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MODELING NONINTERSECTIVE ADJECTIVES USING OPERATOR LOGICS

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ABSTRACT. Our topic is one that involves the interface between natural language and mathematical logic. First-order predicate language/logic does a good job approximating many parts of (English) speech, i.e., nouns, verbs and prepositions, but fails decidedly when it comes to, say, adjectives. In particular, it cannot account for the quite different ways in which the adjectives *green* and *big* modify a noun such as *chair*. In the former case, we can easily view a world in which the class of green chairs is the intersection of the class of green things with the class of chair-things. By contrast, the way *big* modifies a noun depends on the noun itself: a big chair is microscopic when compared to the smallest of galaxies. We investigate logical languages inspired by this phenomenon; particularly those with variables ranging over individuals and with variable-binding operators akin to generalized quantifiers.

1. The Categorization Problem.

We take an "applied mathematics" view of natural language (NL), in the sense that one may use mathematical techniques to model some of its observed behaviors. The behavior of interest to us here is adjective-noun (AN) combinations in English; i.e., those noun phrases in which the adjective is in attributive position relative to the noun. In order to be as concrete as possible at the outset, we focus our attention first on two simple examples of AN combinations. While they have the same syntactic appearance, they exhibit very different semantic behaviors.

(A) green chair

(B) big chair

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My own interest in this subject began in 1971, when I was a graduate student, studying mathematical logic at the University of Wisconsin. I found myself one day having a discussion with a friend, a linguistics undergraduate, concerning parts of speech. I carefully explained that first-order logic does not generally distinguish between common nouns, intransitive verbs, and adjectives; they may all be represented as (unary) predicates. I then cited an example like (A) above to make the point that adjectives behave semantically like common nouns. The class of green chairs is just the intersection of the classes of green entities and chair-like ones, and one may translate AN combinations easily into the language of first-order logic. The grammatical distinction between adjectives and nouns as lexical categories, then, is purely arbitrary, existing only at the syntactic level.

My friend's almost immediate response was an example like (B). Being a big chair does not entail being a big entity; indeed a big chair is small when compared to, say, the smallest known galaxy. So while the noun phrases *green chair* and *chair-like green-thing* might well be regarded as synonymous, the same cannot be said for the phrases *big chair* and *chair-like big-thing*. In short, my hasty theory that all adjectives are **intersective** had to be reexamined.

At the time of this discussion, I happened to be participating in H. J. Keisler's seminar on generalized quantifiers (*à la* [8]) and was thus familiar with the idea that (in the context of extensional logic) one may view the interpretation of a quantifier on a domain A simply as a subset of the power set $\wp(A)$. So that, if Q is a generalized quantifier symbol, x is an individual variable and φ is a formula, then $Qx\varphi$ means (roughly) that the set of elements a of A such that φ holds when a is substituted for each free occurrence of x in φ is a member of the interpretation of Q in A. A fellow student, J. Sgro, had already begun the study of topological interpretations of the symbol Q (where $\{a \in A : \varphi[a/x]$ holds in $A\}$ is an open set in a given topology on A, see [15], [16]), so shortly after my linguistics friend pointed out my faulty theory of adjectives, I was able to see that adjectives like *big* behave not like unary relations on a domain, but rather like set-to-set operators (much the way interior and closure operators do in topology). (Of course, if A is a domain (of individuals) and U is a fixed subset, then the operator $M\colon \wp(A) \to \wp(A)$, defined by $M(X) := U \cap X$, shows how to interpret intersective adjectives like *green* as adjectives in the higher-order–operator sense.)

Adjectives do not necessarily become nonintersective because of any degree of vagueness or ambiguity on their part. Even in informal mathematics, arguably the least vague and ambiguous fragment of NL, there are examples of adjectives that behave nonintersectively. Consider the following two phrases, paralleling (A) and (B) above:

(A′) commutative group

(B′) free group

To see how (A′) and (B′) indeed parallel (A) and (B), we need to be specific about our universe of discourse. For this we take the class of individuals to be the *binary algebras*; for the present purposes, those algebraic systems with one binary operation, one unary operation, and one constant (i.e., nullary operation). Then it makes sense to say of a binary algebra that it is commutative, independently of whatever else one may say about it; e.g., that it is a group. Thus the individuals that are commutative groups are all and only those binary algebras that satisfy the commutativity identity as well as the identities defining a group. Alternatively, an individual is free *relative to a class to which it belongs*; a free group with more than one element is never free as a binary algebra. This shows that the adjective *free* is nonintersective, while having a clear and mathematically precise meaning.

I hasten to say that this discovery about the behavior of adjectives was just my own repondering of some of the thinking that philosophers of language had been doing for some time (see [7], [11], [12], and [17] (esp., "English as a formal language")). What those people had formulated for adjectives like *big* was the **operator approach**, a partial solution to what one might call the **categorization problem**; i.e., the assignment of semantic categories to parts of speech in NL. This is an important first step toward the modeling of (fragments of) NL, in such a way that linguistic observations and intuitions are reasonably reflected in the context of a formal language/logic.

I also wish to remark that the operator approach is **extensional**: If, relative to a model, two formulas are synonymous, then the result of applying the same operator (in a suitable formulation of syntax, which we take up below) to both formulas should lead to synonymous formulas. Adjectives like *big* seem to work this way; however, there are some adjectives in NL that behave nonextensionally. Consider, for example, the following AN combination:

(C) skillful writer

One can easily imagine a world in which there are no neurological anomalies, and all and only those who can read are those who can write. But even then it is a stretch to expect that all and only the skillful readers are skillful writers: Evans may be very good at reading, but quite clumsy when it comes to writing. (This point, using other examples, is made also in [7].)

As with nonintersectivity, nonextensional behavior in adjectives is not a mark of vagueness or ambiguity; nonextensional adjectives also occur (albeit rarely) in mathematical language. Consider the following companion to (C):

(C′) initial frame

The word *frame* refers to the objects in a category that consists of certain bounded lattices (including open-set lattices for topological spaces); arrows (morphisms) in this category are certain lattice homomorphisms. Now apparently for reasons of terminological convenience, the dual of this category (i.e., same objects, arrows reversed) has come to be known as the category of locales; hence the nouns *frame* and *locale* are coextensive. However, the class of initial frames consists of those frames F such that there is just one frame morphism from F into any frame. This is just the class of two-element frames. On the other hand, the class of initial locales is precisely the class of terminal frames; i.e., those frames F such that there is just one frame morphism from any frame into F. This is now the class of one-element frames. What all this tells us is that the adjective *initial* behaves nonextensionally in a mathematical context, without any hint of vagueness or ambiguity.

