Toeplitz operators on Bergman spaces with locally integrable symbols

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Abstract

We study the boundedness of Toeplitz operators T_a with locally integrable symbols on Bergman spaces $A^p(\mathbb{D})$, $1 < p < \infty$. Our main result gives a sufficient condition for the boundedness of T_a in terms of some "averages" (related to hyperbolic rectangles) of its symbol. If the averages satisfy an o–type condition on the boundary of $\mathbb{D}$, we show that the corresponding Toeplitz operator is compact on A^p. Both conditions coincide with the known necessary conditions in the case of nonnegative symbols and $p = 2$. We also show that Toeplitz operators with symbols of vanishing mean oscillation are Fredholm on A^p provided that the averages are bounded away from zero, and derive an index formula for these operators.

1. Introduction

Consider the space $L^p := (L^p(\mathbb{D}, dA), \|\cdot\|_p)$, where dA is the normalized area measure on the unit disc $\mathbb{D}$ of the complex plane, and the Bergman space A^p, which is the closed subspace of L^p consisting of analytic functions. The Bergman projection P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} \, dA(\zeta).$$

It is also known to be a bounded projection of L^p onto A^p, when $1 < p < \infty$. For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, we define the Toeplitz operator T_a with symbol a by setting

$$T_a f = P(af).$$

Since P is bounded, it follows easily that T_a extends to a bounded operator $A^p \to A^p$ for $1 < p < \infty$, whenever a is a bounded measurable function. A considerably more difficult question is the boundedness of T_a on A^p with unbounded symbols. This is a long-standing problem of high interest as Toeplitz operators form one of the most important classes of nonself-adjoint operators. In what follows, we give a sufficient condition for T_a to be bounded on A^p when the symbol a is a *priori* only a locally integrable function. The condition is a rather weak requirement of the boundedness of certain "averages" of a over hyperbolic rectangles (see Theorem 2.3). We remark that not all such symbols are L^1 on the disc. We also deal with compactness and Fredholmness of these operators for a large class of symbols that are not necessarily bounded, see Theorems 2.6 and 2.8.

Conditions for T_a to be bounded are known in some special cases when the operators are acting on Hilbert space. Luecking [3] characterized bounded Toeplitz operators T_a on A^2 with positive symbols in L^1 and Zhu [8] considered the same question in the case of weighted Bergman spaces A_α^2 of bounded symmetric domains. A treatment of radial generating symbols can be found in Vasilevski [7]. The most complete previously known result, due to Zorboska [11], deals with integrable symbols that satisfy the condition of bounded mean oscillation and determines the bounded Toeplitz operators in terms of the boundary behavior of the Berezin transform of their symbols. In contrast, our result gives a complete description in the case of all reflexive Bergman spaces A^p for positive symbols, while for general symbols we prove that the same condition is at least sufficient.

Let $1 < p < \infty$ and denote by $\tau(A^p)$ the closed subalgebra of the Banach algebra of all bounded operators on A^p generated by Toeplitz operators with bounded symbols. Recently Suárez [5] showed that all compact operators on A^p are contained in $\tau(A^p)$ and their Berezin transform necessarily vanishes on the boundary of the unit ball. However, the condition that the Berezin transform of an operator vanishes is not sufficient for compactness; for this and further references we refer the reader to [5]. Our sufficient condition for the compactness of Toeplitz operators, given in the next section, is proven to be also necessary at least when the symbols are positive.

Most previous results on the Fredholm properties of Toeplitz operators have been concerned with the Hilbert space case. A thorough discussion on Fredholmness of Toeplitz operators on the Bergman space A^2 can be found in [7], which deals with several classes of bounded symbols and also with the class of radial symbols. A partial description of the essential spectra of operators in $\tau(A^2)$ is given in [5], which also shows that the description is complete for essentially normal operators. For the case $p = 1$, see [6]. Here we give a sufficient condition for T_a to be Fredholm on A^p

with $1 < p < \infty$ when the symbol a is of vanishing mean oscillation but not necessarily bounded; in addition, we derive an index formula for these Fredholm operators.

2. Main results and preliminaries

In the following we consider various function spaces, all of which are defined on $\mathbb{D}$, unless otherwise stated. We first recall a well-known result which can be found, for example, in [10, Corollary 7.6].

