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Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations

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Abstract

Multi-soliton solutions, i.e. solutions behaving as the sum of N given solitons as $t \to +\infty$, were constructed for the L^2 critical and subcritical (NLS) and (gKdV) equations in previous works (see [23], [16] and [20]). In this paper, we extend the construction of multi-soliton solutions to the L^2 supercritical case both for (gKdV) and (NLS) equations, using a topological argument to control the direction of instability.

1. Introduction

1.1. The generalized KdV equation

We consider the generalized Korteweg-de Vries equations:

$$\text{(gKdV)} \qquad u_t + (u_{xx} + u^p)_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where $p \geq 2$ is an integer. See Section 3.1 for more general nonlinearities.

Recall that the Cauchy problem for (gKdV) in the energy space H^1 has been solved by Kenig, Ponce and Vega [14]: for all $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in \mathcal{C}([0,T], H^1(\mathbb{R}))$ to (gKdV) satisfying $u(0) = u_0$, unique in some sense. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and then $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution).

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For such solutions, the mass and energy are conserved:

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{1.1}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{1.2}$$

Now, we define $Q \in H^1$, $Q > 0$ the unique solution (up to translations) to

$$Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)}\right)^{\frac{1}{p-1}}.$$

Let $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$ and let

$$R_{c_0,x_0}(t,x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c}(x - c_0 t - x_0))$$

be the family of soliton solution of the (gKdV) equation.

It is well-known that the stability properties of a soliton R_{c_0,x_0} depend on the sign of $\frac{d}{dc}\int Q^2_{c\,|c=c_0}$. Since $\int Q_c^2 = c^{\frac{5-p}{2(p-1)}}\int Q^2$, we distinguish the following three cases:

- For $p < 5$ (L^2 subcritical case), solitons are stable and asymptotically stable in H^1 in some suitable sense: see Cazenave and Lions [3], Weinstein [30] , Grillakis, Shatah and Straus [12], for orbital stability, and Pego and Weintein [27], Martel and Merle [17] for asymptotic stability.
- In the L^2 critical case, i.e. $p = 5$, solitons are unstable, and blow up occur for a large class of solutions initially arbitrarily close to a soliton, see Martel and Merle [18], [19].
- In the case $p > 5$ (L^2 supercritical case), solitons are unstable (see Grillakis, Shatah and Straus [12] and Bona, Souganidis and Strauss [2]).

Now, we focus on multi-soliton solutions. Given $2N$ parameters defining N solitons with different speeds,

$$0 < c_1 < \cdots < c_N, \quad x_1, \ldots x_N \in \mathbb{R}, \tag{1.3}$$

we call multi-soliton a solution $u(t)$ to (gKdV) such that

$$\lim_{t\to+\infty} \left\| u(t) - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{H^1} = 0. \tag{1.4}$$

Let us recall known results on multi-solitons:

- For $p = 2$ and 3 (KdV and mKdV), multi-solitons are well-known to exist for any set of parameters (1.3), as a consequence of the inverse scattering method. Moreover, these special explicit solutions describe the elastic collision of the solitons (see e.g. Miura [24]).

- In the L^2-subcritical and critical cases, i.e. for (gKdV) with $p \leq 5$ (or for some more general nonlinearities under the stability assumption $\frac{d}{dc} \int Q^2_{c|c=c_j} > 0$ for all j), Martel [16] constructed multi-solitons for any set of parameters (1.3). The proof of this result follows the strategy of Merle [23] (compactness argument) and relies on monotonicity properties developed in [17] (see also [21]). Recall that Martel, Merle and Tsai [21] proved stability and asymptotic stability of a sum of N solitons for large time for the subcritical case. A refined version of the stability result of [21] shows that for a given set of parameters, there exists a *unique* multi-soliton soliton satisfying (1.4), see Theorem 1 in [16].

In the present paper, we extend the multi-soliton existence result to the L^2-supercritical case, i.e., in a situation where solitons are known to be unstable.

Theorem 1 (Existence of multi-solitons for L^2-supercritical (gKdV))**.** *Let $p > 5$. Let $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $u \in \mathcal{C}([T_0, \infty), H^1)$ to* (gKdV) *such that*

$$\forall t \in [T_0, \infty), \qquad \left\| u(t) - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{H^1} \leq Ce^{-\sigma_0^{3/2} t}.$$

Remark 1. As in the subcritical case, the proof of Theorem 1 is based on a compactness argument and on some large time uniform estimates, however, it also involves an additional topological argument to control an instable direction of the linearized operator around each Q_{c_j}. The proof relies decisively on the introduction of L^2 eigenfunctions of the linearized operator, constructed by Pego and Weinstein [26] by ODE techniques. Note that in [26], the existence of such eigenfunctions for Q_{c_0} is proved to be equivalent to $\frac{d}{dc} \int Q^2_{c|c=c_0} < 0$.

It is possible that other methods of construction work for some range of parameters $0 < c_1 < \cdots < c_N$, but due to the instable directions, the use of such a topological argument is probably necessary to treat the general case (1.3).

Finally, note that the solution $u(t)$ of Theorem 1 belongs to H^s, and that the convergence to $\sum_{j=1}^{N} R_{c_j,x_j}(t)$ holds in H^s, for any $s \geq 1$ (see Proposition 5 of [16]).

We refer to Section 3.1 for a similar existence result for (gKdV) equations with general nonlinearities.

1.2. The non linear Schrödinger equations

Now we turn to the case of the non linear Schrödinger equations:

$$\text{(NLS)} \qquad iu_t + \Delta u + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad u(t,x) \in \mathbb{C},$$

where $p > 1$, for any space dimension $d \geq 1$. Concerning the local well-posedness of the Cauchy problem in H^1, we refer to Ginibre and Velo [10]. Recall that H^1 solutions satisfy the conservation laws

$$\int |u|^2(t) = \int |u_0|^2, \quad \operatorname{Im} \int (\bar{u}\nabla u)(t) = \operatorname{Im} \int \bar{u}_0 \nabla u_0,$$

$$\frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{p+1}\int |u|^{p+1}(t) = \frac{1}{2}\int |\nabla u_0(t)|^2 - \frac{1}{p+1}\int |u_0|^{p+1}.$$

Consider the radial positive solution $Q \in H^1(\mathbb{R}^d)$ to

$$\Delta Q + Q^p = Q, \tag{1.5}$$

which is the unique positive solution of this equation up to translations. We refer to [9], [1] and [15] for classical existence and uniqueness results on equation (1.5). Given $v_0, x_0 \in \mathbb{R}^d$, $\gamma_0 \in \mathbb{R}$ and $c_0 > 0$, the function

$$R_{c_0,\gamma_0,v_0,x_0}(t,x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}(x - v_0 t - x_0)) e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}\|v_0\|^2 t + c_0 t + \gamma_0)}$$

is a soliton solution to (NLS), moving on the line $x = x_0 + v_0 t$.

We recall the following classical results (for any $d \geq 1$):

- For $1 < p < 1 + 4/d$, (L^2 subcritical case) Cazenave and Lions [3] proved that solitons are orbitally stable in H^1. Multi-solitons (defined in a similar way as for (gKdV)) were constructed in this setting by Martel and Merle [20].
- In the L^2 critical case, $p = 1 + 4/d$, solitons are unstable, however multi-solitons were constructed by Merle [23], as a consequence of the construction of special solutions of (NLS) blowing up in finite time at N prescribed points.
- For $p \in (1 + \frac{4}{d}, \frac{d+2}{d-2})$ (for $d = 1, 2$, $p > 1 + \frac{4}{d}$): solitons are unstable (see [12]). Recall that $p = \frac{d+2}{d-2}$ corresponds to the critical $\dot{H}^1$ case.

We claim the following analogue of Theorem 1 in the context of the L^2 supercritical (NLS) equation.

Theorem 2 (Multi-solitons for L^2 supercritical (NLS))**.** *Let $p \in (1+\frac{4}{d}, \frac{d+2}{d-2})$ ($p > 1+\frac{4}{d}$ for $d = 1, 2$). Let $c_1, \dots, c_N > 0$, $\gamma_1, \dots, \gamma_N \in \mathbb{R}$, $x_1, \dots, x_N \in \mathbb{R}^d$, and $v_1, \dots, v_N \in \mathbb{R}^d$ be such that*

$$\forall k \neq j, \quad v_k \neq v_j.$$

Then there exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $u \in \mathcal{C}([T_0, \infty), H^1)$ to (NLS) *such that*

$$\forall t \in [T_0, \infty), \qquad \left\| u(t) - \sum_{j=1}^{N} R_{c_j,\gamma_j,v_j,x_j}(t) \right\|_{H^1} \leq Ce^{-\sigma_0^{3/2} t}.$$

Remark 2. The condition on p means that the problem is L^2 supercritical but $\dot{H}^1$ subcritical (for $d \geq 3$). In the present paper, we do not treat the $\dot{H}^1$ critical case – recall that solitons then have only algebraic decay.

The proof of Theorem 2 is completely similar to the one of Theorem 1, see Section 3.2. Note that similarly to the (gKdV) case, we need eigenfunctions for the linearized operator around Q. For the (NLS) case, see Weinstein [29], Grillakis [11] and Schlag [28].

In Section 1.3, we present an outline of the proof of Theorem 1. A complete proof of Theorem 1 is given in Section 2. Next, extensions of this result to (gKdV) equations with general nonlinearities are presented without proof in Section 3.1. Finally, a sketch of the proof of Theorem 2 is given in Section 3.2. In the Appendix, we gather the proof of two technical lemmas.

1.3. Outline of proof of Theorem 1

For simplicity, we consider only positive solitons and pure power nonlinearities for (gKdV). The proof follows a similar initial strategy as in the works of Merle [23] or Martel [16]. Set

$$R_j(t,x) = R_{c_j,x_j}(t,x), \qquad R(t,x) = \sum_{j=1}^{N} R_j(t,x),$$

and consider a sequence $S_n \to +\infty$.

In the subcritical case ([16] and [20]), one considers the sequence (u_n) of solutions to (gKdV) such that $u_n(S_n) = R(S_n)$. The goal is then to obtain backwards uniform estimates on $u_n(t) - R(t)$ on some time interval $t \in [T_0, S_n]$, where T_0 does not depend on n. From these estimates, one can construct the multi-soliton soliton by compactness arguments. To obtain the uniform estimates, one uses monotonicity properties of local conservation laws and coercivity property of the Hessian of the energy around a soliton:

$$Lv = -v_{xx} - pQ^{p-1}v + v.$$

Indeed, in the subcritical case, it is well-known (see [30]) that $(Lv, v) \geq \lambda\|v\|_{H^1}^2$ ($\lambda > 0$) provided that $(v, Q) = (v, Q_x) = 0$. These two directions are then controlled by modulation with respect to scaling and translation.