While it is possible to modify the extensional operator approach to account for nonextensional AN behaviors, we do not take up the issue here; but rather postpone a treatment to a later paper. (The interested reader is referred to [7] and [17] (esp. "Pragmatics" and "Pragmatics and intensional logic") for other views on the subject.)

In [14] W. V. O. Quine suggests that while adjectives like *green* and *commutative* can have semantic categories (i.e., unary predicates) assigned to them, others, like *big* and *free* cannot. Instead the latter must play a *syncategorematic* role in language, acquiring meaning only in the presence of lexical items that can have such assignments. It should be noted that quantifier symbols were, at one time, also regarded as playing a syncategorematic role in predicate logic. A symbol like $\exists$ did not stand for anything by itself; only in expressions like $\exists x\varphi$ did it have any hope of interpretation. The advent of the notion of *generalized quantifier* [10] changed all that, however; the classic quantifier symbols, as well as others, could now be regarded as special "higher-order unary predicates;" i.e., as subsets of the power set of a domain of individuals. A similar thing could well be said for adjectives, viewed as operators on a power set $\wp(A)$. The trivial observation that they may equally well be regarded as subsets of $A \times \wp(A)$ shows a kinship with generalized quantifiers, at least on the surface. To summarize this section, then, we believe that the operator approach to the solution of the categorization problem for extensional adjectives (as formalized below) offers an alternative to Quine's view.

2. Operator Languages.

We now propose a kind of finitary formal language that reflects at least some of the nonintersective behavior of (extensional) adjectives in NL. The logic associated with languages of this kind will be easily seen to be weaker than full second-order logic (in the sense of *expressive power*, to be made precise below). For the purposes of readability, it will be convenient to use the nonexponential notation $[A \to B]$ for the family of maps from the set A to the set B. (In this way, the power set $\wp(A)$ of a set A may also be denoted $[A \to 2]$, where $2 := \{0, 1\}$ is the usual set of classical truth values, 0 for *false* and 1 for *true*). Given a set A of individuals and a triple (m, k, n) of nonnegative integers, we define an (m, k, n)**-ary truth-value operator on** A to be a subset of $A^m \times (\wp(A^k))^n$; equivalently, a member of $[(A^m \times [A^k \to 2]^n) \to 2]$. Similarly, one may define an (m, k, n)**-ary individual-value operator on** A to be a member of $[(A^m \times [A^k \to A]^n) \to A]$. Among the familiar truth-value operators one finds: (i) m-ary relations, of arity $(m, k, 0)$ (k is irrelevant if $n = 0$); (ii) interpretations of quantifiers like $\exists$ and Q_1 ("there exist uncountably many"), of arity $(0, 1, 1)$; and (iii) interpretations of n-ary logical connectives, of arity $(0, 0, n)$. Among the familiar individual-value operators one finds: (i) m-ary operations, of arity $(m, k, 0)$ (or arity $(0, 0, m)$); (ii) the interpretation of function abstraction in models of λ-calculus [2], of arity $(0, 1, 1)$; and (iii) the differentiation operator in Freshman Calculus, of arity $(1, 1, 1)$. The $(0, 0, 0)$-ary truth-value operators are just the truth values, while the $(0, 0, 0)$-ary individual-value operators are the elements of a given domain of individuals.

For the sake of simplicity of discussion we focus on truth-value operators in this paper, and refer to them just as operators. In order to be able to talk formally about operators, we introduce a variable-binding language $\mathcal{L}$ equipped with operator symbols. The language we have in mind has individual variables and function symbols, just like first-order predicate language, and terms are built up in the usual way. Then, for each triple (m, k, n) of nonnegative integers, we have a family of (m, k, n)*-ary operator symbols.* The atomic formulas are all of the form $Pt_1 \dots t_m$, where P is an m-ary predicate (i.e., an $(m, k, 0)$-ary operator) symbol, and each t_i is a term. The formation rule for a typical (m, k, n)-ary operator symbol H yields the string $H\bar{t}\,\overline{x}\,\overline{\varphi}$, where: $\bar{t}$ is an m-tuple of terms, $\overline{x}$ is a k-tuple of *distinct* variables, and $\overline{\varphi}$ is an n-tuple of formulas. (See [6] for similar syntax; also [13] for another account of variable-binding operators.) The substring $\overline{x}$, in the position displayed above, is a **binding** occurrence of these variables, and all of their occurrences from that point onward are considered as **bound** occurrences. Every other occurrence of a variable is considered **free**. For example, in the formula $HxxPx$, where H is $(1, 1, 1)$-ary and P is $(1, 0, 0)$-ary, the first occurrence of x is free, the second is a binding occurrence (therefore bound), and the third is bound by that binding occurrence.

If φ is a (term or) formula, x is a variable, and t is a term, then we say that t **is free for** x **in** φ if no free occurrence of x in φ falls within the scope of a binding occurrence of any variable occurring in t. We then define $\varphi[t/x]$ to be the simultaneous substitution of each free occurrence of x in φ by t, if t is free for x in φ, and to be φ otherwise. If $\overline{x}$ is a string of k distinct variables and $\bar{t}$ is a string of k terms, then we may easily define $\varphi[\bar{t}/\overline{x}]$ in the same way, by induction on the complexity of φ. (Caution: $(x = y)[xy/yx]$ is $y = x$, while $((x = y)[x/y])[y/x]$ is $y = y$.).

Note that we are making no commitment to specific meanings of symbols in defining the syntax and semantics of the operator language $\mathcal{L}$. In particular, the notion of operator *logic* is not yet on the agenda. Our immediate aim is to provide a “ballpark” semantics that specifies *ranges* of semantic values. Each triple (m, k, n) encodes a syntactic category, the category of (m, k, n)-ary operator symbols; as well as a semantic category, the category of (m, k, n)-ary operators on sets. So, for example, the semantic category encoded by the triple $(0, 0, 2)$ is simply the set of all (sixteen) binary operations on the set 2 of truth values.

Now, to define the semantics, let $\mathcal{L}$ be a set of operator symbols. (We often identify $\mathcal{L}$ with the corresponding language of terms and formulas.) Given a set A of individuals, an **$\mathcal{L}$-structure with domain** A is an assignment $\mathfrak{A}$ that takes operator symbols in $\mathcal{L}$ to same-arity operators on A. The operator assigned to H in $\mathfrak{A}$ is denoted $H^{\mathfrak{A}}$.