Theorem 2.1. *Assume that $a \in L^1$ and that a is nonnegative. Then $T_a : A^2 \to A^2$ is bounded if and only if*

$$\text{(2.1)} \qquad \hat{a}_r \in L^\infty.$$

Here $\hat{a}_r$ of a is the averaging function $\hat{a}_r(z) := |B(z,r)|^{-1} \int_{B(z,r)} a \, dA$, and $B(z,r) \subset \mathbb{D}$ is the hyperbolic disc with center $z \in \mathbb{D}$ and radius $r > 0$, and $|B(z,r)| := \int_{B(z,r)} dA$.

We want to deal with more general symbols, which a *priori* only belong to the space L^1_{loc} of locally integrable functions on $\mathbb{D}$. Let us introduce a collection $\mathcal{D}$ of subsets of $\mathbb{D}$ which are rectangles in polar coordinates and have a hyperbolic radius bounded from above and below.

Definition 2.2. Denote by $\mathcal{D}$ the family that consists of the sets $D := D(r, \theta)$ defined by

$$\text{(2.2)} \qquad D = \{\rho e^{i\phi} \mid r \le \rho \le 1 - \frac{1}{2}(1-r) \, , \, \theta \le \phi \le \theta + \pi(1-r)\}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$.

Given $D = D(r, \theta) \in \mathcal{D}$ and $\zeta = \rho e^{i\phi} \in D$, we denote

$$\text{(2.3)} \qquad \hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \, \varrho \, d\varphi \, d\varrho.$$

Theorem 2.3. *Assume that $a \in L^1_{\text{loc}}$ and that there exists a constant $C > 0$ such that*

$$\text{(2.4)} \qquad |\hat{a}_D(\zeta)| \le C$$

for all $D \in \mathcal{D}$ and all $\zeta \in D$. Then $T_a : A^p \to A^p$ is well defined and bounded for all $1 < p < \infty$, and there is a constant $C_1 > 0$ such that

$$\text{(2.5)} \qquad \|T_a\| \le C_1 \sup_{D \in \mathcal{D}, \, \zeta \in D} |\hat{a}_D(\zeta)|.$$

In the following we shall denote

$$\mathbf{C}_a := \sup_{D \in \mathcal{D}, \zeta \in D} |\hat{a}_D(\zeta)|.$$

Remark 2.4. 1) There exist symbols a satisfying (2.4) but which are not L^1 on the disc. The function

$$(2.6) \qquad a(re^{i\theta}) := \begin{cases} \dfrac{1}{r(1-r)} \sin \dfrac{1}{1-r} \,, & r \geq \dfrac{1}{2} \\[2ex] 1 \,, & r < \dfrac{1}{2} \end{cases}$$

is an example. Obviously, it is not L^1. However, given $D = D(1-2\delta, \theta)$ with a small enough δ and $\zeta = \rho e^{i\phi} \in D$, we have, using the change of variable $y = 1/(1-\varrho)$

$$(2.7) \quad |D||\hat{a}_D(\zeta)| = \int_\theta^\phi d\varphi \left| \int_{1-2\delta}^\rho \frac{1}{1-\varrho} \sin \frac{1}{1-\varrho} d\varrho \right| \leq \pi\delta \left| \int_{1/(2\delta)}^{1/(1-\rho)} \frac{1}{y} \sin y \, dy \right|$$

Dividing the integration interval to subintervals of the form $J_n := [2\pi n, 2\pi(n+1)]$, $n \in \mathbb{N}$, one can replace (because of the periodicity of the sinus) on J_n the function y^{-1} by $y^{-1} - (2\pi(n+1))^{-1}$, which has the bound Cn^{-2} on J_n. Summing over n and bounding $|\sin y|$ by 1, the modulus of the integral $\int_{1/(2\delta)}^{1/(1-\rho)} y^{-1} \sin y \, dy$ is seen to have the bound $C\delta$. Since $|D|$ is of order δ^2, the requirement (2.4) is seen to hold.

