}cd$[or $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}d{=}b \vdash \mathsf{I}ab$. In the second case, $\Gamma \vdash \Delta$ would include $\mathsf{I}cd, \mathsf{A}c{=}x, \mathsf{A}c{=}y \vdash \mathsf{I}xy$, which is impossible by hypothesis. The last case is similar.
- There is a c such that some $\mathsf{I}cd$ [or $\mathsf{I}dc$] is in Γ, and $c \preceq a, c \preceq b$. This is impossible, by $(*)$ and by hypothesis, since we can have neither $\mathsf{I}cd$ [or $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, nor $\mathsf{I}cd$ [or $\mathsf{I}dc$], $\mathsf{A}c{=}x, \mathsf{A}c{=}y \vdash \mathsf{I}xy$ included in $\Gamma \vdash \Delta$.
- There is a c such that, for some d, $c \not\preceq d$ and $c \preceq a, c \preceq b$. Then, by definition of α, no $\mathsf{A}c{=}x, \mathsf{A}c{=}y \vdash \mathsf{I}xy$ is included in $\Gamma \vdash \Delta$. Therefore, by $(*)$, some $\mathsf{A}c{=}a, \mathsf{A}c{=}b$ is included in Γ. But this is plainly impossible, as $\mathsf{I}ab$ is in Δ.

2. We suppose that the affirmative sequent $\Gamma \vdash \Delta$ is not derivable, and that neither $\mathsf{A}aa$ nor $\mathsf{I}aa$ are in Δ. By theorem 13, it does not include an affirmative correct syllogism. We will show that $\Gamma \vdash \Delta$ is not valid for reflexive Aristotelian algebras.

Define $a\alpha b$ and $a\iota b$ as $\mathsf{A}ab \in \Gamma$ and $\mathsf{I}ab \in \Gamma$, respectively. And let $\mathcal{C}$ be the reflexive algebra $\langle G, \preceq, \frown^C \rangle$, generated by $\langle G, \alpha, \iota \rangle$, with $\mathcal{C}(a) = a$, for a in G (the set of constants occurring in Γ, Δ).

As we obviously have $\mathcal{C} \models \varphi$, for all φ in Γ, we show that $\mathcal{C} \not\models \varphi$, for all φ in Δ.

Suppose that $\mathsf{A}ab$ is in Δ, and that $\mathcal{C} \models \mathsf{A}ab$, *i.e.*, $a \preceq b$. We have $a \neq b$, because $\mathsf{A}aa$ is not in Δ. Hence some correct syllogism $\mathsf{A}a\_b \vdash \mathsf{A}ab$ is included in the sequent $\Gamma \vdash \Delta$.

Suppose that $\mathsf{I}ab$ is in Δ, and that $\mathcal{C} \models \mathsf{I}ab$, *i.e.*, $a \frown^{C} b$. $a \neq b$, since $\mathsf{I}aa$ is not in Δ. Therefore, some correct syllogism of the form $\mathsf{I}cd$ [or $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}d{=}b \vdash \mathsf{I}ab$ or $\mathsf{A}c\_a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$ [or $\mathsf{I}ba$] would be included in $\Gamma \vdash \Delta$. ∎

5.2.4. *Proof theory for class interpretations.* The class system C is obtained by adding $\vdash \mathsf{A}aa$ as initial sequents to the former system of derivations. The free class system FC is C with the following rules added, but the subalternation rule removed:

$$\frac{\Gamma, \mathsf{I}aa \vdash \Delta}{\Gamma \vdash \mathsf{A}ab, \Delta} \qquad\qquad \frac{\Gamma, \mathsf{I}aa \vdash \Delta}{\Gamma, \mathsf{I}ab \vdash \Delta}$$

Every sequent derivable in FC is derivable in C: $\vdash \mathsf{I}aa$ is derivable in C; therefore $\Gamma \vdash \Delta$ results from $\Gamma, \mathsf{I}aa \vdash \Delta$, by the cut rule; and $\Gamma, \mathsf{I}ab \vdash \Delta$ and $\Gamma \vdash \mathsf{A}ab, \Delta$ from $\Gamma \vdash \Delta$, by weakening.

Since $\mathsf{I}cc, \mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$ is F-derivable by proposition 9, $\mathsf{I}cd, \mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$ and $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab, \mathsf{A}cd$ are derivable in FC; and $\mathsf{I}dc$, $\mathsf{A}c{=}a$, $\mathsf{A}c{=}b \vdash \mathsf{I}ab$ derives from $\mathsf{I}dc \vdash \mathsf{I}cd$ and $\mathsf{I}cd, \mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$ by the cut rule.

The completeness of these systems relative to validity for Aristotelian families can now be easily deduced from theorem 17:

If the affirmative sequent $\Gamma \vdash \Delta$ is not FC-derivable, then $\Gamma \vdash \Delta$ is not F-derivable, and neither $\mathsf{I}cd$ [or $\mathsf{I}dc$], $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab$, nor $\mathsf{A}c{=}a, \mathsf{A}c{=}b \vdash \mathsf{I}ab, \mathsf{A}cd$ is included in $\Gamma \vdash \Delta$. Therefore $\Gamma \vdash \Delta$ is not valid for free Aristotelian families.