In the supercritical case, one cannot obtain uniform estimates in the same way, since the previous property of L fails. It is known that $(L\cdot, \cdot)$ is positive definite up to the directions $Q^{\frac{p+1}{2}}$ and Q_x; the direction Q_x can still be handled using modulation in the translation parameter, but the even direction $Q^{\frac{p+1}{2}}$ cannot be controled by the scaling parameter as for the subcritical case (this is of course related to the instable nature of the soliton).

At this point, we need the L^2 eigenfunctions $Z^\pm$ of the operator $L\partial_x$:

$$L(Z_x^\pm) = \pm e_0 Z^\pm, \quad e_0 > 0,$$

constructed by Pego and Weinstein [26]. Following Duyckaerts and Merle [5], we prove that $(L\cdot, \cdot)$ is positive definite up to the directions $Z^\pm$ and Q_x (see Lemma 1 in the present paper). The direction Z^- being in some sense a stable direction, it does not create any difficulty. For the instable direction Z^+, we do need an extra parameter, which cannot be controlled by a scaling argument. Thus, instead of considering the final data $u_n(S_n) = R(S_n)$, as in [16], we look at solutions u_n to (gKdV) with final data:

$$u_n(S_n) = R(S_n) + \sum_{j,\pm} \mathfrak{b}_{j,n}^\pm Z_j^\pm, \quad \text{where} \quad Z_j^\pm(t,x) = c^{\frac{1}{p-1}} Z^\pm(\sqrt{c_j}(x - c_j t - x_j)),$$

and $\mathfrak{b}_n = (\mathfrak{b}_{j,n}^\pm)_{j=1,\ldots N;\pm}$ belongs to some small neighborhood of 0 in $\mathbb{R}^{2N}$. A topological argument then allows us to select, for all n, a value of $\mathfrak{b}_n$ so that a uniform estimate of $\|u_n(t) - R(t)\|_{H^1}$ holds on some interval $[T_0, S_n]$.

2. Proof of Theorem 1

2.1. Preliminary results

Consider the operator

$$Lv = -v_{xx} - pQ^{p-1}v + v.$$

For $p > 5$, it is known from the work of Pego and Weinstein [26] that the operator $\partial_x L$ has two eigenfunctions Y^+ and Y^- (related by $Y^-(x) = Y^+(-x)$) such that

$$(LY^\pm)_x = \pm e_0 Y^\pm, \text{ where } e_0 > 0.$$

In contrast with the (NLS) case (see references in section 3.2), the existence of $Y^\pm$ is not obtained by variational arguments, but by sharp ODE techniques. Note that [26] provides a complete description of the spectrum of $\partial_x L$

in L^2 for any $p>1$; in particular, the existence of such eigenfunctions related to $\pm e_0$ with $e_0>0$ is proved to be equivalent to super criticality (i.e. $p>5$ in the present case).

Next, we observe that $Z^\pm = LY^\pm$ are eigenfunctions of $L\partial_x$ (adjoint to $-\partial_x L$). Indeed,

$$L(Z_x^\pm) = \pm e_0 Z^\pm.$$

The functions $Z^\pm$ are normalized so that $\|Z^\pm\|_{L^2}=1$. Moreover, we recall from [26] (standard ODE arguments) that $Z^\pm, Y^\pm \in \mathscr{S}(\mathbb{R})$ and have exponential decay, along with their derivatives. Let $\eta_0>0$ such that

$$\forall x\in\mathbb{R},\qquad |Z^+(x)|+|Z^-(x)|+|Z_x^+(x)|+|Z_x^-(x)|\le Ce^{-\eta_0|x|}.$$

Following [5] (concerning the (NLS) case), we claim the following coercivity property of L (for $f,g\in L^2$, $(f,g)=\int fg$ denotes the scalar product in L^2).

Lemma 1. *There exists* $\lambda>0$ *such that*

$$\forall v\in H^1,\quad (Lv,v)\ge \lambda\|v\|_{H^1}^2-\frac{1}{\lambda}\left((v,Z^+)^2+(v,Z^-)^2+(v,Q_x)^2\right).$$

Proof. The proof is similar to the one of [5, Lemma 5.2]. It is given here for the reader's convenience.

First we recall the following well-known result.

Claim. There exists $\nu>0$ such that

$$\forall v\in H^1,\quad (Lv,v)\ge \nu\|v\|_{H^1}^2-\frac{1}{\nu}\left((v,Q_x)^2+(v,Q^{\frac{p+1}{2}})^2\right). \tag{2.1}$$

Indeed, Q_x and $Q^{\frac{p+1}{2}}$ are two eigenfunctions for L, namely

$$LQ_x=0\quad\text{ and }\quad LQ^{\frac{p+1}{2}}=\mu_0 Q^{\frac{p+1}{2}},\text{ where } \mu_0=1-\left(\frac{p+1}{2}\right)^2<0.$$

The claim then follows from Sturm-Liouville theory.

To prove the Lemma, it suffices to show that

$$\text{if } (v,Z^+)=(v,Z^-)=(v,Q_x)=0 \text{ then } (Lv,v)\ge\lambda\|v\|_{H^1}^2. \tag{2.2}$$

Let v satisfy the orthogonality conditions in (2.2) and decompose the functions v, $Y^\pm$ L^2 orthogonaly in $\mathrm{Span}(Q_x,Q^{\frac{p+1}{2}})^\perp$ and $\mathrm{Span}(Q_x,Q^{\frac{p+1}{2}})$ (the notation $\perp$ is here related to L^2 orthogonality).

$$v=w+\alpha Q^{\frac{p+1}{2}},\quad Y^+=y^++\beta Q_x+\gamma Q^{\frac{p+1}{2}},\quad Y^-=y^-+\delta Q_x+\eta Q^{\frac{p+1}{2}}.$$

By the symmetry $Y^+(-x)=Y^-(x)$ and uniqueness of the orthogonal decomposition, note that $\delta=-\beta$, $\eta=\gamma$ and $y^+(-x)=y^-(x)$.

We claim that the functions y^+, y^- are linearly independent. Indeed, decompose into even and odd parts

$$y^+ = y^e + y^o, \quad y^- = y^e - y^o.$$

As $Y^\pm$, the functions $y^\pm$, y^e and y^o have exponential decay, along with their derivatives.

Let us prove that $y^e \neq 0$ and $y^o \neq 0$; we observe from $(LY^+)_x = e_0 Y^+$ that

$$(Ly^e)_x + (Ly^o)_x - \mu_0\gamma(Q^{\frac{p+1}{2}})_x = e_0 y^e + e_0 y^o + e_0\beta Q_x + e_0\gamma Q^{\frac{p+1}{2}}, \text{ i.e.}$$
$$(Ly^e)_x = e_0(y^o + \beta Q_x) - \mu_0\gamma(Q^{\frac{p+1}{2}})_x, \quad (Ly^o)_x = e_0(y^e + \gamma Q^{\frac{p+1}{2}}).$$

If $y^o = 0$, then $y^e = 0$ and $\gamma = 0$, hence $\beta = 0$, and thus $Y^+ = Y^- = 0$, which is a contradiction. Now, if we assume $y^e = 0$, by $(Ly^o)_x = e_0(y^e + \gamma Q^{\frac{p+1}{2}})$ and $\int Q^{\frac{p+1}{2}} \neq 0$, we obtain $\gamma = 0$. Thus, from $0 = (Ly^e)_x = e_0(y^o + \beta Q_x)$, we get $y^o = 0$ and $\beta = 0$, so that $Y^+ = Y^- = 0$, a contradiction. From the property $y^e \neq 0$ and $y^o \neq 0$, one deduces that $ay^+ + by^- = 0$ implies $a = b = 0$, hence y^+ and y^- are linearly independent.

We now go back to the proof of coercivity. Note that

$$(LY^\pm, Y^\pm) = \pm e_0^{-1}(LY^\pm, (LY^\pm)_x) = 0.$$

We compute

$$\begin{aligned}
0 &= (v, Z^+) = (v, LY^+) = (Lv, Y^+) = (Lw, y^+) + \alpha\mu_0\gamma\|Q^{\frac{p+1}{2}}\|_{L^2}^2,\\
0 &= (v, Z^-) = (v, LY^-) = (Lv, Y^-) = (Lw, y^-) + \alpha\mu_0\gamma\|Q^{\frac{p+1}{2}}\|_{L^2}^2,\\
0 &= (LY^+, Y^+) = (Ly^+, y^+) + \gamma^2\mu_0\|Q^{\frac{p+1}{2}}\|_{L^2}^2,\\
0 &= (LY^-, Y^-) = (Ly^-, y^-) + \gamma^2\mu_0\|Q^{\frac{p+1}{2}}\|_{L^2}^2.
\end{aligned}$$

Hence

(2.3)
$$(Lv, v) = (Lw, w) + \mu_0\alpha^2\|Q^{\frac{p+1}{2}}\|_{L^2}^2 = (Lw, w) - \frac{(Lw, y^+)(Lw, y^-)}{\sqrt{(Ly^+, y^+)}\sqrt{(Ly^-, y^-)}}.$$

Consider

$$a = \sup_{\omega \in \mathrm{Span}(y^+, y^-)\setminus\{0\}} \left| \frac{(L\omega, y^+)}{\sqrt{(L\omega, \omega)(Ly^+, y^+)}} \cdot \frac{(L\omega, y^-)}{\sqrt{(L\omega, \omega)(Ly^-, y^-)}} \right|.$$

Recall $(L\cdot, \cdot)$ is positive definite on $\mathrm{Span}(Q_x, Q^{\frac{p+1}{2}})^\perp$; applying Cauchy-Schwarz inequality to each of the two terms of the product above, we find $a \leq 1$. Furthermore, if $a = 1$, there exists $\omega \neq 0$ such that these two Cauchy-Schwarz inequalities are actually equalities, but this is not possible since y^+ and y^- are independent.

Therefore, we have proved that $a < 1$. Using L orthogonal decomposition on $\mathrm{Span}(Q_x, Q^{\frac{p+1}{2}})^\perp$, we also obtain that for all $\omega \in \mathrm{Span}(Q_x, Q^{\frac{p+1}{2}})^\perp$,

$$\left|\frac{(L\omega, y^+)(L\omega, y^-)}{\sqrt{(Ly^+, y^+)}\sqrt{(Ly^-, y^-)}}\right| \le a(L\omega, \omega).$$

Hence, by (2.3) and next (2.1),

$$(Lv, v) \ge (1-a)(Lw, w) \ge \nu(1-a)\|w\|_{H^1}^2 > 0,$$

and so $(Lw, w) \ge |\mu_0|\alpha^2\|Q^{\frac{p+1}{2}}\|_{L^2}^2$.