However, we are still able to define the Toeplitz operator under the condition (2.4) by introducing a decomposition $\mathbb{D} = \cup_{n \in \mathbb{N}} D_n$ of the disc into disjoint sets of type (2.2), and showing that for the following functions of z,

$$(2.8) \qquad F_n(z) := \int_{D_n} \frac{a(\zeta)f(\zeta)}{(1-z\bar{\zeta})^2} \, dA(\zeta),$$

the series $\sum_{n=1}^\infty F_n(z)$ converges absolutely for almost every z and defines an L^p–function. Hence, $T_a f$ can be defined as the sum of this series, and the definition coincides with the conventional one in case $a \in L^1$. The details will be presented in the proof of Theorem 2.3.

2) There is some hope that our condition (2.4) may also turn out necessary, since there is no modulus of a in the integral (2.3). (The condition obtained by replacing a by $|a|$ in (2.3) is of course far from being necessary.) On the other hand, it is quite obvious that the conditions (2.1) and (2.4) are equivalent for *positive* symbols. Namely, in that case, given $D(r, \theta)$ the assumption (2.4) is equivalent to assuming it only for $\zeta = \rho e^{i\phi}$ with $\rho = 1 - (1-r)/2$ and $\phi = \theta + \pi(1-r)$. Then $\hat{a}_D(\zeta)$ is just the average of a over D, and D on the other hand is essentially a hyperbolic disc with constant radius.

3) For nonpositive symbols a the exact form of the sets D is of crucial importance: it can be used to estimate indefinite integrals of a, and on the other hand, we need a decomposition of $\mathbb{D}$ as a disjoint union of sets of the form D. These facts and an integration by parts argument will be used to obtain our main result in Section 3.

As a corollary of our result one can generalize the statement of Theorem 2.1 for all $1 < p < \infty$. We do this via the Berezin transform $B(T)$ of $T \in \mathcal{L}(A^p)$ defined by

$$(2.9) \qquad B(T)(z) = (1 - |z|^2)^2 \langle TK_z, K_z \rangle = \langle Tk_{z,p}, k_{z,q} \rangle$$

where

$$K_z(\zeta) = \frac{1}{(1 - \zeta\bar{z})^2} \quad (\zeta \in \mathbb{D}).$$

is the reproducing kernel function and

$$(2.10) \qquad k_{z,p}(\zeta) := K_z(\zeta)/\|K_z\|_2^{2/q} \quad (\zeta \in \mathbb{D}) \quad \text{with} \quad \|k_{z,p}\|_p = 1.$$

Recall also the Berezin transform $B(f)$ of a function $f \in L^1$ defined by

$$B(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^4} \, dA(\zeta).$$

Corollary 2.5. *Let $1 < p < \infty$ and let $a \geq 0$ be locally integrable. Then $T_a : A^p \to A^p$ is bounded if and only if $\hat{a}_r \in L^\infty$ (equivalently, if and only if $B(a) \in L^\infty$).*

Proof. Note first that $B(a)$ is bounded on $\mathbb{D}$ if and only if $\hat{a}_r$ is bounded on $\mathbb{D}$ (see [10]). Now sufficiency follows from Theorem 2.3 and Remark 2.4. Conversely, it is easy to see that the Berezin transform of a Toeplitz operator T_a is the Berezin transform of its symbol, that is, $B(T_a) = B(a)$. Therefore, if T_a is bounded on A^p, then necessarily $B(a)$ is bounded on $\mathbb{D}$, see (2.9) and (2.10). ∎

Next we consider the compactness of Toeplitz operators. For a nonnegative symbol $a \in L^1$, the Toeplitz operator T_a is compact if and only if $\hat{a}_r(z) \to 0$ as $|z| \to 0$ (which is equivalent to $B(a)(z) \to 0$ as $|z| \to 1$); see [10]. According to [11], for $a \in BMO_\partial^1$ (see (2.12) below), the Toeplitz operator T_a is compact on A^2 if and only if $B(a)(z) \to 0$ as $|z| \to 1$. However, as stated in the same article, given a general unbounded a, it is unknown whether a bounded Toeplitz operator $T_a : A^2 \to A^2$ is compact when $B(a)$ vanishes on the boundary. If T is in the Toeplitz algebra $\tau(A^p)$ and if $B(T)(z) \to 0$ as $|z| \to 1$, then T is compact; see [5].