If the affirmative sequent $\Gamma \vdash \Delta$ is not C-derivable, then $\Gamma \vdash \Delta$ is not derivable, and neither $\mathsf{A}aa$ nor $\mathsf{I}aa$ is in Δ. Therefore $\Gamma \vdash \Delta$ is not valid for Aristotelian families.

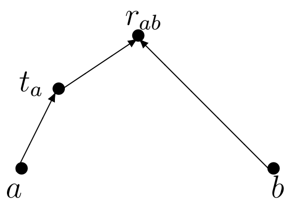
6. Decidability

The systems are all decidable, by the theorems 13 and 7: to see whether $\Gamma \vdash \Delta$ is correct [or freely correct], one checks whether its affirmative transform includes a general Barbara or Darii [or Barbari or Darapti]. Moreover if we embed the syllogistic theories within propositional logic in the manner of [4], we deduce the known decidability for these systems: to see whether a sentence is true in all Aristotelian algebras [or free Aristotelian algebras], one writes it down as a conjunction of disjunctions, so that the problem reduces to the correctness [or free correctness] of a finite set of sequents in our original language.

If we embed syllogistic theories within predicate logic, by taking appropriate subsets of the most inclusive system, generated by the following non-logical axioms: $\forall x \forall y\, (\mathsf{A}xy \leftrightarrow \neg \mathsf{O}xy)$, $\forall x \forall y\, (\mathsf{I}xy \leftrightarrow \neg \mathsf{E}xy)$, $\forall x \forall y \forall z\, (\mathsf{A}xy \rightarrow (\mathsf{A}yz \rightarrow \mathsf{A}xz))$, $\forall x \forall y \forall z\, (\mathsf{I}xy \rightarrow (\mathsf{A}yz \rightarrow \mathsf{I}xz))$, $\forall x \forall y\, (\mathsf{A}xy \rightarrow \mathsf{I}xy)$, $\forall x \forall y\, (\mathsf{I}xy \rightarrow \mathsf{I}yx)$, $\forall x\, \mathsf{A}xx$, then the first-order resulting theories are shown to be undecidable, using Rabin's [5] method in the following way.

Let $\mathcal{M} = \langle M, R \rangle$ be a model of the predicate calculus with a single binary relation, whose undecidability we will use. Let A be M to which new distinct elements t_a and r_{ab} are added, for each $\langle a, b \rangle \in R$.

Define $\prec$ as being the transitive and reflexive closure of the relation on A that holds between x and y iff, for some a, b, aRb; and $x = a, y = t_a$, or $x = t_a, y = r_{ab}$, or $x = b, y = r_{ab}$:



Let $a \frown b$ iff there is a c such that $c \prec a$ and $c \prec b$.

Then $\mathcal{A} = \langle A, \prec, \frown \rangle$ is a reflexive Aristotelian algebra.

Let $D(x)$ be the formula saying that x is $\prec$-minimal:

$$\forall y\, (\mathsf{A}yx \rightarrow \mathsf{A}xy),$$

and $F(x, y)$ the formula

$$\exists r \exists t\, (\mathsf{A}xt \wedge \mathsf{A}tr \wedge \neg \mathsf{A}tx \wedge \neg \mathsf{A}rt \wedge (\mathsf{A}xy \vee \mathsf{A}yr))$$

Then $\langle \{\, x \mid \mathcal{A} \models D(x) \,\}, \{\, \langle x, y \rangle \mid \mathcal{A} \models D(x) \wedge D(y) \wedge F(x, y) \,\} \rangle = \mathcal{M}$.

The situation is known to be different in the case of the richer theory with complements and meets, *viz.*, Boolean algebra, which was shown decidable by Tarski.

References

[1] Clark, Michael, *The place of syllogistic in logical theory*, Nottingham: Nottingham University Press, 1980.

[2] Corcoran, John, "Completeness of an ancient logic," *The Journal of Symbolic Logic* **37** no. 4 (1972), pp.696–702.

[3] Hodges, Wilfrid, "The laws of distribution for syllogisms," *Notre Dame Journal of Formal Logic* **39** (1998), pp.221–230.

[4] Łukasiewicz, Jan, *Aristotle's syllogistic from the standpoint of modern formal logic*, Oxford, London and New York: Clarendon Press, Oxford University Press, 1951. 2nd edition, 1957.

[5] Rabin, Michael O., "A simple method for undecidability proofs and some applications," in *Logic, Methodology and the Philosophy of Science. Proceedings of the 1964 International Congress*, edited by Yehoshua Bar-Hillel, Amsterdam: North-Holland, 1965, pp.58–68.

[6] Shepherdson, John C., "On the interpretation of Aristotelian syllogistic," *The Journal of Symbolic Logic* **21** no. 2 (1956), pp.137–147.

[7] Smiley, Jonathan, "What is a syllogism," *The Journal of Philosophical Logic* **2** no. 1 (1973), pp.136–154.

Université catholique de Louvain, Département de Philosophie, Place Mercier, 14, B–1348 Louvain-la-Neuve

E-mail address: `crabbe@risp.ucl.ac.be`