Thus, for $C = \frac{4}{1-a}\max(\frac{1}{\nu}, \frac{1}{|\mu_0|}\|Q^{\frac{p+1}{2}}\|_{H^1}^2\|Q^{\frac{p+1}{2}}\|_{L^2}^{-2})$ we get

$$\begin{aligned} C(Lv, v) &\ge C(1-a)(Lw, w) \ge C\frac{1-a}{2}(Lw, w) + C\frac{1-a}{2}|\mu_0|\alpha^2\|Q^{\frac{p+1}{2}}\|_{L^2}^2 \\ &\ge 2\|w\|_{H^1}^2 + 2\alpha^2\|Q^{\frac{p+1}{2}}\|_{H^1}^2 \ge \|w + \alpha Q^{\frac{p+1}{2}}\|_{H^1}^2 = \|v\|_{H^1}^2. \end{aligned}$$

■

2.2. Main Proposition and proof of Theorem 1

We denote

$$(2.4) \qquad \begin{aligned} &R_j(t,x) = c_j^{\frac{1}{p-1}} Q(\sqrt{c_j}(x - c_j t - x_j)), \quad R(t,x) = \sum_{j=1}^N R_j(t,x), \\ &Z_j^\pm(t,x) = c_j^{\frac{1}{p-1}} Z^\pm(\sqrt{c_j}(x - c_j t - x_j)). \end{aligned}$$

Let $S_n \to \infty$ be a increasing sequence of time, $\mathfrak{b}_n = (\mathfrak{b}_{j,n}^\pm)_{j,\pm} \in \mathbb{R}^{2N}$ be a sequence of parameters to be determined, and let u_n be the solution to

$$(2.5) \qquad \begin{cases} u_{nt} + (u_{nxx} + u_n^p)_x = 0, \\ u_n(S_n) = R(S_n) + \displaystyle\sum_{j \in \{1,\ldots,N\},\pm} \mathfrak{b}_{j,n}^\pm Z_j^\pm(S_n). \end{cases}$$

Let

$$(2.6) \qquad \sigma_0 = \frac{1}{4}\min\left\{\eta_0\sqrt{c_1}, e_0^{2/3}c_1, c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\right\}.$$

Proposition 1. *There exist $n_0 \ge 0$, $T_0 > 0$ and $C > 0$ (independent of n) such that the following holds: For each $n \ge n_0$, there exists $\mathfrak{b}_n = (\mathfrak{b}_{j,n}^\pm)_{j,\pm} \in \mathbb{R}^{2N}$ with*

$$\left(\sum_{j,\pm} {\mathfrak{b}_{j,n}^{\pm}}^2\right)^{1/2} \le Ce^{-\sigma_0^{3/2}S_n},$$

and such that the solution u_n to (2.5) is defined on the interval $[T_0, S_n]$, and satisfies

$$\forall t \in [T_0, S_n], \quad \|u_n(t) - R(t)\|_{H^1} \le Ce^{-\sigma_0^{3/2}t}.$$

Assuming this Proposition, we now deduce the proof of Theorem 1. The proof of Proposition 1 is postponed to Section 2.3.

Proof of Theorem 1 assuming Proposition 1. It follows closely the proof of Theorem 1 in [16]. We may assume $n_0 = 0$ in Proposition 1 without loss of generality.

Step 1: Compactness argument. From Proposition 1, there exists a sequence $u_n(t)$ of solutions to (gKdV), defined on $[T_0, S_n]$ and $C_0, \sigma_0 > 0$ such that the following uniform estimates hold:

$$\forall n \in \mathbb{N},\ \forall t \in [T_0, S_n], \qquad \|u_n(t) - R(t)\|_{H^1} \le C_0 e^{-\sigma_0^{3/2} t}. \tag{2.7}$$

We claim the following compactness result on the sequence $u_n(T_0)$.

Claim.

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\ge A} u_n^2(T_0, x)dx = 0.$$

Proof. Let $\varepsilon > 0$, $T(\varepsilon) \ge T_0$ be such that $C_0 e^{-\sigma_0^{3/2} T(\varepsilon)} \le \sqrt{\varepsilon}$ and n large enough so that $S_n \ge T(\varepsilon)$. Then

$$\int (u_n(T(\varepsilon)) - R(T(\varepsilon))^2 \le \varepsilon.$$

Let $A(\varepsilon)$ be such that $\int_{|x|\ge A(\varepsilon)} R(T(\varepsilon))^2(x)dx \le \varepsilon$; we get

$$\int_{|x|\ge A(\varepsilon)} u_n^2(T(\varepsilon), x)dx \le 4\varepsilon.$$

Let $g(x) \in C^3$ be such that $g(x) = 0$ if $x \le 0$, $g(x) = 1$ if $x \ge 2$, and furthermore $0 \le g'(x) \le 1$, $|g'''(x)| \le 1$.

Recall that for $f(x) \in C^3$, we have (Kato's identity [13])

$$\frac{d}{dt}\int u_n^2 f = -3\int (u_{nx})^2 f_x + \int u_n^2 f_{xxx} + \frac{2p}{p+1}\int u_n^{p+1} f_x.$$

For $C(\varepsilon) > 1$ to be determined later, we thus have:

$$\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) = -\frac{3}{C(\varepsilon)}\int (u_{nx})^2 g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)$$
$$+ \frac{1}{C(\varepsilon)^3}\int u_n^2 g'''\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) + \frac{2p}{(p+1)C(\varepsilon)}\int u_n^{p+1} g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right).$$

For $t \geq T_0 \geq 0$, u_n satisfies $\|u_n(t)\|_{H^1} \leq C_0 + \sum_{j=1}^N \|Q_{c_j}\|_{H^1} \leq C^0$, and so:

$$\begin{aligned}\left|\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| & \\ &\leq \frac{1}{C(\varepsilon)}\left(3\int {u_n}_x^2(t) + \int u_n^2(t) + \frac{2p}{p+1}\|u_n\|_{L^\infty}^{p-1}\int u_n^2(t)\right)\\ &\leq \frac{1}{C(\varepsilon)}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right).\end{aligned}$$

Now choose $C(\varepsilon) = \max\left\{1, \frac{T(\varepsilon)-T_0}{\varepsilon}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right)\right\}$, so that

$$\left|\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{\varepsilon}{T(\varepsilon)-T_0}.$$

By integration on $[T_0, T(\varepsilon)]$:

$$\int_{x\geq 2C(\varepsilon)+A(\varepsilon)} u_n^2(T_0,x) \leq \int u_n^2(T_0,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) \leq 5\varepsilon.$$

Now considering $\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{-A(\varepsilon)-x}{C(\varepsilon)}\right)$, we get in a similar way

$$\int_{x\leq -2C(\varepsilon)-A(\varepsilon)} u_n^2(T_0,x) \leq 5\varepsilon.$$

Therefore, setting $A_\varepsilon = 2C(\varepsilon/10) + A(\varepsilon/10)$, we obtain:

$$\forall n\in\mathbb{N}, \qquad \int_{|x|\geq A_\varepsilon} u_n^2(T_0,x) \leq \varepsilon.$$

∎

By (2.7), the sequence $(u_n(T_0))$ is bounded in H^1, thus we can extract a subsequence (still denoted by (u_n)) which converges weakly to $\varphi_0 \in H^1(\mathbb{R})$. The previous compactness result ensures that the convergence is strong in $L^2(\mathbb{R})$. Indeed, let $\varepsilon > 0$ and let A be such that $\int_{|x|\geq A}\varphi_0^2(x)dx \leq \varepsilon$ and

$$\forall n\in\mathbb{N}, \qquad \int_{|x|\geq A} u_n^2(T_0,x) \leq \varepsilon.$$

By the compact embedding $H^1([-A,A]) \to L^2([-A,A])$, $\int_{|x|\leq A}|u_n(T_0,x) - \varphi_0(x)|^2dx \to 0$ as $n\to+\infty$. We thus derive that

$$\limsup_{n\in\mathbb{N}} \|u_n(T_0)-\varphi_0\|_{L^2(\mathbb{R})}^2 \leq 4\varepsilon.$$

Since this is true for all $\varepsilon > 0$, $u_n(T_0) \to \varphi_0$ in $L^2(\mathbb{R})$ as $n\to+\infty$. By interpolation, $u_n(T_0)$ converges strongly to φ_0 in H^s for all $s\in[0,1)$.

Step 2. Construction of the multi-soliton u^*. Denote $u^*(t)$ the solution to

$$\begin{cases} u^*_t + (u^*_{xx} + (u^*)^p)_x = 0, \\ u^*(T_0) = \varphi_0. \end{cases}$$

Due to [14], the Cauchy problem for (gKdV) is locally well-posed in H^s for $s \geq 1/2$: we will work in $H^{1/2}$ (which is not a critical space) and H^1. Let $u^* \in \mathcal{C}([T_0, T^*), H^1)$ be the maximal solution to (gKdV). Recall the blow up alternative: either $T^* = +\infty$ or $T^* < \infty$ and then $\|u^*(t)\|_{H^1} \to \infty$ as $t \uparrow T^*$.

Since the flow is continuous in $H^{1/2}$, for any $t \in [T_0, T^*)$, $u_n(t)$ is defined for n large enough and $u_n(t) \to u^*(t)$ in $H^{1/2}$ as $n \to +\infty$. By the uniform H^1 bound, we also obtain $u_n(t) \rightharpoonup u^*(t)$ in H^1-weak. Hence, using Proposition 1,

$$\forall t \in [T_0, T^*), \qquad \|u^*(t) - R(t)\|_{H^1} \leq \liminf_{n\to\infty} \|u_n(t) - R(t)\|_{H^1} \leq Ce^{-\sigma_0^{3/2} t}.$$

In particular, we deduce that

$$\forall t \in [T_0, T^*), \quad \|u^*(t)\|_{H^1} \leq Ce^{-\sigma_0^{3/2} t} + \|R(t)\|_{H^1} \leq C + \sum_{j=1}^{N} \|Q_{c_j}\|_{H^1}.$$

The blow-up alternative implies $T^* = \infty$. Hence $u^* \in \mathcal{C}([T_0, \infty), H^1)$ and moreover $\|u^*(t) - R(t)\|_{H^1} \leq Ce^{-\sigma_0^{3/2} t}$ for all $t \geq T_0$. ■

2.3. Proof of Proposition 1

The proof proceeds in several steps. For the sake of simplicity, we will drop the index n for the rest of this section (except for S_n). We possibly drop the first terms of the sequence S_n, so that for all n, S_n is large enough for our purposes.

Step 1. Choice of a set of initial data.