We give the following sufficient condition for T_a to be compact on A^p.

Theorem 2.6. *Assume that $a \in L^1_{\text{loc}}$ and that*

$$(2.11) \qquad\qquad \lim_{d(D)\to 0} \sup_{\zeta \in D} |\hat{a}_D(\zeta)| = 0 \,,$$

where

$$d(D) := \text{dist}(D, \partial\mathbb{D}) := \inf\{|z - w| \mid z \in D, |w| = 1\}.$$

Then $T_a : A^p \to A^p$ is a compact operator for all $1 < p < \infty$.

The proof is a modification of the proof of Theorem 2.3 and is outlined in Section 4.

Corollary 2.7. *Let $1 < p < \infty$ and let $a \geq 0$ be locally integrable. Then $T_a : A^p \to A^p$ is compact if and only if $\hat{a}_r(z) \to 0$ as $|z| \to 1$ (or, which is the same, if and only if $B(a)(z) \to 0$ as $|z| \to 1$.)*

Proof.   The proof is similar to that of Corollary 2.5. Just note that since $k_{z,p} \to 0$ weakly, we have $B(a)(z) \to 0$ as $|z| \to 1$ whenever T_a is compact. ∎

To state our result on Fredholmness, we first recall the notion of the mean oscillation $MO^p_r(f)$ of a function f in L^p (here $r > 0$ is fixed for a moment): it is defined by

$$(2.12) \qquad MO^p_r(f)(z) = \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} |f(\zeta) - \hat{f}_r(z)|^p \, dA(\zeta) \right)^{1/p} .$$

The space of bounded mean oscillation BMO^p_r consists of all locally L^p integrable functions for which $\|f\|^p_{r,p} := \sup_{z\in\mathbb{D}} MO^p_r(f)(z) < \infty$. If, in addition, $MO^p_r(f)(z) \to 0$ as $|z| \to 1$, we say that f is in VMO^p_r. The definition of BMO^p_r depends on p (unlike in the case of the classical BMO for the unit circle) and $BMO^p_r \subset BMO^q_r$ properly for $q < p$. Since the definition is independent of r, we write BMO^p_∂ for BMO^p_r and VMO^p_∂ for VMO^p_r. The following decompositions can be found in [9] (for the case $p = 2$ see also [10, Chapter 7]). For $p \geq 1$,

$$BMO^p_\partial = BO + BA^p \quad \text{and} \quad VMO^p_\partial = VO + VA^p \,,$$

where BO is the space of continuous functions on $\mathbb{D}$ for which $\omega(f)(z) := \sup\{|f(z) - f(w)| : w \in B(z,1)\}$ is a bounded function of z, and BA^p is the space of all functions f on $\mathbb{D}$ so that $\widehat{|f|^p_r}$ is bounded; the spaces VO and VA^p are closed subspaces of BO and BA^p, respectively, consisting of all f for which $\omega(f)(z)$ and $\widehat{|f|^p_r}(z)$ vanish, respectively, as $|z| \to 1$.

A bounded linear operator T on a Banach space X is said to be Fredholm if both its kernel and cokernel are finite dimensional. The index of T is then the difference of these dimensions.

Theorem 2.8. *Assume that $a \in VMO_{\partial}^1$ and that in addition to (2.4) it satisfies for some $\delta > 0$, $C > 0$,*

$$(2.13) \qquad\qquad |\hat{a}_D(\zeta)| \geq C$$

for all $D \in \mathcal{D}$ with $d(D) \leq \delta$, for all $\zeta \in D$. Then T_a is Fredholm, and there is a positive number $R < 1$ such that

$$\operatorname{Ind} T_a = -\operatorname{ind}(B(a){\upharpoonright}s\mathbb{T}) = -\operatorname{ind}(\hat{a}_r{\upharpoonright}s\mathbb{T})$$

for any $s \in [R, 1)$, where $\operatorname{ind} h$ stands for the winding number of a function h, and $h{\upharpoonright}s\mathbb{T}$ stands for the restriction of h into the set $s\mathbb{T}$.

The proof is given in Section 5.

3. Proof of Theorem 2.3

We need to fix an explicit decomposition of $\mathbb{D}$ into sets belonging to the family $\mathcal{D}$.

