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INTRODUCTION TO "CURRENT DIRECTIONS IN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS"

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The papers collected here are among those presented at a conference on Nonlinear Partial Differential Equations in March, 1987, at Provo, Utah. They are representative of some of the more dominant current trends of research in partial differential equations. I would like to give my view of these trends, with special reference to the contributions in this volume.

1. *Singular limits leading to interface problems.*

It is well known that for reaction-diffusion systems, the presence of a small parameter multiplying the highest order spatial derivatives in one or more of the equations may lead to the existence of solutions with internal layers. Layers in this context means that the spatial variation is much sharper in some localized regions of space than in others. These regions are typically small neighborhoods of surfaces which at least locally separate space into two parts.

Interest in internal layers of this sort is not only intrinsic to the mathematical phenomenon, but also derives from their being models for interfacial phenomena in practically all of the natural sciences.

In dynamical problems, layers appear as moving free boundaries; then the laws governing their motion and their stability or instability are highly important considerations. In stationary problems, dealing with singularly perturbed systems of second order elliptic or ordinary differential equations, one obtains a problem for a fixed free boundary in space. Issues here often center on the existence of solutions. In both cases, it is of interest to determine the fine structure of the interface; in fact that type of result may be crucial to the other task of unraveling the law of motion or of determining the interface condition.

Interfacial problems as singular limits of reaction-diffusion equations are well represented in these proceedings.

Caginalp and Chadam consider the stability of moving interfaces in solidification problems which incorporate both surface tension and dynamical undercooling.

Nishiura surveys a wide range of results for a certain type of system of two reaction-diffusion equations in one space variable, exploring the wealth of layer phenomena which have so far been uncovered there, and revealing in particular the powerful SLEP method for investigating the stability of interfaces, and the relatively novel phenomena of breathing (oscillating) interfaces and bifurcating traveling waves.

In more space dimensions, steady problems with singularly perturbed variational structure have suggested the following natural questions: can one define a limit variational problem in some reasonable way? What is this limit problem in specific cases? What is the relation between its minimizers and those of the original problems? These questions have spawned the concept of Γ-convergence, which is the subject of the articles by Owen and by Sternberg. The variational problems which they consider have interfacial properties.

Among the many applications of the techniques of diffusive interfaces, none are more striking than the study of interfaces between phases of a two-phase mixture. The phase field models were devised primarily to account for such interfacial phenomena as curvature, kinetic undercooling, and nucleation effects. The formal singular limits of these models are discussed here by Caginalp.

Chemical interfaces are an intriguing phenomenon and give rise to various kinds of spatial patterning. The paper by Chadam and Ortoleva here presents the case for patterning in the fascinating context of geochemistry.

The "mesa problem," discussed by Sacks, involves a different kind of singular limit, this time for the porous medium equation, in which the solution is constant in the limit in one region of space, and equal to a prescribed function in the rest of space.

Stuart discusses an unusual application of bifurcation theory to the singular limit of a free boundary problem which arises in porous medium combustion.

2. Attractors, multiple time scales, and stable/unstable manifolds.

A very definite current trend in the study of reaction-diffusion systems is to ask the same questions of them which have been asked historically of dynamical systems described by ordinary differential equations. Examples are: what are the attractors, connections, inertial manifolds, stable and unstable manifolds, etc.?

Hale discusses some of these issues in the framework of asymptotically smooth semigroups. An important point which emerges is the significance of multiple time scales, layered solutions, and unstable manifolds in understanding the dynamics of singularly perturbed systems. Mielke successfully addresses the question of centerstable (etc.) manifolds for quasilinear parabolic problems.

3. Solutions which develop singularities.

By this, I mean solutions of evolution problems which become singular at a certain finite time and therefore cease to exist, even in a weak sense, after that time. Possibly the best known example of this is thermal runaway in combustion; but there are others as well. Qualitative details about the solution at the time of breakdown is always an intriguing topic for investigation. In these proceedings we find examples in Le Mesurier's treatment of a nonlinear Schrödinger equation, and in Hulshof's treatment of a problem involving flow in a partially saturated medium.

4. The relevance of special solutions.

Historically, when confronted with a difficult nonlinear partial differential equation, scientists have been prone to look for solutions of special type, often by similarity methods. The simplest example of this is, of course, the search for stationary solutions of an evolution problem, and the second simplest, the search for traveling wave solutions. After such solutions are found, one can ask about their relevance among the set of all solutions. Does a significant class of solutions approach these special solutions as $t \to \infty$? Are they stable?

In the case of the stationary solutions, Weinstein addresses the stability of Neu's vortex solutions, with reference to various possible dynamics. In the case of traveling waves, Angenent demonstrates, in a neat and clean fashion, their global relevance for scalar nonlinear diffusion equations with periodic boundary conditions. In Hale's discussion of motion among an unstable manifold (commonplace, he argues), it is clear that traveling wave types play a predominant role.

Finally, similarity solutions and almost periodic solutions of different problems are treated by Troy and Biroli, respectively.

5. Relations among different models of a given physical phenomenon.

Applied mathematics deals primarily with models of natural phenomena. Most such phenomena of interest are complicated enough to engender a variety of mathematical models. A question of obvious importance, but relatively seldom studied, is that of the rigorous relationship between two models for the same phenomenon. Hornung thoroughly addresses this issue for porous medium models. Models for solidification and melting represent another good example of the multiplicity of models, and the papers of Caginalp and Showalter and Walkington deal with this. An alternate to the usual Stefan problem is the hyperbolic Stefan problem, the subject of the contribution by Showalter and Walkington. As mentioned before in section 1 above, the limits of singularly perturbed problems are sometimes free boundary problems, and Caginalp discusses the way formally to derive alternate models for phase changes.

6. Existence theory for models of natural phenomena which are new or which have gaps in their basic theory.

This more traditional query in nonlinear partial differential equations is represented here not only in the contribution of Showalter and Walkington referred to above, but also in the paper by Kuttler and Hicks, dealing with an established model in compressible flow. The article by Morgan gives a verifiable sufficient condition for a reaction-diffusion-convection system to have global solutions without necessarily having invariant rectangles.