Lemma 2 (Modulation for time independent function)**.** *Let $0 < c_1 < \ldots < c_N$. There exist $C, \varepsilon > 0$ such that the following holds. Given $(\alpha_i)_{i=1,\ldots N}$ such* $\min\{|\alpha_i - \alpha_j| \ \ i \neq j\} \geq 1/\varepsilon$, *if $u(x) \in L^2$ is such that*

$$\left\| u - \sum_{j=1}^{N} Q_{c_j}(x - \alpha_j) \right\|_{L^2} \leq \varepsilon,$$

then there exist modulation parameters $\mathbf{y} = (y_j)_{j=1,\ldots,N}$ such that setting

$$v = u - \sum_{j=1}^{N} Q_{c_j}(x - \alpha_j - y_j),$$

the following holds

$$\|v\|_{L^2} + \sum_{j=1}^{N} |y_j| \le C \Big\| u - \sum_{j=1}^{N} Q_{c_j}(x-\alpha_j) \Big\|_{L^2}, \tag{2.8}$$

$$\text{and } \forall j=1,\dots,N, \quad \int v(x)(Q_{c_j})_x(x-\alpha_j-y_j)dx = 0. \tag{2.9}$$

Furthermore, $u \mapsto (v, \mathbf{y})$ *is a smooth diffeomorphism.*

Notation. For $\mathfrak{b}$ small, from (2.5) and continuity in H^1, $u(t)$ is defined and modulable (in the sense of the previous lemma) for t close to S_n. As long as $u(t)$ is modulable around $R(t)$, we denote by $\mathbf{y}(t) = (y_j(t))_{j=1,\dots,N}$ the modulation parameters, and

$$\tilde{R}_j(t) = R_j(t, x - y_j(t)), \quad \tilde{R}(t) = \sum_{j=1}^{N} \tilde{R}_j(t), \quad \tilde{Z}_j^{\pm}(t,x) = Z_j^{\pm}(t, x - y_j(t)),$$

$$v(t) = u(t) - \tilde{R}(t) \quad \text{so that} \quad \forall j=1,\dots,N, \quad \int v(t)(\tilde{R}_j)_x(t) = 0,$$

$$\mathbf{a}^{\pm}(t) = (a_j^{\pm}(t))_{j=1,\dots,N}, \quad \text{where} \quad a_j^{\pm} = \int v(t,x)\tilde{Z}_j^{\pm}(t,x)dx.$$

We also set $L_j = -\partial_{xx} - p\tilde{R}_j^{p-1}(t) + c_j$.

We consider $\mathbb{R}^N$ equipped with the ℓ^2 norm. We denote by $B_{\mathcal{B}}(P,r)$ the *closed* ball of the Banach space $\mathcal{B}$, centered at P and of radius $r \ge 0$. If $P = 0$, we simply write $B_{\mathcal{B}}(r)$. Finally, $\mathbb{S}_{\mathbb{R}^N}(r)$ denotes the sphere of radius r in $\mathbb{R}^N$.

In view of Lemma 1, we have to control the functions $\mathbf{a}^{\pm}(t)$ on some time interval $[T_0, S_n]$. Since Z^+ and Z^- are not orthogonal and because of the interactions between the various solitons, the values of $\mathbf{a}^{\pm}(S_n)$ are not directly related to $\mathfrak{b}$. The next lemma allows us to establish a one-to-one mapping between the choice of $\mathfrak{b}$ in (2.5) and the suitable constraints $\mathbf{a}^+(S_n) = \mathfrak{a}^+$, $\mathbf{a}^-(S_n) = 0$, for any choice of $\mathfrak{a}^+$.

Lemma 3 (Modulated final data)**.** *There exists* $C > 0$ *(independent of* n*) such that for all* $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2} S_n})$ *there exists a unique* $\mathfrak{b}$ *with* $\|\mathfrak{b}\| \le C\|\mathfrak{a}^+\|$ *and such that the modulation* $(v(S_n), \mathbf{y}(S_n))$ *of* $u(S_n)$ *satisfies*

$$\mathbf{a}^+(S_n) = \mathfrak{a}^+ \quad \text{and} \quad \mathbf{a}^-(S_n) = 0.$$

Proof. See Appendix. ■

Let T_0 to be determined later in the proof, independent of n. Let $\mathfrak{a}^+$ to be chosen, $\mathfrak{b}$ be given by Lemma 3 and let u be the corresponding solution of (2.5). We now define the maximal time interval $[T(\mathfrak{a}^+), S_n]$ on which suitable exponential estimates hold.

Definition 1. Let $T(\mathfrak{a}^+)$ be the infimum of $T \geq T_0$ such that the following properties hold for all $t \in [T, S_n]$:

- Closeness to $R(t)$:
$$\|u(t) - R(t)\|_{H^1} \leq \varepsilon.$$
In particular, this ensures that $u(t)$ is modulable around $R(t)$ in the sense of Lemma 2.

- Estimates on the modulation parameters:
$$e^{\sigma_0^{3/2} t} v(t) \in B_{H^1}(1), \quad e^{\sigma_0^{3/2} t} \mathbf{y}(t) \in B_{\mathbb{R}^N}(1),$$
$$e^{(3/2)\sigma_0^{3/2} t} \mathbf{a}^-(t) \in B_{\mathbb{R}^N}(1), \quad e^{(3/2)\sigma_0^{3/2} t} \mathbf{a}^+(t) \in B_{\mathbb{R}^N}(1).$$

Observe that Proposition 1 is proved if for all n, we can find $\mathfrak{a}^+$ such that $T(\mathfrak{a}^+) = T_0$. The rest of the proof is devoted to prove the existence of such a value of $\mathfrak{a}^+$.

We claim the following preliminary results on the modulation parameters of $u(t)$.

Claim.

$$v_t + \left(v_{xx} + (v + \tilde{R})^p - \sum_{j=1}^N \tilde{R}_j^p \right)_x - \sum_{j=1}^N \frac{dy_j}{dt} \tilde{R}_{j\,x} = 0, \tag{2.10}$$

$$\forall t \in [T(\mathfrak{a}^+), S_n], \qquad \left\| \frac{d\mathbf{y}}{dt}(t) \right\| \leq C\|v(t)\|_{L^2} + Ce^{-2\sigma_0^{3/2} t}. \tag{2.11}$$

(2.12)
$$\forall t \in [T(\mathfrak{a}^+), S_n], \ \forall j, \qquad \left| \frac{da_j^\pm}{dt}(t) \pm e_0 c_j^{3/2} a_j^\pm(t) \right| \leq C\|v(t)\|_{H^1}^2 + Ce^{-3\sigma_0^{3/2} t}.$$

Proof. The equation of $v(t)$ is obtained by elementary computations from the equation of $u(t)$. Taking the scalar product of this equation with $\tilde{R}_{j\,x}$, we see that $y_j(t)$ satisfy

$$\frac{dy_j}{dt} \|Q_{c_j\,x}\|_{L^2}^2 = \int \left(v_{xx} + (v + \tilde{R})^p - \sum_{k=1}^N \tilde{R}_k^p \right)_x \tilde{R}_{j\,x} - \left(c_j + \frac{dy_j}{dt} \right) \int v \tilde{R}_{j\,xx}.$$

For $t \geq T_0$ large enough $\left| \int v \tilde{R}_{j\,xx} \right| \leq \|Q_{c_j\,xx}\|_{L^2} \|v(t)\|_{L^2} \leq Ce^{-\sigma_0^{3/2} t} \leq \frac{1}{2}\|Q_{c_j\,x}\|_{L^2}^2$. Using integration by parts to have all the derivatives on $\tilde{R}_{j\,x}$ and using Cauchy-Schwarz inequality, we get (2.11).

Now, we prove (2.12). First, note that $\int \tilde{R}_{j\,x} \tilde{Z}_j^\pm = 0$ follows from

$$\int Q_x Z^\pm = \pm e_0^{-1} \int Q_x L(Z_x^\pm) = \pm e_0^{-1} \int L(Q_x) Z_x^\pm = 0. \tag{2.13}$$

Using the equation of $v(t)$ and next the equations of $Z^\pm$,

$$\begin{aligned}
\frac{da_j^\pm}{dt}(t) &= \int v_t \tilde{Z}_j^\pm + \int v \tilde{Z}_{j\ t}^\pm \\
&= -\int (v_{xx} + (v+\tilde{R})^p - \sum_k \tilde{R}_k^p)_x \tilde{Z}_j^\pm \\
&\qquad + \sum_k \frac{dy_k}{dt} \int \tilde{R}_{kx} \tilde{Z}_j^\pm - (c_j + \frac{dy_j}{dt}) \int v \tilde{Z}_{j\ x}^\pm \\
&= -\int (v_{xx} + p\tilde{R}_j^{p-1} v)_x \tilde{Z}_j^\pm - c_j \int v \tilde{Z}_{j\ x}^\pm \\
&\qquad - \int ((v+\tilde{R})^p - \sum_k \tilde{R}_k^p - p\tilde{R}_j^{p-1} v)_x \tilde{Z}_j^\pm \\
&\qquad + \sum_{k\neq j} \frac{dy_k}{dt} \int \tilde{R}_{kx} \tilde{Z}_j^\pm - \frac{dy_j}{dt} \int v \tilde{Z}_{j\ x}^\pm \\
&= -\int v L_j(\tilde{Z}_{j\ x}^\pm) + \int ((v+\tilde{R})^p - \sum_k \tilde{R}_k^p - p\tilde{R}_j^{p-1} v) \tilde{Z}_{j\ x}^\pm \\
&\qquad + \sum_{k\neq j} \frac{dy_k}{dt} \int \tilde{R}_{kx} \tilde{Z}_j^\pm - \frac{dy_j}{dt} \int v \tilde{Z}_{j\ x}^\pm \\
&= \mp e_0 c_j^{3/2} a_j^\pm(t) + \int ((v+\tilde{R})^p - \sum_k \tilde{R}_k^p - p\tilde{R}_j^{p-1} v) \tilde{Z}_{j\ x}^\pm \\
&\qquad + \sum_{k\neq j} \frac{dy_k}{dt} \int \tilde{R}_{kx} \tilde{Z}_j^\pm - \frac{dy_j}{dt} \int v \tilde{Z}_{j\ x}^\pm .
\end{aligned}$$

Using (2.6), for $k \neq j$,

$$\begin{aligned}
|\tilde{R}_k(t,x)|(|\tilde{Z}_j^\pm(t,x)| + |\tilde{Z}_{j\ x}^\pm(t,x)|) &\leq C e^{-2\sqrt{\sigma_0}(|x-c_kt|+|x-c_jt|)} \\
&\leq C e^{-3\sigma_0^{3/2} t} e^{-\sqrt{\sigma_0}|x-c_jt|}.
\end{aligned} \tag{2.14}$$

Hence we have

$$\left| \int (|v+\tilde{R}|^{p-1}(v+\tilde{R}) - \sum_{k=1}^N \tilde{R}_k^p - p\tilde{R}_j^{p-1} v) \tilde{Z}_{j\ x}^\pm \right| \leq C\|v(t)\|_{H^1}^2 + C e^{-3\sigma_0^{3/2} t},$$

$$\begin{aligned}
\left| \sum_{k\neq j} \frac{dy_k}{dt} \int \tilde{R}_{kx} \tilde{Z}_j^\pm \right| &\leq C e^{-3\sigma_0^{3/2} t} \|v(t)\|_{H^1} + C e^{-4\sigma_0^{3/2} t} \\
&\leq C\|v(t)\|_{H^1}^2 + C e^{-4\sigma_0^{3/2} t}.
\end{aligned}$$

The term $\frac{dy_j}{dt} \int v \tilde{Z}_{j\ x}^\pm$ is controlled using (2.11). ■

Step 2. Conditional stability of v and $\mathbf{y}$ under the control of $\mathbf{a}^\pm$.

We claim the following improvement of the estimates for $v(t)$ and $\mathbf{y}$ on $[T(\mathfrak{a}^+), S_n]$.