Definition 3.1. Let us denote by $n := n(m, \mu)$ a bijection from the set $\{(m, \mu) \mid m \in \mathbb{N}, \mu = 1, \ldots, 2^{-m}\}$ onto $\mathbb{N}$ which preserves the order in the sense that $m < l \Rightarrow n(m, \mu) < n(l, \lambda)$ for all μ and λ, and $\mu < \lambda \Rightarrow n(m, \mu) < n(m, \lambda)$ for all m.

We define the sets $D_n := D_{n(m,\mu)} := D\big(1 - 2^{-m+1}, 2\pi(\mu - 1)2^{-m}\big) \in \mathcal{D}$, that is,

$$
\begin{aligned}
D_n := \big\{ z = re^{i\theta} \mid & 1 - 2^{-m+1} < r \leq 1 - 2^{-m}, \\
& \pi(\mu - 1)2^{-m+1} < \theta \leq \pi\mu 2^{-m+1} \big\},
\end{aligned}
$$

$$(3.1)$$

where $m \in \mathbb{N}$ and $\mu = 1, \ldots, 2^m$. We also denote for $n = n(m, \mu)$,

$$
\begin{aligned}
& r_n = 1 - 2^{-m+1} \,, r_n' := 1 - 2^{-m} \\
& \theta_n = \pi(\mu - 1)2^{-m+1} \,,\ \theta_n' := \pi\mu 2^{-m+1},
\end{aligned}
$$

$$(3.2)$$

hence for all $n \in \mathbb{N}$, the set D_n is of the form

$$(3.3) \qquad D_n = \{ z = re^{i\theta} \mid r_n < r \leq r_n', \theta_n < \theta \leq \theta_n' \}.$$

We denote for all $\zeta = \rho e^{i\phi} \in D_n$

$$(3.4) \qquad I_n(\zeta) := \hat{a}_{D_n}(\zeta) = \frac{1}{|D_n|} \int_{r_n}^{\rho} \int_{\theta_n}^{\phi} a(\varrho e^{i\varphi})\, \varrho\, d\varrho\, d\varphi.$$

To prove the theorem we assume that (2.4) holds; then, for all n and all $\zeta \in D_n$,

$$(3.5) \qquad\qquad |I_n(\zeta)| \leq \mathbf{C}_a,$$

where $\mathbf{C}_a$ is as after (2.5).

Let $f \in A^p$. Let us fix an $n(m,\mu)$ and evaluate the following integral using integration by parts:

$$
\begin{aligned}
\int_{D_n} \frac{a(\zeta)f(\zeta)}{(1-z\bar\zeta)^2}\, dA(\zeta) &= \int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} \frac{a(re^{i\theta})f(re^{i\theta})}{(1-zre^{-i\theta})^2}\, r\, dr\, d\theta \\
&= \int_{r_n}^{r'_n} \left(\int_{\theta_n}^{\theta'_n} ra(re^{i\varphi})d\varphi \right) \frac{f(re^{i\theta'_n})}{(1-zre^{-i\theta'_n})^2}\, dr \\
&\quad - \int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} \left(\int_{\theta_n}^{\theta} a(re^{i\varphi})d\varphi \right) \partial_\theta \frac{f(re^{i\theta})}{(1-zre^{-i\theta})^2}\, d\theta\, r\, dr \\
&= \left(\int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} a(\varrho e^{i\varphi})\, \varrho\, d\varphi\, d\varrho \right) \frac{f(r'_n e^{i\theta'_n})}{(1-zr'_n e^{-i\theta'_n})^2} \\
&\quad - \int_{r_n}^{r'_n} \left(\int_{r_n}^{r} \int_{\theta_n}^{\theta'_n} a(\varrho e^{i\varphi})\, \varrho\, d\varphi\, d\varrho \right) \partial_r \frac{f(re^{i\theta'_n})}{(1-zre^{-i\theta'_n})^2}\, dr \\
&\quad - \int_{\theta_n}^{\theta'_n} \left(\int_{r_n}^{r'_n} \int_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\, \varrho\, d\varphi\, d\varrho \right) \partial_\theta \frac{f(r'_n e^{i\theta})}{(1-zr'_n e^{-i\theta})^2}\, d\theta \\
&\quad + \int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} \left(\int_{r_n}^{r} \int_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\varrho\, d\varphi\, d\varrho \right) \partial_r \partial_\theta \frac{f(re^{i\theta})}{(1-zre^{-i\theta})^2}\, d\theta\, dr
\end{aligned}
$$