We define the satisfaction relation between $\mathcal{L}$-structures and $\mathcal{L}$-formulas in a manner that straightforwardly extends the Tarski semantics for first-order logic; in the interests of precision and the establishment of some notation, we detail some of the highlights in defining the concept of “φ is true in $\mathfrak{A}$ in the environment ρ.” First of all, given an

$\mathcal{L}$-structure $\mathfrak{A}$, we borrow from computer science and define an **environment** (on $\mathfrak{A}$) to be simply a function from the set of variables to A. We use the notation $\rho[\overline{a}/\overline{x}]$, where $\overline{x}$ and $\overline{a}$ are same-length strings of distinct variables and (not necessarily distinct) individuals, respectively, to indicate the new environment that: (i) agrees with ρ for all variables not occurring in the string $\overline{x} = x_1 \dots x_k$, and (ii) takes each x_i in this string to a_i.

Any environment ρ induces a semantic map $\|\cdot\|^{\mathfrak{A}}_\rho$ that takes $\mathcal{L}$-terms to members of A (in the conventional way) and $\mathcal{L}$-formulas to truth values. Not unexpectedly, the definition of $\|\cdot\|^{\mathfrak{A}}_\rho$ uses induction on the complexity of terms and formulas; the clause governing operators reads as follows:

$$\|Ht_1 \dots t_m x_1 \dots x_k \varphi_1 \dots \varphi_n\|^{\mathfrak{A}}_\rho := \\ H^{\mathfrak{A}}(\|t_1\|^{\mathfrak{A}}_\rho, \dots, \|t_m\|^{\mathfrak{A}}_\rho, \|\lambda\overline{x}\varphi_1\|^{\mathfrak{A}}_\rho, \dots, \|\lambda\overline{x}\varphi_n\|^{\mathfrak{A}}_\rho),$$

where $\|\lambda\overline{x}\varphi_i\|^{\mathfrak{A}}_\rho$ is shorthand for the map from A^k to 2 that takes $\overline{a}$ to $\|\varphi_i\|^{\mathfrak{A}}_{\rho[\overline{a}/\overline{x}]}$.

It is easy to prove that the semantic values $\|\varphi\|^{\mathfrak{A}}_\rho$ and $\|\varphi\|^{\mathfrak{A}}_{\rho'}$ are equal if ρ and ρ' agree on the variables that occur free in φ. If $\|\varphi\|^{\mathfrak{A}}_\rho = 1$ (i.e., φ is true in $\mathfrak{A}$ in the environment ρ), we sometimes write $\mathfrak{A} \models_\rho \varphi$; and we write $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \models_\rho \varphi$ for every environment ρ. ($\mathfrak{A}$ is a **model** of φ.) As usual, we define an $\mathcal{L}$-formula φ to be a **sentence** if it has no free variables; in which case $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \models_\rho \varphi$ for some environment ρ.

The following is the result one should expect about the semantics of free substitution. We include the proof to illustrate the semantics just introduced.

Proposition 1. *Let φ be a term or formula from $\mathcal{L}$, x a variable, and t a term. If $\mathfrak{A}$ is an $\mathcal{L}$-structure and ρ is an environment, then $\|\varphi[t/x]\|^{\mathfrak{A}}_\rho = \|\varphi\|^{\mathfrak{A}}_{\rho'}$; where ρ' is either $\rho[\|t\|^{\mathfrak{A}}_\rho/x]$ or ρ, depending, respectively, upon whether or not t is free for x in φ.*

Proof. Consider the case φ is a formula; argue using induction on formula complexity. There is no problem at the base stage; at the H-introduction stage, we make the simplifying (but not compromising) assumption that H is $(1,1,1)$-ary. Also, since the model $\mathfrak{A}$ remains the same throughout the argument, we suppress its mention in much of the notation. We thus wish to show that $\|(Hsy\varphi)[t/x]\|_\rho = \|Hsy\varphi\|_{\rho'}$, as stated above.

First assume that y and x are the same variable. Then we have to show

$$\|(Hsx\varphi)[t/x]\|_\rho = \|Hsx\varphi\|_{\rho'}.$$

If t is not free for x in $Hsx\varphi$, then there is nothing to prove. Otherwise, the left-hand side becomes

$$\|Hs[t/x]x\varphi\|_\rho = H^{\mathfrak{A}}(\|s[t/x]\|_\rho, \|\lambda x\varphi\|_\rho) = H^{\mathfrak{A}}(\|s\|_{\rho[a/x]}, \|\lambda x\varphi\|_\rho),$$

by the semantic definition and the induction hypothesis (at the level of terms), where $a := \|t\|_\rho$.

At the same time, the right-hand side is equal to

$$\|Hsx\varphi\|_{\rho[a/x]} = H^{\mathfrak{A}}(\|s\|_{\rho[a/x]}, \|\lambda x\varphi\|_{\rho[a/x]}).$$

Now, for any $b \in A$, $\rho[b/x] = \rho[a/x][b/x]$. Hence $\|\lambda x\varphi\|_\rho = \|\lambda x\varphi\|_{\rho[a/x]}$, and the induction step in this case is proved.

Finally assume y and x are distinct variables. Again, if t is not free for x in $Hsy\varphi$, there is nothing to prove. Otherwise, the left-hand side becomes

$$\|Hs[t/x]y\varphi[t/x]\|_\rho = H^{\mathfrak{A}}(\|s\|_{\rho[a/x]}, \|\lambda y\varphi[t/x]\|_\rho),$$

while the right-hand side is equal to $H^{\mathfrak{A}}(\|s\|_{\rho[a/x]}, \|\lambda y\varphi\|_{\rho[a/x]})$.

It remains to show that, for any $b \in A$,

$$\|\varphi[t/x]\|_{\rho[b/y]} = \|\varphi\|_{(\rho[a/x])[b/y]}.$$

But $(\rho[a/x])[b/y] = (\rho[b/y])[a/x]$; and, because t is free for x in $Hsy\varphi$, y does not occur in t. Thus $a = \|t\|_{\rho[b/y]}$. By the induction hypothesis, then, we have the desired semantic equality, and the proof is complete. □

Here are some examples of NL expressions, with their possible translations into the formalism introduced above.

(D) Many are chosen. $\to MxCx$
(where M is $(0,1,1)$-ary)

(E) Bobo is a small elephant. $\to SbxEx)$
(where S is $(1,1,1)$-ary

(F) Evans would rather eat than fight. $\to RexExFx$
(where R is $(1,1,2)$-ary)

(G) Evans is more likely than Bobo to be studious. $\to LebxSx$
(where L is $(2,1,1)$-ary)

Two $\mathcal{L}$-formulas φ and ψ are **synonymous**, relative to an $\mathcal{L}$-structure $\mathfrak{A}$, if $\|\varphi\|^{\mathfrak{A}}_\rho = \|\psi\|^{\mathfrak{A}}_\rho$ for every environment ρ. (In first-order logic, then, φ and ψ are synonymous, relative to $\mathfrak{A}$, if and only if $\mathfrak{A} \models (\varphi \leftrightarrow \psi)$.) Two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are **semantically equivalent** if the same $\mathcal{L}$-sentences are true in both structures, and we write $\mathfrak{A} \equiv \mathfrak{B}$ when this relationship (the obvious extension of elementary equivalence from first-order logic) holds.