Lemma 4 (Control of v and $\mathbf{y}$)**.** *For T_0 large enough (independent of n) and for all $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$, the following holds*

$$\forall t \in [T(\mathfrak{a}^+), S_n], \quad \|u(t) - R(t)\|_{H^1} \leq Ce^{-\sigma_0^{3/2}t} \leq \varepsilon_0/2, \tag{2.15}$$

$$e^{\sigma_0^{3/2}t}\|v(t)\|_{H^1} \leq 1/2, \quad e^{\sigma_0^{3/2}t}\|\mathbf{y}(t)\| \leq 1/2. \tag{2.16}$$

The proof of Lemma 4 is postponed to the end of this section. It is very similar to the proofs in the subcritical case (see [16] or [20]).

Step 3. Control of $\mathbf{a}^-(t)$.

Lemma 5 (Control of $\mathbf{a}^-(t)$)**.** *For T_0 large enough (independent of n) and for all $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$, the following holds*

$$\forall t \in [T(\mathfrak{a}^+), S_n], \quad e^{(3/2)\sigma_0^{3/2}t}\|\mathbf{a}^-(t)\| \leq 1/2.$$

Proof. It follows from (2.12), (2.16) and $a_j^-(S_n) = 0$ that for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$|a_j^-(t)| \leq Ce^{e_0 c_j^{3/2}t} \int_t^{S_n} e^{-e_0 c_j^{3/2}s}\left(e^{-2\sigma_0^{3/2}s} + e^{-3\sigma_0^{3/2}s}\right) ds \leq Ce^{-2\sigma_0^{3/2}t}.$$

Hence, for T_0 large enough,

$$\forall t \in [T(\mathfrak{a}^+), S_n], \quad \|\mathbf{a}^-(t)\| \leq Ce^{-2\sigma_0^{3/2}t} \leq \frac{1}{2}\, e^{-(3/2)\sigma_0^{3/2}t}.$$

■

Step 4. Control of $\mathbf{a}^+(t)$ by a topogical argument.

Finally we turn to the control of $\mathbf{a}^+(t)$ which will provide us with a suitable value of $\mathfrak{a}^+$. This is the new key argument of this paper.

Lemma 6 (Control of $\mathbf{a}^+(t)$)**.** *For T_0 large enough, there exists*

$$\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n}) \quad \textit{such that} \quad T(\mathfrak{a}^+) = T_0.$$

Proof. We argue by contradiction.

Assume that for all $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$, one has $T(\mathfrak{a}^+) > T_0$. From Lemmas 4 and 5

$$u(T(\mathfrak{a}^+)) - R(T(\mathfrak{a}^+)) \in B_{H^1}(\varepsilon_0/2), \quad e^{\sigma_0^{3/2}T(\mathfrak{a}^+)}v(T(\mathfrak{a}^+)) \in B_{H^1}(1/2),$$
$$e^{\sigma_0^{3/2}T(\mathfrak{a}^+)}\mathbf{y}(T(\mathfrak{a}^+)) \in B_{\mathbb{R}^N}(1/2), \quad e^{(3/2)\sigma_0^{3/2}T(\mathfrak{a}^+)}\mathbf{a}^-(T(\mathfrak{a}^+)) \in B_{\mathbb{R}^N}(1/2).$$

Hence by definition of $T(\mathfrak{a}^+)$ and continuity of the flow, one must have

$$e^{(3/2)\sigma_0^{3/2}T(\mathfrak{a}^+)}\mathbf{a}^+(T(\mathfrak{a}^+)) \in \mathbb{S}_{\mathbb{R}^N}(1). \tag{2.17}$$

Let $T < T(\mathfrak{a}^+)$ be close enough to $T(\mathfrak{a}^+)$ so that the solution $u(t)$ and its modulation are well-defined on $[T, S_n]$. For $t \in [T, S_n]$, let

$$\mathcal{N}(\mathfrak{a}^+(t)) = \mathcal{N}(t) = \left\| e^{(3/2)\sigma_0^{3/2}t}\mathbf{a}^+(t)\right\|^2. \tag{2.18}$$

Then, by (2.12) and (2.16), we have

$$\begin{aligned}\frac{d\mathcal{N}}{dt}(t) &= e^{3\sigma_0^{3/2}t}\sum_j \Big(3\sigma_0^{3/2} - 2\frac{da_j^+}{dt}\Big)a_j^+ \\ &= e^{3\sigma_0^{3/2}t}\sum_j \Big(3\sigma_0^{3/2} - 2e_0 c_j^{3/2}\Big)|a_j^+|^2 \\ &\qquad + O\Big(e^{3\sigma_0^{3/2}t}\|\mathbf{a}^+\|(\|v(t)\|_{H^1}^2 + e^{-3\sigma_0^{3/2}t})\Big).\end{aligned}$$

In view of the definition of σ_0 (see (2.6)), for all j,

$$2e_0 c_j^{3/2} - 3\sigma_0^{3/2} \geq e_0 c_1^{3/2} \geq 4e_0\sigma_0^{3/2}.$$

For $t \in [T(\mathfrak{a}^+), S_n]$, due to the bound on $\|v\|_{H^1}$, we have

$$e^{3\sigma_0^{3/2}t}\|\mathbf{a}^+\|(\|v(t)\|_{H^1}^2 + e^{-3\sigma_0^{3/2}t}) \leq Ce^{-(1/2)\sigma_0^{3/2}t}\sqrt{\mathcal{N}(t)}.$$

Hence we get

$$\frac{d\mathcal{N}}{dt}(t) \leq -4e_0\sigma_0^{3/2}\mathcal{N}(t) + Ce^{-(1/2)\sigma_0^{3/2}t}\sqrt{\mathcal{N}(t)}.$$

We consider this estimate at $t = T(\mathfrak{a}^+) \geq T_0$ (so large that $Ce^{-(1/2)\sigma_0^{3/2}t} \leq 2e_0\sigma_0^{3/2}$), and using $\mathcal{N}(T(\mathfrak{a}^+)) = 1$, we get

$$\forall \mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n}), \quad \frac{d\mathcal{N}}{dt}(T(\mathfrak{a}^+)) \leq -2e_0\sigma_0^{3/2}. \tag{2.19}$$

From (2.19), a standard argument says that the map $\mathfrak{a}^+ \mapsto T(\mathfrak{a}^+)$ is continuous. Indeed, by (2.19), for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{N}(T(\mathfrak{a}^+) - \varepsilon) > 1 + \delta$ and, for all $t \in [T(\mathfrak{a}^+) + \varepsilon, S_n]$ (possibly empty), $\mathcal{N}(t) < 1 - \delta$. By continuity of the flow of the (gKdV) equation, it follows that there exist $\eta > 0$ such that for all $\|\tilde{\mathfrak{a}}^+ - \mathfrak{a}^+\| \leq \eta$, the corresponding $\tilde{\mathbf{a}}^+(t)$ satisfies $|\mathcal{N}(\tilde{\mathbf{a}}^+(t)) - \mathcal{N}(\mathbf{a}^+(t))| \leq \delta/2$ for all $t \in [T(\mathfrak{a}^+) - \varepsilon, S_n]$. In particular, $T(\mathfrak{a}^+) - \varepsilon \leq T(\tilde{\mathfrak{a}}^+) \leq T(\mathfrak{a}^+) + \varepsilon$.

Now, we consider the continuous map

$$\begin{aligned}\mathcal{M}: \quad B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n}) &\to \mathbb{S}_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n}), \\ \mathfrak{a}^+ &\mapsto e^{-(3/2)\sigma_0^{3/2}(S_n - T(\mathfrak{a}^+))}\mathbf{a}^+(T(\mathfrak{a}^+)).\end{aligned}$$

Let $\mathfrak{a}^+ \in \mathbb{S}_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$. From (2.19), it follows that $T(\mathfrak{a}^+) = S_n$ and so $\mathcal{M}(\mathfrak{a}^+) = \mathfrak{a}^+$, which means that $\mathcal{M}$ restricted to $\mathbb{S}_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$ is the identity. But the existence of such a map $\mathcal{M}$ contradicts Brouwer's fixed point theorem. In conclusion, there exists $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(e^{-(3/2)\sigma_0^{3/2}S_n})$ such that $T(\mathfrak{a}^+) = T_0$. ∎

The end of this section is devoted to the proof of Lemma 4.

Proof of Lemma 4. Let ψ be defined by

$$\psi(x) = 0 \text{ for } x \leq -1, \quad \psi(x) = 1 \text{ for } x \geq 1,$$

$$\psi(x) = \frac{1}{c_0} \int_{-1}^{x} e^{-\frac{1}{1-y^2}} dy \quad \text{ for } x \in (-1,1),$$

where $c_0 = \int_{-1}^{1} e^{-\frac{1}{1-y^2}} dy$. Then $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is non-decreasing and $0 \leq \psi \leq 1$. Define $m_j(t) = \frac{1}{2}((c_j + c_{j+1})t + x_j + x_{j+1})$ for $j = 1, \dots, N-1$ and

$$\text{for } j = 1, \dots, N-1, \ \psi_j(t,x) = \psi\left(\frac{1}{\sqrt{t}}(x - m_j(t))\right), \quad \psi_N(t) = 1;$$

$$\text{for } j = 2, \dots, N \quad \phi_j = \psi_j - \psi_{j-1}, \quad \phi_1 = \psi_1;$$

$$M_j(t) = \int u^2(t)\phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1}\right)(t)\phi_j(t). \tag{2.20}$$

By the decay properties of $\tilde{R}_k(t)$, and the support properties of ϕ_j and its derivatives, we have

$$\forall j \neq k, \quad (|\tilde{R}_k(t,x)| + |(\tilde{R}_k)_x(t,x)|)|\phi_j(t,x)| \leq Ce^{-3\sigma_0^{3/2}t}e^{-\sqrt{\sigma_0}|x-y_j(t)|}, \tag{2.21}$$

$$\forall j, \quad (|\tilde{R}_j(t,x)| + |(\tilde{R}_j)_x(t,x)|)|\phi_j(t,x) - 1| \leq Ce^{-3\sigma_0^{3/2}t}e^{-\sqrt{\sigma_0}|x-y_j(t)|}, \tag{2.22}$$

$$\forall j,k, \quad |\tilde{R}_k(t,x)|(|\phi_{j_x}(t,x)| + |\phi_{j_{xxx}}(t,x)| + |\phi_{j_t}(t,x)|) \leq \leq \frac{C}{\sqrt{t}}e^{-3\sigma_0^{3/2}t}e^{-\sqrt{\sigma_0}|x-y_k(t)|}, \tag{2.23}$$

$$\forall j, \quad |\phi_{j_x}(t,x)| + |\phi_{j_{xxx}}(t,x)| + |\phi_{j_t}(t,x)| \leq \frac{C}{\sqrt{t}}. \tag{2.24}$$

We begin with some technical claims.

Claim.

$$\left|\frac{d}{dt}M_j(t)\right| \leq \frac{C}{\sqrt{t}}\|v(t)\|_{H^1}^2 + Ce^{-3\sigma_0^{3/2}t}, \tag{2.25}$$

$$\left|\frac{d}{dt}\sum_{j=1}^{N}\left(E_j(t) + \frac{c_j}{2}M_j(t)\right)\right| \leq \frac{C}{\sqrt{t}}\|v(t)\|_{H^1}^2 + Ce^{-3\sigma_0^{3/2}t}. \tag{2.26}$$

Proof. By direct computations,

$$\frac{d}{dt}\int u^2\phi_j = -3\int u_x^2\phi_{j_x} + \int u^2\left(\phi_{j_{xxx}} + \phi_{j_t}\right) + \frac{2p}{p+1}\int u^{p+1}\phi_{j_x}.$$

Thus, expanding $u(t) = \tilde{R}(t) + v(t)$, and using (2.23) and (2.24), the first two integrals are estimated as desired. For the last term it suffices to observe additionally that $\|u(t)\|_{L^\infty} \leq C(\|v(t)\|_{H^1} + \|\tilde{R}(t)\|_{H^1}) \leq C$. This proves (2.25).

Estimate (2.26) is a consequence of (2.25), the conservation of energy and $\sum_{j=1}^N \phi_j = 1$. ■

Claim.

$$\left|\Big(E_j(t) + \frac{c_j}{2}M_j(t)\Big) - \Big(E(Q_{c_j}) + \frac{c_j}{2}\int Q_{c_j}^2\Big) - \frac{1}{2}H_j(t)\right| \leq Ce^{-3\sigma_0^{3/2}t} + Ce^{-\sigma_0^{3/2}t}\|v(t)\|_{L^2}^2, \tag{2.27}$$

where $H_j(t) = \int (v_x^2(t) - p\tilde{R}_j^{p-1}(t)v^2(t) + c_j v^2(t))\phi_j(t)$.

Proof. First, we claim

$$\left|M_j(t) - \Big(\int Q_{c_j}^2 + 2\int v(t)\tilde{R}_j(t) + \int v^2(t)\phi_j(t)\Big)\right| \leq Ce^{-3\sigma_0^{3/2}t}, \tag{2.28}$$

$$\left|E_j(t) - E(Q_{c_j}) - \Big(\frac{1}{2}\int (v_x^2(t) - p\tilde{R}_j^{p-1}(t)v^2(t))\phi_j(t) - c_j\int v(t)\tilde{R}_j(t)\Big)\right| \leq Ce^{-3\sigma_0^{3/2}t} + Ce^{-\sigma_0^{3/2}t}\|v(t)\|_{L^2}^2, \tag{2.29}$$

Indeed, expanding $u(t) = v(t) + \sum_k \tilde{R}_k(t)$ in $M_j(t)$ and $E_j(t)$, we get

$$M_j(t) = \int u^2\phi_j(t) = \int \left(v^2 + 2v\tilde{R} + \Big(\sum_{k=1}^N \tilde{R}_k\Big)^2\right)\phi_j(t),$$

$$\begin{aligned} E_j(t) &= \int \left(\frac{1}{2}(v_x^2 + 2v_x\tilde{R}_x + \tilde{R}_x^2) - \frac{1}{p+1}(v+\tilde{R})^{p+1}\right)\phi_j(t) \\ &= \int \left(\frac{1}{2}(v_x^2 - p\tilde{R}^{p-1}v^2)\right)\phi_j + \int \left(\frac{1}{2}\tilde{R}_x^2 - \frac{1}{p+1}\tilde{R}^{p+1}\right)\phi_j(t) \\ &\quad - \int v(\tilde{R}_{xx} + \tilde{R}^p)\phi_j - \int \tilde{R}_x v\phi_{j_x} \\ &\quad + \frac{1}{p+1}\int \left((-(v+\tilde{R})^{p+1} + \tilde{R}^{p+1}) + (p+1)v\tilde{R}^p + \frac{p(p+1)}{2}\tilde{R}^{p-1}v^2\right)\phi_j(t). \end{aligned}$$

Now, estimates (2.21) and (2.22) give (for $k \neq j$)

$$\left|\int \tilde{R}_j^2\phi_j(t) - \int Q_{c_j}^2\right| + \int \tilde{R}_k^2\phi_j(t) + \left|\int \left(\frac{1}{2}\tilde{R}_x^2 - \frac{1}{p+1}\tilde{R}^{p+1}\right)\phi_j(t) - E(Q_{c_j})\right| \leq Ce^{-3\sigma_0^{3/2}t}.$$

By $Q_{xx} + Q^p = Q$, we have

$$\int v(t)(\tilde{R}_{xx} + \tilde{R}^p)\phi_j = c_j \int v(t)\tilde{R}_j(t) + O(e^{-3\sigma_0^{3/2}t}).$$

Using also (2.23) and for $k \geq 3$

$$\int |v(t)|^k \phi_j(t) \leq \|v(t)\|_{L^\infty}^{k-2} \int v(t)^2 \phi_j(t) \leq Ce^{-\sigma_0^{3/2}t}\|v\|_{L^2}^2,$$

we obtain (2.28) and (2.29).

Estimate (2.27) is obtained by summing (2.28) and (2.29). Note that in particular that the scalar products $\int v(t)\tilde{R}_j(t)$ cancel. ■

Claim. For T_0 large enough, there exists $K > 0$ such that for all $t \in [T_0, S_n]$,

$$\|v(t)\|_{H^1}^2 \leq K \sum_j H_j(t) + K^2 \sum_j \left(\left(\int v(t)\tilde{Z}_j^+(t)\right)^2 + \left(\int v(t)\tilde{Z}_j^-(t)\right)^2\right). \tag{2.30}$$

Proof. Estimate (2.30) is a standard consequence of Lemma 1 and $\int v\tilde{R}_{j_x} = 0$. See e.g. [21, Lemma 4]. ■

Now, we finish the proof of Lemma 4. Let $t \in [T(\mathfrak{a}^+), S_n]$. Integrating (2.26) on $[t, S_n]$,

$$\left| \sum_{j=1}^N \left\{ \left(E_j(S_n) + \frac{c_j}{2}M_j(S_n)\right) - \left(E_j(t) + \frac{c_j}{2}M_j(t)\right) \right\} \right| \leq$$
$$\leq Ce^{-3\sigma_0^{3/2}t} + C\int_t^{S_n} \|v(s)\|_{H^1}^2 \frac{ds}{\sqrt{s}}.$$

From (2.27), we get:

$$\left| \sum_{j=1}^N (H_j(S_n) - H_j(t)) \right| \leq$$
$$\leq Ce^{-3\sigma_0^{3/2}t} + Ce^{-\sigma_0^{3/2}t}(\|v(t)\|_{L^2}^2 + \|v(S_n)\|_{L^2}^2) + C\int_t^{S_n} \|v(s)\|_{H^1}^2 \frac{ds}{\sqrt{s}}.$$

Note that from Lemmas 2 and 3, and from the definition of $T(\mathfrak{a}^+)$,

$$|H_j(S_n)| \leq C\|v(S_n)\|_{H^1}^2 \leq C\|\mathfrak{b}\|^2 \leq Ce^{-3\sigma_0^{3/2}t} \quad \text{and} \quad \|v(t)\|_{H^1}^2 \leq e^{-2\sigma_0^{3/2}t}.$$

By (2.30) and the above estimates

$$\begin{aligned}
\|v(t)\|_{H^1}^2 &\leq K\sum_{j=1}^N H_j(t) + K^2 \sum_{j,\pm} a_j^{\pm}(t)^2 \\
&\leq Ce^{-3\sigma_0^{3/2}t} + C\sum_{j,\pm} a_j^{\pm}(t)^2 + C\int_t^{S_n} \|v(s)\|_{H^1}^2 \frac{ds}{\sqrt{s}} \\
&\leq C_0 e^{-3\sigma_0^{3/2}t} + \frac{C_0}{\sqrt{t}} e^{-2\sigma_0^{3/2}t}. \qquad (2.31)
\end{aligned}$$

Hence, for T_0 large enough so that $C_0 e^{-\sigma_0^{3/2} T_0} \leq 1/8$ and $C_0/\sqrt{T_0} \leq 1/8$ we get

$$e^{\sigma_0^{3/2} t} \|v(t)\|_{H^1} \leq 1/2.$$

By (2.11) and (2.31),

$$\|\mathbf{y}_t(t)\| \leq C e^{-2\sigma_0^{3/2} t} + C\|v(t)\|_{L^2},$$

$$\|\mathbf{y}(t)\| \leq \|\mathbf{y}(S_n)\| + C \int_t^{S_n} \left(e^{-(3/2)\sigma_0^{3/2} s} + \frac{e^{-\sigma_0^{3/2} s}}{s^{1/4}} \right) ds$$

$$\leq C e^{-(3/2)\sigma_0^{3/2} t} + \frac{C}{t^{1/4}} e^{-\sigma_0^{3/2} t}, \tag{2.32}$$

and we deduce $e^{\sigma_0^{3/2} t}\|\mathbf{y}(t)\| \leq 1/2$ by possibly taking a larger T_0. Finally, we have:

$$\|u(t) - R(t)\|_{H^1} \leq \|R(t) - \tilde{R}(t)\|_{H^1} + \|v(t)\|_{H^1} \leq C\|\mathbf{y}(t)\| + \|v(t)\|_{H^1}$$

$$\leq C e^{-\sigma_0^{3/2} t} \leq \varepsilon_0/2, \tag{2.33}$$

by possibly taking a larger T_0. This concludes the proof of Lemma 4. ■

3. Generalizations

3.1. The gKdV equations with general nonlinearities

We now present extensions of Theorem 1 to a more general form of the KdV equation, i.e.

$$u_t + (u_{xx} + f(u))_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{gKdV}$$

In order to have both local well-posedness in H^1 from [14] and the existence of eigenvalues for the linearized operator in the instable case from [26], we assume

$$f \text{ is } C^3, \text{ convex on } \mathbb{R}_+, \text{ and } f(0) = f'(0) = 0, \tag{3.1}$$

but these assumptions can probably be relaxed. Concerning the solitons, we consider velocities $c_j > 0$ such that

$$\begin{aligned} &\text{a solution } Q_c \text{ of } (Q_c)_{xx} + f(Q_c) = cQ_c \text{ exists for all } c \text{ close to } c_j \\ &\text{and } \frac{d}{dc} \int Q_c^2{}_{|c=c_j} \neq 0. \end{aligned} \tag{3.2}$$

Then, combining the proof of Theorem 1 and [16], we claim the following extension of Theorem 1.