Given an $\mathcal{L}$-structure $\mathfrak{A}$ and a relation $R \in [A^k \to 2]$ on A, we say R is **definable** if there is an $\mathcal{L}$-formula φ, a string $\overline{x}$ of k distinct variables, and an environment ρ such that $R = \|\lambda\overline{x}\varphi\|^{\mathfrak{A}}_{\rho}$. The **essential part** $\mathfrak{A}^e$ of $\mathfrak{A}$ is then obtained from $\mathfrak{A}$ by redefining each $H^{\mathfrak{A}}\colon (A^m \times [A^k \to 2]^n) \to 2$ to take an $(m+n)$-tuple $(a_1, \ldots, a_m, R_1, \ldots, R_n)$ to 1 just in case $H^{\mathfrak{A}}(a_1, \ldots, a_m, R_1, \ldots, R_n) = 1$ and each R_i is a definable element of $[A^k \to 2]$. The following is proved using straightforward induction on formula complexity (see, e.g., [3],[9]).

Proposition 2. $\mathfrak{A} \equiv \mathfrak{A}^e$

3. Alternative Syntax.

Returning briefly to my personal 1971 insight concerning how to categorize nonintersective adjectives like *big* as $(1,1,1)$-ary operators, I was soon led to a formulation of an *interior-operator logic* for topology, paralleling J. Sgro's *open-set quantifier logic* $L(Q)$ [15]. Designated $L(I)$ (where L indicates a "base" vocabulary of predicate and function symbols), this was an ordinary first-order language with one additional $(1,1,1)$-ary operator symbol I. The formation rule for this operator symbol was not $Itx\varphi$ as prescribed in Section 2; rather it was $Ix\varphi$, a syntax that unfortunately muddies the distinction between free and bound occurrences of variables. In $IxPx$, for example, the truth value in a model may vary with what elements of the domain are substituted for x; so the occurrences of x in this formula cannot be bound. At the same time, treating those occurrences as free and defining $(IxPx)[c/x]$ to have c being directly subsituted for x is not unlike the confused Calculus student's substituting $x = c$ *before* performing the differentiation. In order to get the desired meaning, namely "c is in the interior of the set of x such that Px holds," it becomes necessary to circumlocute, using the availability of first-order logic. $\varphi[t/x]$ is defined as usual for free occurrences of x not within the scope of an occurrence of Ix, and $(Ix\psi)[t/x]$ is defined to be $\exists x((x = t) \wedge Ix\psi)$. (We abandon what amounts to Polish notation in our syntax for logical connectives, in favor of the more traditional infix notation.)

I had conversations with Sgro on this topic at the time; I had already been able to prove a completeness theorem for $L(I)$ that was analogous to, but much easier than, the one Sgro had proved for $L(Q)$. Also I had shown that $L(I)$ is strictly stronger than $L(Q)$ in expressive power (in the sense that: (i) there is a faithful translation from $L(Q)$-formulas to $L(I)$-formulas; but (ii) one can find two topological L-structures that satisfy the same $L(Q)$-sentences, but not the same $L(I)$-sentences). Despite this, there did not seem to be a clear way to convert a completeness theorem for one logic into one for the other. (We return to this theme in Section 5 below.)

It was not long after these conversations that I happened across C. C. Chang's article [3]. Being inspired by R. Montague's "Pragmatics" article in [17], Chang developed a syntax with formulas $Nx\overline{\varphi}$, the intended semantic category for N being encoded by $(1, 1, n)$. (He, too, stipulated that the variable after the operator symbol should be free, but never needed to address the issue of defining $\varphi[t/x]$ as a syntactic object.) One possible interpretation of N is as an indexed necessity operator: "x thinks φ is necessarily true." Chang argued that his *modal model theory* contained Montague's "Pragmatics" language; in particular, he claimed it subsumed most versions of modal, temporal, and intensional logic. This claim is questionable, however, because of the essentially extensional nature of Chang's (and our) approach: from the synonymy of φ and ψ, one must infer the synonymy of $Nx\varphi$ and $Nx\psi$. Modal logics (*et al*) simply do not allow this inference in general.

Sgro also considered extensions of $L(Q)$ in [15], designed to treat product topologies. In the language $L(Q, Q^2, \dots)$, each Q^k, $k \geq 1$, is a $(0, k, 1)$-ary operator symbol, interpreted as the open set quantifier in the k-fold cartesian product topology. ($Q^1 = Q$.) In the later paper [16], he did the same thing for the interior operator. Again, the variables occurring after the $(m, m, 1)$-ary operator symbol I^m were taken to be free, and the substitution problem was solved using the circumlocution mentioned above. Interestingly enough, at about the same time as Sgro's article [16], there appeared the article [9] of J. Makowski and M. Ziegler. They were also interested in Chang's operators, especially as a source of new languages for topological model theory. However, when they formulated their interior-operator language, their choice of syntax was the less problematic one given in Section 2.

4. Constrained Semantics.

Up to this point, except for historical asides, we have been concerned with operator languages whose symbols are given the widest possible interpretation. From here on, we focus on languages where certain symbols are given narrow or *constrained* interpretations. If $\mathcal{H}$ is a set of operator symbols whose meanings are to be considered as constrained (including equality, as well as the connective and quantifier symbols of first-order predicate logic), and L denotes an additional (disjoint) set of predicate and function symbols whose meanings are to be considered unconstrained, we denote by $L(\mathcal{H})$ the operator language with symbols from $L \cup \mathcal{H}$. In practice, the notation $L(\mathcal{H})$ refers to a family of operator languages, with L acting as a parameter. In keeping with the notation ($L(I)$, $L(Q)$, etc.) introduced in the last section, we often write $L(H_1, H_2, \dots)$ instead of $L(\mathcal{H})$ when $H_1, H_2, \dots$ is a list of the symbols of $\mathcal{H}$ that are not the logical symbols from first-order logic.