Theorem 3. *Let* $0 < c_1 < \cdots < c_N$ *and* $x_1, \ldots, x_N \in \mathbb{R}$ *be such that for all* j, (3.2) *holds. There exist* $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, *and a solution* $u \in \mathcal{C}([T_0,\infty), H^1)$ *to* (gKdV) *such that*

$$\forall t \in [T_0,\infty), \qquad \left\| u(t) - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{H^1} \le Ce^{-\sigma_0^{3/2} t}.$$

Remark 3. The critical case $\frac{d}{dc}\int Q^2_{c|c=c_j} = 0$ is treated in [16] for the pure power case. We leave open the case where for a general $f(u)$, $\frac{d}{dc}\int Q^2_{c|c=c_j} = 0$ for some c_j, but it probably can be treated by similar techniques.

From the techniques developped in [25], [7] and [8] concerning the (BBM) equation

$$\text{(BBM)} \qquad (u - u_{xx})_t + (u + u^p)_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

and from the construction of suitable eigenfunctions of the linearized equation by Pego and Weinstein [26] (see page 74), one can also extend the results obtained in this paper to the (BBM) equation for any $p > 1$.

3.2. The non linear Schrödinger equations

In this section, we sketch the proof of Theorem 2. It is an extension of the proof of Theorem 1 in the present paper and of the main result in [20].

3.2.1. Preliminaries

Let $v = v_1 + iv_2$, we define the operator $\mathcal{L}$ by

$$\mathcal{L}v = -L_- v_2 + iL_+ v_1, \tag{3.3}$$

where the self-adjoint operators L_+ and L_- are defined by

$$L_+ v_1 := -\Delta v_1 + v_1 - pQ^{p-1} v_1, \quad L_- v_2 := -\Delta v_2 + v_2 - Q^{p-1} v_2, \tag{3.4}$$

From [29], [11] and [28], there exist $e_0 > 0$, $Y^\pm \in \mathcal{S}(\mathbb{R})$ ($\bar{Y}^+ = Y^-$), normalized so that $\|Y^\pm\|_{L^2} = 1$ and such that

$$\mathcal{L}Y^\pm = \pm e_0 Y^\pm; \tag{3.5}$$

moreover, for some $K > 0$, for any $v = v_1 + iv_2 \in H^1$ ($(f,g) = \mathrm{Re}\int f\bar{g}$)

$$\begin{aligned} \|v\|_{H^1}^2 \le K(L_+ v_1, v_1) + K(L_- v_2, v_2) + K^2 \left(\int (\nabla Q) v_1 \right)^2 \\ + K^2 \left(\int Q v_2 \right)^2 + K^2 \left(\mathrm{Im} \int Y^+ \bar{v} \right)^2 + K^2 \left(\mathrm{Im} \int Y^- \bar{v} \right)^2. \end{aligned} \tag{3.6}$$

See [5, 6] for the proof of (3.6).

3.2.2. Proof of Theorem 2 assuming uniform estimates

We denote

$$R(t,x)=\sum_{j=1}^{N} R_j(t,x) \quad \text{where} \quad R_j(t,x)=R_{c_j,\gamma_j,v_j,x_j}(t,x),$$
$$Y_j^{\pm}(t,x)=c_j^{\frac{1}{p-1}}Y^{\pm}(\sqrt{c_j}(x-v_jt-x_j))e^{i(\frac12 v_j\cdot x-\frac14\|v_j\|^2t+c_jt+\gamma_j)}. \tag{3.7}$$

Let $S_n \to \infty$ be an increasing sequence of time. We claim the existence of final data giving suitable uniform estimates.

Proposition 2. *There exist* $n_0\geq 0$, $\sigma_0>0, T_0>0, C>0$ *(independent of* n*) such that the following holds. For each* $n\geq n_0$*, there exists* $\mathfrak{b}=(\mathfrak{b}_{j,n}^{\pm})_{j,\pm}\in\mathbb{R}^{2N}$ *with* $\|\mathfrak{b}\|\leq e^{-\sigma_0^{3/2}S_n}$*, and such that the solution* u_n *to*

$$\begin{cases} iu_{nt}+\Delta u_n+|u_n|^{p-1}u_n=0,\\ u_n(S_n)=R(S_n)+i\displaystyle\sum_{j\in\{1,\ldots,N\},\pm}\mathfrak{b}_{j,n}^{\pm}Y_j^{\pm}(S_n) \end{cases} \tag{3.8}$$

is defined on the interval $[T_0,S_n]$*, and satisfies*

$$\forall t\in[T_0,S_n],\quad \|u_n(t)-R(t)\|_{H^1}\leq Ce^{-\sigma_0^{3/2}t}.$$

The proof of Theorem 2 assuming Proposition 2 is completely similar to Section 2.2 in the present paper and to Section 2 in [20], thus it is omitted (note that for this part, as in [20], we use the local H^s Cauchy theory due to Cazenave and Weissler [4]).

3.2.3. Proof of the uniform estimates

We are reduced to prove Proposition 2. We only sketch the proof since it is very similar to Section 2.3 of the present paper combined with Section 3 in [20].

The first step of the proof is to reduce (without loss of generality) to the special case where

$$v_{1,1}<v_{2,1}<\cdots<v_{N,1},$$

where $v_{j,k}$ ($j\in\{1,\ldots,N\}$, $k\in\{1,\ldots,d\}$) represents the $k-th$ component of the velocity vector $v_j\in\mathbb{R}^d$. It is a simple observation, based on the invariance by rotation of the (NLS) equation, see [20, Claim 1, page 855].

Next, in the (NLS) case, modulation theory for $u(t)$ close to $R(t)$ says that there exist parameters $\mathbf{y}(t)=(y_1(t),\ldots,y_N(t))\in(\mathbb{R}^d)^N$ and $\mu(t)=$

$(\mu_1(t),\dots,\mu_N(t))\in\mathbb{R}^N$ such that

$$\tilde R_j(t)=R_j(t,x-y_j(t))e^{i\mu_j(t)},\quad \tilde R(t)=\sum_{j=1}^N \tilde R_j(t),$$
$$\tilde Y_j^{\pm}(t,x)=Y_j^{\pm}(t,x-y_j(t))e^{i\mu_j(t)},$$

and $v(t)=u(t)-\tilde R(t)$ satisfies

$$\forall j=1,\dots,N,\quad \operatorname{Re}\int v(t)(\nabla\tilde R_j)(t)=\operatorname{Im}\int v(t)\tilde R_j(t)=0.$$

Note that the parameter $\mu_j(t)$ is used to control the direction $\operatorname{Im}\int v(t)\tilde R_j(t)$.

In view of (3.6), we are led to set

$$\mathbf{a}^{\pm}(t)=(a_j^{\pm}(t))_{j=1,\dots,N},\quad \text{where}\quad a_j^{\pm}(t)=\operatorname{Im}\int \tilde Y_j^{\mp}(t,x)\bar v(t,x)dx.$$

For given $\mathfrak{a}^+\in\mathbb{R}^N$, we define $\mathfrak{b}\in\mathbb{R}^{2N}$ as for the (gKdV) case in Lemma 3. We define $T(\mathfrak{a}^+)$ as in Definition 1, with the additional requirement $e^{\sigma_0^{3/2}t}\mu(t)\in B_{\mathbb{R}^N}(1)$. By standard computations, the following holds on $[T(\mathfrak{a}^+),S_n]$.

Claim. For some $\sigma_0>0$,

$$\left\|\frac{d\mathbf{y}}{dt}(t)\right\|+\left\|\frac{d\mu}{dt}(t)\right\|\le C\|v(t)\|_{L^2}+Ce^{-2\sigma_0^{3/2}t}, \tag{3.9}$$

$$\left|\frac{da_j^{\pm}}{dt}(t)\pm e_0c_j^{3/2}a_j^{\pm}(t)\right|\le C\|v(t)\|_{L^2}^2+Ce^{-3\sigma_0^{3/2}t}. \tag{3.10}$$

Proof. The proof follows from the equation of v

$$iv_t+\Delta v+\sum_j\left(|\tilde R_j|^{p-1}v+(p-1)|\tilde R_j|^{p-2}\operatorname{Re}(\tilde R_j v)\right)$$
$$+O(\|v\|_{H^1}^2+e^{-4\sigma_0^{3/2}t})-\sum_j\frac{dy_j}{dt}\tilde R_{jx}-\sum_j\frac{d\mu_j}{dt}\tilde R_j=0,$$

and direct computations using the definition of $Y^{\pm}$. ■

Now we follow exactly the same strategy as in the proof of Theorem 1, by proving analogues of Lemmas 4, 5 and 6.

For the proof of the estimate on $v(t)$, we use a functional adapted to the (NLS) equations, as in [20] and [22]:

$$\mathcal{G}(t)=\sum_j\Bigg[\int\left(\frac12|\nabla u|^2-\frac{1}{p+1}|u|^{p+1}\right)\phi_j$$
$$+\left(c_j+\frac{|v_j|^2}{4}\right)\int|u|^2\phi_j-v_j\cdot\operatorname{Im}\int\bar u\nabla u\phi_j\Bigg],$$

where

$$\psi_j(t,x) = \psi\left(\frac{1}{\sqrt{t}}(x_1 - m_j(t))\right),\ m_j(t) = \frac{1}{2}((v_{j,1}+v_{j+1,1})t + x_{j,1} + x_{j+1,1});$$
$$\phi_1 = \psi_1, \quad \phi_j = \psi_j - \psi_{j-1}.$$

Note that $\mathcal{G}(t)$ controls the size of $v(t)$ in H^1 up to $\mathbf{a}^{\pm}(t)$ as a consequence of (3.6). As for (gKdV), the following claim allows us to prove the estimate on $\|v(t)\|_{H^1}$.

Claim.

$$\left|\frac{d\mathcal{G}}{dt}(t)\right| \leq \frac{C}{\sqrt{t}}\|v(t)\|_{H^1}^2 + Ce^{-3\sigma_0^{3/2}t}.$$

The estimates of $\mathbf{a}^{\pm}(t)$ are exactly the same as in Lemmas 5 and 6, using (3.10).

A. Appendix

Proof of Lemma 2. We use the following notation $\mathbf{y} = (y_j)_{j=1,\dots,N}$ and

$$R_j(x) = Q_{c_j}(x-\alpha_j), \quad R(x) = \sum_{j=1}^N R_j(x),$$
$$\tilde{R}_j(x) = R_j(x-y_j), \quad \tilde{R}(x) = \sum_{j=1}^N \tilde{R}_j(x).$$

Let $w = u - R$ small in L^2. Consider

$$\begin{array}{rccl} \Phi: & L^2\times\mathbb{R}^N & \to & \mathbb{R}^N, \\ & (w,\mathbf{y}) & \mapsto & \left(\int (w+R-\tilde{R})\tilde{R}_{j_x}\right)_{j=1,\dots,N}. \end{array}$$

Let $\mathbf{z} = (z_j)_{j=1,\dots,N}$. By the decay properties of $\tilde{R}_j$,

$$\begin{aligned}(d_{\mathbf{y}}\Phi(w,\mathbf{y}).\mathbf{z})_j &= \sum_{k=1}^N z_k \int \tilde{R}_{kx}\tilde{R}_{j_x} - z_j\int (w+R-\tilde{R})\tilde{R}_{j_{xx}} \\ &= z_j\|Q_{c_{j_x}}\|_{L^2}^2 + O\Big(\sum_{k\neq j} e^{-\sigma_0|\alpha_k-\alpha_j|}|z_k|\Big) + O(|z_j|\|w\|_{L^2}) + O(|z_j|\|\mathbf{y}\|).\end{aligned}$$

Hence

$$d_{\mathbf{y}}\Phi(w,\mathbf{y}) = \mathrm{diag}(\|Q_{c_{j_x}}\|_{L^2}^2) + O(\sum_{k\neq j} e^{-\sigma_0|\alpha_k-\alpha_j|}) + O(\|w\|_{L^2}) + O(\|\mathbf{y}\|). \tag{A.1}$$

Therefore, if $\min\{|\alpha_k - \alpha_j|, i \neq j\}$ is large enough then $d_{\mathbf{y}}\Phi(0,0)$ is invertible. Since $\Phi(0,0) = 0$, by the implicit function theorem, it follows that there exists $\varepsilon > 0$, $\varepsilon \leq \eta$ and a C^1 function $\phi : B_{L^2}(0,\varepsilon) \to B_{\mathbb{R}^N}(0,\eta)$ such that $\Phi(w,\mathbf{y}) = 0$ in $B_{L^2}(0,\varepsilon) \times \phi(B_{L^2}(0,\varepsilon))$ is equivalent to $\mathbf{y} = \phi(w)$. Finally we set $v = v(w) = w + R - \sum_{j=1}^N R_j(\cdot - \phi(w)_j)$. ■

Proof of Lemma 3. Consider the maps:

$$\begin{array}{lll} \mathcal{I} : \mathbb{R}^{2N} \to H^1 & \Theta : \mathcal{V} \to H^1 \times \mathbb{R}^N & \mathcal{S} : H^1 \times \mathbb{R}^N \to \mathbb{R}^{2N} \\ \quad \mathfrak{b} \mapsto \sum_{j,\pm} \mathfrak{b}_j^\pm Z_j^\pm(S_n) & \quad w \mapsto (v,\mathbf{y}) & \quad (v,\mathbf{y}) \mapsto \left(\int v\tilde{Z}_j^\pm\right)_{j,\pm} \end{array}$$

where, in the definition of Θ, $(v,\mathbf{y})$ represents the modulation of $u = w + R(S_n)$ and $\mathcal{V} = B_{H^1}(\varepsilon)$ (ε being defined in the proof Lemma 2), and in the definition of $\mathcal{S}$, we have set $\tilde{Z}_j^\pm(x) = Z_j^\pm(S_n, x - y_j)$.

Then $\mathcal{I}(0) = 0$, $\Theta(0) = (0,0)$ and $\mathcal{S}(0,0) = 0$. Recall also from Lemma 2 that

$$\|v\|_{L^2} + \|\mathbf{y}\| + \|R_j(S_n) - \tilde{R}_j(S_n)\|_{H^1} \leq C\|w\|_{L^2}.$$

To prove Lemma 3, we claim that $\Psi = \mathcal{S} \circ \Theta \circ \mathcal{I}$ is a diffeomorphism on a fixed neighbourhood of $0 \in \mathbb{R}^{2N}$ by computing $d\Psi = d\mathcal{S} \circ d\Theta \circ d\mathcal{I}$. Indeed, we claim

Claim.

$$d\Psi(\mathfrak{b}) = \begin{pmatrix} A & (\int Z^+Z^-)A \\ (\int Z^+Z^-)A & A \end{pmatrix} + O(e^{-\sigma_0^{3/2} S_n} + \|\mathfrak{b}\|),$$

where $A = \operatorname{diag}((\|Z_j\|_{L^2}^2)_j) = \operatorname{diag}((c_j^{\frac{5-p}{2(p-1)}})_j)$.

Remark 4. Note that if $N = 1$ (only one soliton), with e.g. $c_1 = 1$, then the map Ψ is represented by the matrix

$$B = \begin{pmatrix} \int (Z^+)^2 & \int Z^+Z^- \\ \int Z^+Z^- & \int (Z^-)^2 \end{pmatrix} = \operatorname{Gramm}(Z^\pm).$$

Indeed, the functions $Z^\pm$ are orthogonal to Q_x, so that $\mathbf{y} = 0$ in this case and Ψ is linear. Since $Z^\pm$ are linearly independent (see proof of Lemma 1), the matrix B is invertible.

The claim means that for the general case $N \geq 2$, we obtain a similar behavior around each soliton plus small terms due to the interaction of the various solitons.

Proof. We start with the computation of differentials of $\mathcal{I}$, Θ and $\mathcal{S}$. First, $\mathcal{I}$ is affine so that $d\mathcal{I}(\mathfrak{b}) = \mathcal{I}$ for all $\mathfrak{b}$. Second, for $h \in H^1$, $\mathbf{z} \in \mathbb{R}^N$,

$$(d\mathcal{S}(v,\mathbf{y}).(h,\mathbf{z}))_{j,\pm} = -z_j \int v{\tilde{Z}_j^\pm}_x + \int h\tilde{Z}_j^\pm.$$

Finally, we consider Θ. Let Φ and ϕ be defined as in the proof of the Lemma 2 above for $R(S_n)$. Then, by (A.1), $d_{\mathbf{y}}\Phi(w,\mathbf{y})$ is a diagonally dominant matrix and thus it is invertible. Denoting by M its inverse, it follows from (A.1) that

$$M = \operatorname{diag}((\|Q_{c_j x}\|_{L^2}^{-2})_j) + O(\|w\|_{L^2} + \|\mathbf{y}\| + e^{-\sigma_0^{3/2} S_n}).$$

Differentiating $\Phi(w,\phi(w)) = 0$ with respect to w, we find $d\phi = -M \circ d_w\Phi$. Since $(d_w\Phi(w,\mathbf{y}).h)_j = \int h\tilde{R}_{j_x}(S_n)$ and

$$\Theta(w) = \Big(w + R - \sum_j R_j(S_n, \cdot - \phi(w)_j), \phi(w)\Big),$$

we obtain

$$\begin{aligned}
d\Theta(w).h &= (h - \sum_j (\tilde{R}_{j_x}(S_n)((M \circ d_w\Phi).h)_j, -M \circ d_w\Phi.h)\\
&= \left(h + \sum_{j=1}^N \|\tilde{R}_{j_x}(S_n)\frac{\int h\tilde{R}_{j_x}(S_n)}{Q_{c_j x}\|_{L^2}^2}, \left(-\frac{\int h\tilde{R}_{j_x}(S_n)}{\|Q_{c_j x}\|_{L^2}^2}\right)_{j=1,\dots,N}\right)\\
&\quad + O(\|h\|_{L^2}(e^{-\sigma_0^{3/2} S_n} + \|w\|_{L^2})).
\end{aligned}$$

Let $\tilde{\mathfrak{b}} \in \mathbb{R}^{2N}$. Since $\mathcal{I}$ is linear, we have

$$d\Psi(\mathfrak{b}).\tilde{\mathfrak{b}} = d\mathcal{S}(\Theta(\mathcal{I}(\mathfrak{b}))).(d\Theta(\mathcal{I}(\mathfrak{b})).\mathcal{I}(\tilde{\mathfrak{b}})).$$

By the previous computations, we derive

$$\begin{aligned}
&d\Theta(\mathcal{I}(\mathfrak{b})).\mathcal{I}(\tilde{\mathfrak{b}}) =\\
&\quad = \left(\mathcal{I}(\tilde{\mathfrak{b}}) + \sum_{j=1}^N \tilde{R}_{j_x}(S_n)\frac{\int \mathcal{I}(\tilde{\mathfrak{b}})\tilde{R}_{j_x}(S_n)}{\|Q_{c_j x}\|_{L^2}^2}, \left(-\frac{\int \mathcal{I}(\tilde{\mathfrak{b}})\tilde{R}_{j_x}(S_n)}{\|Q_{c_j x}\|_{L^2}^2}\right)_{j=1,\dots,N}\right)\\
&\qquad + O(\|\tilde{\mathfrak{b}}\|(e^{-\sigma_0^{3/2} S_n} + \|\mathfrak{b}\|)).
\end{aligned}$$

Inserting the expression of $\mathcal{I}(\tilde{\mathfrak{b}})$, using $\|\mathbf{y}\| \le C\|\mathfrak{b}\|$, $\int Z^{\pm}Q_x = 0$ and the decay properties of the functions Q and Z, we get

$$d\Theta(\mathcal{I}(\mathfrak{b})).\mathcal{I}(\tilde{\mathfrak{b}}) = (\mathcal{I}(\tilde{\mathfrak{b}}), 0) + O(\|\tilde{\mathfrak{b}}\|(e^{-\sigma_0^{3/2} S_n} + \|\mathfrak{b}\|)).$$

Therefore, using the expression of $d\mathcal{S}$, we finally obtain

$$d\Psi(\mathfrak{b}) = \operatorname{Gramm}((Z_j^{\pm})_{j,\pm}) + O(e^{-\sigma_0^{3/2} S_n} + \|\mathfrak{b}\|) = P + O(e^{-\sigma_0^{3/2} S_n} + \|\mathfrak{b}\|)$$

where $\operatorname{Gramm}((Z_j^{\pm})_{j,\pm})$ is the Gramm matrix of the family $(Z_j^{\pm})_{j,\pm}$

$$\operatorname{Gramm}((Z_j^{\pm})_{j,\pm})_{(j_1,\pm_1),(j_2,\pm_2)} = \int Z_{j_1}^{\pm_1} Z_{j_2}^{\pm_2},$$

and

$$P = \begin{pmatrix} A & (\int Z^+ Z^-)A \\ (\int Z^+ Z^-)A & A \end{pmatrix},$$

where $A = \mathrm{diag}((\|Z_j\|_{L^2}^2)_j) = \mathrm{diag}((c_j^{\frac{5-p}{2(p-1)}})_j)$ (recall that $\|Z^\pm\|_{L^2} = 1$). This finishes the proof of the claim. ∎

Since P is invertible (Z^+ and Z^- are independent, see proof of Lemma 1), we deduce that $d\Psi$ is invertible on some ball $B_{\mathbb{R}^{2N}}(\eta)$ ($\eta > 0$ independent of n for $n \geq n_0$ large enough). As a consequence, Ψ is a diffeomorphism from $B_{\mathbb{R}^{2N}}(\eta)$ to some neighbourhood $\mathcal{W}$ of $0 \in \mathbb{R}^{2N}$. Let $\delta > 0$ be such that $B_{\mathbb{R}^{2N}}(\delta) \subset \mathcal{W}$. For any $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(\delta)$, there exist a unique $\mathfrak{b} = \mathfrak{b}(\mathfrak{a}^+) \in B_{\mathbb{R}^{2N}}(\eta)$ such that $\Psi(\mathfrak{b}(\mathfrak{a}^+)) = (\mathfrak{a}^+, 0)$ and $\|\mathfrak{b}(\mathfrak{a}^+)\| \leq C\|\mathfrak{a}^+\|$. ∎

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