When we are considering a fixed set $\mathcal{H}$ of "special" operator symbols as described above, we use the notation (A, α) to indicate an $\mathcal{H}$-structure with domain A (i.e., α is a map that interprets each symbol from $\mathcal{H}$ in the set A). If $\mathbb{K}$ is a class of $\mathcal{H}$-structures in which all the logical symbols are interpreted in the usual manner, then we call the pair $(\mathcal{H}, \mathbb{K})$ a **constraint system**. For any set L of extra predicate and function symbols, $\mathbb{K}_L$ is the class of **constrained** $L(\mathcal{H})$-structures, defined to be all pairs $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is an arbitrary L-structure over A and $(A, \alpha) \in \mathbb{K}$. If $(\mathcal{H}, \mathbb{K})$ is a constraint system and $\mathcal{A} \subseteq \mathcal{H}$, we call the pair $(\mathcal{A}, \mathbb{K})$ a **subsystem** of $(\mathcal{H}, \mathbb{K})$.

Given a constraint system $(\mathcal{H}, \mathbb{K})$, we may define a partial order on $\wp(\mathcal{H})$, extending the inclusion order, and based on the notion of *expressive power*. If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{H}$, define $\mathcal{A}$ to be **expressively weaker** than $\mathcal{B}$ (in symbols, $\mathcal{A} \leq \mathcal{B}$) if, for each L and formula φ of $L(\mathcal{A})$, there is a formula φ^* of $L(\mathcal{B})$ such that for each $(\mathfrak{A}, \alpha) \in \mathbb{K}_L$ and each environment ρ on A, $\|\varphi\|_\rho^{(\mathfrak{A},\alpha)} = \|\varphi^*\|_\rho^{(\mathfrak{A},\alpha)}$. If $\mathcal{A} \leq \mathcal{B}$ holds and *vice versa*, we say $\mathcal{A}$ and $\mathcal{B}$ are **intertranslatable**; we write $\mathcal{A} < \mathcal{B}$ to indicate that $\mathcal{A} \leq \mathcal{B}$ holds, but not *vice versa*.

When indicating relative expressive power, we often abuse notation slightly and write $L(\mathcal{A}) \leq L(\mathcal{B})$. Such a relation is usually established by way of a uniform translation procedure, defined by induction on the complexity of formulas (see below). On the other hand, to refute the relation, one frequently uses an argument based on semantic equivalence. Let $(\mathfrak{A}, \alpha), (\mathfrak{B}, \beta) \in \mathbb{K}_L$, $\mathcal{A} \subseteq \mathcal{H}$. Write $(\mathfrak{A}, \alpha) \equiv_{\mathcal{A}} (\mathfrak{B}, \beta)$ to mean that $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are **semantically equivalent over** $\mathcal{A}$, that the same sentences of $L(\mathcal{A})$ are true in both structures. The following assertion is immediate.

Proposition 3. *Suppose $L(\mathcal{A}) \leq L(\mathcal{B})$. If $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are in $\mathbb{K}_L$ and $(\mathfrak{A}, \alpha) \equiv_{\mathcal{B}} (\mathfrak{B}, \beta)$, then $(\mathfrak{A}, \alpha) \equiv_{\mathcal{A}} (\mathfrak{B}, \beta)$.*

An interesting case study of relative expressive power, already in the literature, concerns topological operator languages. To set the stage, let $\mathcal{H} = \{Q, I, Q^2, Q_2\}$, where: Q and Q^2 are Sgro's open set quantifiers, I is the interior operator (of respective arities $(0,1,1)$, $(0,2,1)$, and $(1,1,1)$, see Section 3 above), and Q_2, of arity $(0,1,2)$, whose intended reading is that the first of two sets meets the interior of the second. To complete the picture, $\mathbb{K}$ may be viewed as consisting of pairs (A, τ), where τ is a topology with underlying set A. So τ itself is the interpretation of the symbol Q in (A, τ), but the interpretation of I is the associated interior operator. Likewise, the informal interpretations of the other two symbols may be easily given their formal counterparts, all relative to the given topology. The following shows that, while topological structure may be expressed interdefinably in terms of open sets and interior operators at the set-theoretic level, there is a breakdown at the level of operator language. Here is a summary of how the four operator languages $L(H)$, $H \in \mathcal{H}$, relate to one another in expressive power. (This is also proved in [9]; we had a proof in 1971 of the first half of Proposition 4(iii), as mentioned earlier.)

Proposition 4. (i) *$L(Q) \leq L(Q^2)$.*
(ii) *$L(Q^2)$ and $L(I)$ have incomparable expressive power.*
(iii) *$L(Q) < L(I)$ and $L(Q) < L(Q^2)$.*
(iv) *$L(Q_2)$ and $L(I)$ have the same expressive power.*

Proof. Ad (i). As a rule, translation definitions use induction on formula complexity. Often (but not always) they fix atomic formulas, and commute with certain of the operator symbols (e.g., $(\exists x\varphi)^* := \exists x\varphi^*$). So when we specify a translation, we will focus, without further comment, on those parts that involve anything new. The translation $(\cdot)^*$ from $L(Q)$ to $L(Q^2)$ is then given by $(Qx\varphi)^* := Q^2 xy\varphi^*$, where y is the first variable not occurring in φ^* (assuming a preassigned well-ordering on the set of individual variables). It is a straightforward induction to see that this works.

Ad (ii). In [9] it is proved that every countable T_1 topological $L(I)$-structure (i.e., for any two distinct points, each is contained in an open set not containing the other) is semantically equivalent to a countable T_1 topological $L(I)$-structure with a countable base of closed open sets. An immediate consequence of this is that no $L(I)$ can formulate the T_2 axiom, saying that for any two points, there are disjoint neighborhoods of those points. The $L(Q^2)$ sentence $Q^2xy(x \neq y)$, however, does formulate this axiom because of the well-known fact that a space is a T_2 space if and only if the diagonal is closed in the square of the space. Thus it is not the case that $L(Q^2) \leq L(I)$.

The $L(I)$-sentence $\exists xyIxz(z \neq y)$ expresses that the topology is nontrivial. Nontriviality cannot be expressed using $L(Q^2)$ because of the following example. Let L consist of just one unary predicate P. Let A be an infinite set, and form $\mathfrak{A}$ by interpreting P as an infinite set B with infinite complement in A. Let τ be the trivial topology, and let $\tau' := \tau \cup \{X\}$, where X is a proper infinite subset of B. Then $(\mathfrak{A}, \tau')$ is a model of $\exists xIxxPx$, while $(\mathfrak{A}, \tau)$ is not. On the other hand, it is easy to show that the essential parts, relative to $L(Q^2)$, of both $(\mathfrak{A}, \tau)$ and $(\mathfrak{A}, \tau')$ are the same. By Proposition 2 it follows that $(\mathfrak{A}, \tau) \equiv_{\{Q^2\}} (\mathfrak{A}, \tau')$; thus it is not the case that $L(I) \leq L(Q^2)$.

Ad (iii). We already know $L(Q) \leq L(Q^2)$, and the induction clause $(Qx\varphi)^* := \forall x(\varphi^* \rightarrow Ixx\varphi^*)$ shows how to establish $L(Q) \leq L(I)$. If there were translations in the reverse direction in either case, then we would be contradicting (ii).

Ad (iv). We were introduced to Q_2 in [9], and we use their translations. (Note that they treat Q_2 as a quantifier with two binding variables, but there is really only one because the two formulas following are regarded as having only one free variable. So when they write $Q_2xy(\varphi(x), \psi(y))$, we write $Q_2x\varphi\psi$.) What is interesting in this result is the fact that topological operators of different arities can be equivalent to one another.

The translation $(\cdot)^*$ that takes $L(I)$ to $L(Q_2)$ has $(Itx\varphi)^* := Q_2y(y = t)\varphi^*$ as its major defining clause; the reverse translation $(\cdot)^{\#}$ has $(Q_2x\varphi\psi)^{\#} := \exists y(\varphi^{\#} \wedge Iyx\psi^{\#})$ as its major clause. $\square$

Another way to see that $L(Q)$ is strictly weaker than $L(I)$ is to consider the T_0 axiom, the statement that for any two points, at least one is in an open set not containing the other. This is easy to express in $L(I)$ (i.e., $\forall x\forall y((x = y) \vee Ixz(z \neq y) \vee Iyz(z \neq x))$, but cannot be expressed in $L(Q)$. (By contrast, the T_1 axiom *can* be so formulated. This is because the T_1 axiom holds if and only if each singleton set is closed; so we have the $L(Q)$-sentence $\forall xQy(y \neq x)$ expressing this.)

The reader interested in highly expressive topological languages is referred to the monograph [4]. There the parametrized language L^t, a fragment of monadic second-order logic, is studied in depth.

5. Completeness Issues.

The question of the completeness of a particular logic concerns whether a given proof theory is enough to include exactly the true statements as theorems. To make things more precise, suppose we are given a subsystem $(\mathcal{A}, \mathbb{K})$ of a constraint system $(\mathcal{H}, \mathbb{K})$. We want to be able to specify a proof-theoretic mechanism, uniform over languages $L(\mathcal{A})$, valid for models in $\mathbb{K}_L$, and sufficient to provide finitary proofs for all semantic consequences of sets of sentences in $L(\mathcal{A})$. The uniformity aspect is achieved via axiom/rule schemata, expressed in terms of generalized formulae that avoid mention of particular extra predicate and function symbols. These include the axiom/rule schemata from first-order logic with equality. When, say, an axiom schema is interpreted in $L(\mathcal{A})$, the result is a set of formulae of a particular syntactic shape. Each such formula may be viewed as a sentence by taking its universal closure.

For our basic proof theory, then, one that ignores the particulars of $\mathbb{K}$, we include first-order logic, plus the following axiom schemata:

(Bound Substitution Schema)

$$H\overline{y}\,\overline{x}\varphi_1 \ldots \varphi_n \leftrightarrow H\overline{y}\,\overline{z}\varphi_1[\overline{z}/\overline{x}] \ldots \varphi_n[\overline{z}/\overline{x}]$$

(Extensionality Schema)

$$\bigwedge_{i=1}^{n}(\varphi_i \leftrightarrow \psi_i) \rightarrow (H\overline{y}\,\overline{x}\varphi_1 \ldots \varphi_n \leftrightarrow H\overline{y}\,\overline{x}\psi_1 \ldots \psi_n)$$

In each schema, H is a generic (m, k, n)-ary operator symbol $(n > 0)$, $\overline{y}$ is a string of m variables, and $\overline{x}$ and $\overline{z}$ are strings of k distinct variables such that no z_i occurs in any φ_j.

As usual, we define a set Σ of $L(\mathcal{A})$-sentences to be **consistent** if no contradiction is provable from Σ, relative to this proof-generating framework. Clearly every $L(\mathcal{A})$-structure satisfies all sentences that are instances of these schemata, so soundness is not a problem. The first step in proving completeness results for operator languages is the following "weak completeness theorem," which may easily be proved using the method of witnesses (pioneered by L. Henkin [5]; see also [8], [15], [16] for extensions of this method to other operator languages).

Proposition 5. *Let Σ be a set of sentences from an operator language $L(\mathcal{H})$. Then Σ is consistent if and only if Σ has a model (among all possible $L(\mathcal{H})$-structures).*

The second step in proving completeness results for operator languages is model-theoretic: Let $(\mathcal{A}, \mathbb{K})$ be a subsystem of the constraint system $(\mathcal{H}, \mathbb{K})$. Define $(\mathcal{A}, \mathbb{K})$ to be **complete** if it is possible to specify for each L, a set Θ_L of $L(\mathcal{A})$-sentences such that: (i) every sentence in Θ_L is true in every model in $\mathbb{K}_L$; and (ii) every $L(\mathcal{A})$-structure that is a model of Θ_L is semantically equivalent, over $\mathcal{A}$, to some model in $\mathbb{K}_L$. The following is immediate.

Proposition 6. *Let $(\mathcal{H}, \mathbb{K})$ be a complete constraint system, with Θ_L witnessing the fact for each L. Let Σ be a set of sentences from $L(\mathcal{H})$. Then Σ is consistent with Θ_L if and only if Σ has a model in $\mathbb{K}_L$.*

So for example, in [8], a constrained $L(Q_1)$-structure is defined to be just an L-structure in which the uncountably-many quantifier Q_1 has the interpretation as the set of uncountable subsets of the domain of that structure. A special set Θ_L of four simple $L(Q_1)$-schemata is given, it being easy to show that each instance of these schemata is true in any constrained structure. The hard part of the $L(Q_1)$ completeness theorem, originally conjectured by W. Craig and G. Fuhrken in 1962, is showing that every model of Θ_L is semantically equivalent to a constrained structure; i.e., that this constrained operator language is complete (via Θ_L). (In [15] and [16], a similar program is carried out for the topological languages $L(Q)$ and $L(I)$, *inter alia*.)

An easy corollary of (finitary) completeness is the model-theoretic phenomenon of compactness. Define a subsystem $(\mathcal{A}, \mathbb{K})$ of a constraint system $(\mathcal{H}, \mathbb{K})$ to be **compact** if, for any L and any set Σ of $L(\mathcal{A})$-sentences: if Σ has no model in $\mathbb{K}_L$, then some finite subset of Σ has no model in $\mathbb{K}_L$.

Proposition 7. *Every complete constraint (sub)system is compact.*

An easy example of a constraint system $(\mathcal{H}, \mathbb{K})$ that is *not* compact is any one where $\mathbb{K}$ consists of finite structures of arbitrarily large cardinality. The next two results connect completeness/compactness notions with translation.

Proposition 8. *Let $(\mathcal{H}, \mathbb{K})$ be a constraint system, with $\mathcal{A} \leq \mathcal{B} \subseteq \mathcal{H}$. If $(\mathcal{B}, \mathbb{K})$ is compact, so is $(\mathcal{A}, \mathbb{K})$.*

Proof. Fix L, and let Σ be a set of $L(\mathcal{A})$-sentences, such that every finite subset of Σ has a model in $\mathbb{K}_L$. Let $\varphi \mapsto \varphi^*$ be a translation that witnesses $L(\mathcal{A}) \leq L(\mathcal{B})$. Let $\Sigma^* := \{\varphi^* : \varphi \in \Sigma\}$. If $\{\varphi_1^*, \ldots, \varphi_n^*\}$ is any finite subset of Σ^*, then this set has a model in $\mathbb{K}_L$, by the definition of translation and the fact that $\{\varphi_1, \ldots, \varphi_n\}$ has a model in $\mathbb{K}_L$. Since $(\mathcal{B}, \mathbb{K})$ is compact, Σ^* has a model in $\mathbb{K}_L$. That structure must also be a model of Σ. $\square$

It would be nice to have an analogue of Proposition 8 with *complete* replacing *compact*. We do not know whether this is true; however, we have the following weaker result.

Proposition 9. *Let $(\mathcal{H}, \mathbb{K})$ be a constraint system, and let $\mathcal{A}$ and $\mathcal{B}$ be intertranslatable subsets of $\mathcal{H}$. If $\varphi \mapsto \varphi^*$ witnesses $L(\mathcal{A}) \leq L(\mathcal{B})$, and if Θ_L witnesses the completeness of $L(\mathcal{A})$, then $(\Theta_L)^*$ witnesses the completeness of $L(\mathcal{B})$.*

Proof. We wish to show that if $(\mathfrak{A}, \alpha) \models (\Theta_L)^*$, then $(\mathfrak{A}, \alpha)$ is semantically equivalent, over $\mathcal{B}$, to some member of $\mathbb{K}_L$. By the definition of *translation*, we know $(\mathfrak{A}, \alpha) \models \Theta_L$. Since $L(\mathcal{A})$ is complete, $(\mathfrak{A}, \alpha)$ is semantically equivalent, over $\mathcal{A}$, to a member of $\mathbb{K}_L$. Now, since $L(\mathcal{B}) \leq L(\mathcal{A})$, $(\mathfrak{A}, \alpha)$ is semantically equivalent, over $\mathcal{B}$, to a member of $\mathbb{K}_L$, by Proposition 3. $\square$

Other issues of translatability and completeness are inspired by the phenomenon of intersective adjectives in NL (such as *green* and *commutative*, etc.). In some cases (m, k, n)-ary operators "behave like predicates;" we make this notion clearer, in the case $m = k$, as follows.

For simplicity, let $\mathcal{H}$ consist of the (m, m, n)-ary operator symbol H, as well as the $(m, 0, 0)$-ary operator (predicate) symbol P that H is intended to behave like. In the case of intersective adjectives, we have $Hxx\varphi$ being interpreted as $Px \wedge \varphi$ (see the example above involving green chairs), so there is the added ingredient of a logical connective.

In a more general setting, we specify an $(n+1)$-ary Boolean function $f\colon 2^{n+1} \to 2$, and define $\mathbb{K}^f$ to consist of all $\mathcal{H}$-structures (A, α) such that, for any m-ary relations $R_1, \ldots, R_n$ on A, and any $\overline{a} \in A^m$, we have $H^\alpha(\overline{a}, R_1, \ldots, R_n) = f(P^\alpha(\overline{a}), R_1(\overline{a}), \ldots, R_n(\overline{a}))$. (So in the intersective adjective case, $f\colon 2^2 \to 2$ is the minimization function *min*.)

Proposition 10. *Let $(\mathcal{H}, \mathbb{K}^f)$ be as defined above. Then $L(H) \leq L(P)$.*

Proof. Let B_f be a realization of f in terms of logical connectives, introduced as an abbreviation and expressed in Polish notation. Our proposed translation $\varphi \mapsto \varphi^*$ witnessing $L(H) \leq L(P)$ makes no changes in terms or formulae until the H-introduction clause, where we make the definition $(H\bar{t}\bar{x}\varphi_1 \ldots \varphi_n)^* := B_f P\bar{t}(\varphi_1')^*[\bar{t}/\bar{x}] \ldots (\varphi_n')^*[\bar{t}/\bar{x}]$, where the formula φ_i' results from φ_i via a uniform meaning-preserving replacement of bound variables, in such a way that no variable in $\bar{t}$ occurs bound in φ_i'. (This may be accomplished, e.g., by first assuming a well-ordering on the set of variables, and then using a "pick-the-least-variable-not-previously-seen" type of replacement algorithm. So, for example, if H is (1,1,1)-ary and f is *min*, then (restoring infix notation for logical conjunction)

$$\begin{aligned}
(Hx_2x_1\exists x_2Rx_1x_2)^* &= Px_2 \wedge ((\exists x_2Rx_1x_2)')^*[x_2/x_1] \\
&= Px_2 \wedge (\exists x_3Rx_1x_3)^*[x_2/x_1] \\
&= Px_2 \wedge (\exists x_3Rx_1x_3)[x_2/x_1] \\
&= Px_2 \wedge \exists x_3Rx_2x_3.)
\end{aligned}$$

To see that this works, fix L, let $(\mathfrak{A}, \alpha) \in \mathbb{K}_L$, and let ρ be an environment on A. We prove by induction on formula complexity that, for any $L(H)$-formula φ, $\|\varphi\|_\rho^{(\mathfrak{A},\alpha)} = \|\varphi^*\|_\rho^{(\mathfrak{A},\alpha)}$. The least trivial part is the H-introduction clause; we make the inessential simplification that H is $(1,1,1)$-ary. It remains to show that $\|Htx\varphi\|_\rho = \|B_fPt(\varphi')^*[t/x]\|_\rho$ (where the superscript indicating $(\mathfrak{A}, \alpha)$ is suppressed), under the inductive assumption that $\|(\varphi')^*\|_\sigma = \|\varphi\|_\sigma$ for every environment σ on A. Let $a := \|t\|_\rho$. We may assume, without loss of generality, that no variable in t occurs bound in φ; so φ' may be taken to be φ. Then the left-hand side of our proposed equality becomes $\|H^\alpha(a, \|\lambda x\varphi\|_\rho)\|$, while the right-hand side becomes $f(P^\alpha(a), \|\varphi^*[t/x]\|_\rho)$. Because $(\mathfrak{A}, \alpha) \in \mathbb{K}_L$, the left-hand side then becomes $f(P^\alpha(a), \|\lambda x\varphi\|_\rho(a)) = f(P^\alpha(a), \|\varphi\|_{\rho[a/x]})$. What is left to show, then, is the equality $\|\varphi^*[t/x]\|_\rho = \|\varphi\|_{\rho[a/x]}$. Now t is free for x in φ; by the nature of the translation mechanism $(\cdot)^*$, t is free for x in φ^*. Thus, by Proposition 1, the left-hand side is $\|\varphi^*\|_{\rho[a/x]}$, which, by the inductive hypotheses, is equal to the right-hand side. $\square$

We would now like to show $L(P) \leq L(H)$. This, however, requires a restriction on the Boolean function f. Define $f\colon 2^{n+1} \to 2$ to be **nonsingular** if there is some $\overline{\beta} \in 2^n$ such that the one-variable function $f(x, \overline{\beta})$ is one-one (i.e., either the identity map or the negation map on 2).

Proposition 11. *Let* $(\mathcal{H}, \mathbb{K}^f)$ *be as defined above. If* f *is nonsingular, then* $L(P) \leq L(H)$.

Proof. Our proposed translation $\varphi \mapsto \varphi^\sharp$ witnessing $L(P) \leq L(H)$ makes a change only at the atomic level, and then only with the symbol P. First we pick $\overline{\beta} \in 2^n$ such that $f(x, \overline{\beta})$ is one-one; say it is identically x. We identify $\overline{\beta}$ naturally with the corresponding string of (0,0,0)-ary logical operations, $\perp$ for *false* and $\top$ for *true*. Then $(P\bar{t})^* := H\bar{t}\,\overline{x}\,\overline{\beta}$ (the variable string $\overline{x}$ being fixed, but arbitrary).

To see that this definition works requires only a verification at the atomic level. So let $(\mathfrak{A}, \alpha) \in \mathbb{K}_L$ be given, with ρ an environment on A. Then

$$\begin{aligned} \|(Pt_1 \dots t_m)^*\|_\rho &= \|Ht_1 \dots t_m \overline{x}\,\overline{\beta}\|_\rho \\ &= f(P^\alpha(\|t_1\|_\rho, \dots, \|t_m\|_\rho), \overline{\beta}) \\ &= P^\alpha(\|t_1\|_\rho, \dots, \|t_m\|_\rho) \\ &= \|Pt_1 \dots t_m\|_\rho. \end{aligned}$$

If $f(x, \overline{\beta})$ happens to be identically $1 - x$, then we define $(P\bar{t})^* := \neg H\bar{t}\,\overline{x}\,\overline{\beta}$. $\square$

We close this report with the following completeness result.

Proposition 12. *Let* $(\mathcal{H}, \mathbb{K}^f)$ *be as defined above. Then* $(\{H\}, \mathbb{K}^f)$ *is complete.*

Proof. Let B_f be a realization of $f\colon 2^{n+1} \to 2$, as described earlier, and let L be given. If f is nonsingular, then we have our conclusion, by Propositions 9, 10, and 11 (plus completeness for first-order logic). So suppose f is singular. Then, for all $\overline{\beta} \in 2^n$, the map $x \mapsto f(x, \overline{\beta})$ is constant. Thus let $\overline{\beta} \in \{\perp, \top\}^n$ be fixed, but arbitrary, and define Θ_L to be the set of $L(H)$-formulae of the form $H\overline{x}\,\overline{x}\varphi_1 \dots \varphi_n \leftrightarrow B_f H\overline{x}\,\overline{x}\,\overline{\beta}\varphi_1 \dots \varphi_n$. For simplicity, assume H is $(1,1,1)$-ary, so Θ_L is the set of formulae $Hxx\varphi \leftrightarrow B_f Hxx\beta\varphi$. Let $(\mathfrak{A}, \alpha) \in \mathbb{K}_L^f$, with ρ an environment on A. Set $a := \|x\|_\rho^{(\mathfrak{A},\alpha)} = \|x\|_\rho$. Then

$$\begin{aligned} \|Hxx\varphi\|_\rho &= H^\alpha(a, \|\lambda x\varphi\|_\rho) \\ &= f(P^\alpha(a), \|\lambda x\varphi\|_\rho(a)) \\ &= f(P^\alpha(a), \|\varphi\|_{\rho[a/x]}) \\ &= f(P^\alpha(a), \|\varphi\|_\rho). \end{aligned}$$

But this is $f(\|Hxx\beta\|_\rho, \|\varphi\|_\rho)$ because f is singular, and this is $\|B_f Hxx\beta\varphi\|_\rho$. This tells us that $(\mathfrak{A}, \alpha) \models \Theta_L$.

Now pick an arbitrary $L(H)$-structure $(\mathfrak{A}, \eta)$ that satisfies all sentences in Θ_L. We need to show that $(\mathfrak{A}, \eta)$ is semantically equivalent, over $L(H)$, to a structure in $\mathbb{K}_L^f$. So for all $L(H)$-formulae φ and all environments ρ on A, we have $\|Hxx\varphi\|_\rho = \|B_f Hxx\beta\varphi\|_\rho$. Let $a := \rho(x)$. Then the left-hand side is $H^\eta(a, \|\lambda x\varphi\|_\rho)$, and the right-hand side is $f(\|Hxx\beta\|_\rho, \|\varphi\|_\rho)$. But then this is $f(\|\lambda x Hxx\beta\|_\rho(a), \|\lambda x\varphi\|_\rho(a))$. So by setting $U := \|\lambda x Hxx\beta\|_\rho$, we see: first, that U is independent of ρ; and second, that whenever $a \in A$ and $R \in [A \to 2]$ is definable, then $H^\eta(a, R) = f(U(a), R(a))$. So define the $L(H, P)$-structure $(\mathfrak{A}, \alpha)$ so that $H^\alpha(a, R) = f(U(a), R(a))$ for all $a \in A$ and $R \in [A \to 2]$ and $P^\alpha = U$. Then $(\mathfrak{A}, \alpha) \in \mathbb{K}_L^f$; and, by Proposition 2, $(\mathfrak{A}, \alpha) \equiv_{\{H\}} (\mathfrak{A}, \eta)$. □

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