$$(3.6) \qquad =: F_{1,n}(z) + F_{2,n}(z) + F_{3,n}(z) + F_{4,n}(z) =: F_n(z).$$

We shall find suitable bounds for the terms $F_{j,n}(z)$. Let us first consider the case $j=1$. The integral of a in $F_{1,n}$ has, by (3.5), the bound

$$(3.7) \qquad\qquad \left| \iint a(\varrho e^{i\varphi})\, \varrho\, d\varphi\, d\varrho \right| \leq \mathbf{C}_a |D_n|$$

Moreover, by [10, Proposition 4.13],

$$(3.8) \qquad\qquad |f(r'_n e^{i\theta'_n})| \leq \frac{C}{|D_n|} \int_{D_n} |f(\zeta)|dA(\zeta),$$

and since

$$(3.9) \qquad\qquad |1 - zr'_n e^{i\theta'_n}| \geq C|1-z\bar\zeta|$$

for all $\zeta \in D_n$, we get

$$(3.10) \qquad\qquad |F_{1,n}(z)| \leq C\,\mathbf{C}_a \int_{D_n} \frac{|f(\zeta)|}{|1-z\bar\zeta|^2}\, dA(\zeta).$$

As for $F_{2,n}$, performing the differentiation and using estimates like (3.8) (which also holds for f') and (3.9), we get

$$\left| \partial_r \frac{f(re^{i\theta})}{(1 - zre^{-i\theta})^2} \right| \leq \frac{C}{|D_n|} \int_{D_n} \left(\frac{|f(\zeta)|}{|1 - z\bar\zeta|^3} + \frac{|f'(\zeta)|}{|1 - z\bar\zeta|^2} \right) dA(\zeta)$$

Hence, taking into account (3.7) and $|r'_n - r_n| \leq C|1 - z\bar\zeta|$ and $|r'_n - r_n| \leq C(1 - |\zeta|^2)$ for all $\zeta \in D_n$, we can evaluate $F_{2,n}$:

$$|F_{2,n}(z)| = \left| \int_{r_n}^{r'_n} \left(\int_{r_n}^{r} \int_{\theta_n}^{\theta'_n} a(\varrho e^{i\varphi}) \, \varrho \, d\varphi \, d\varrho \right) \partial_r \frac{f(re^{i\theta'_n})}{(1 - zre^{-i\theta'_n})^2} \, dr \right|$$

$$\leq C \int_{r_n}^{r'_n} \left| \int_{r_n}^{r} \int_{\theta_n}^{\theta'_n} a(\varrho e^{i\varphi}) \varrho \, d\varphi \, d\varrho \right| \frac{1}{|D_n|} \int_{D_n} \left(\frac{|f(\zeta)|}{|1 - z\bar\zeta|^3} + \frac{|f'(\zeta)|}{|1 - z\bar\zeta|^2} \right) dA(\zeta) \, dr$$

$$\leq C' \, \mathbf{C}_a \int_{r_n}^{r'_n} \int_{D_n} \left(\frac{|f(\zeta)|}{|1 - z\bar\zeta|^3} + \frac{|f'(\zeta)|}{|1 - z\bar\zeta|^2} \right) dA(\zeta) \, dr$$

(3.11)

$$\leq C'' \, \mathbf{C}_a \int_{D_n} \left(\frac{|f(\zeta)|}{|1 - z\bar\zeta|^2} + \frac{|f'(\zeta)|(1 - |\zeta|^2)}{|1 - z\bar\zeta|^2} \right) dA(\zeta).$$

The remaining terms with $j = 3, 4$ can be estimated analogously. For $j = 3$ we get the same bound as in (3.11), and for $j = 4$ we get the bound

$$|F_{4,n}(z)| \leq C \, \mathbf{C}_a \int_{D_n} \left(\frac{|f(\zeta)|}{|1 - z\bar\zeta|^2} + \frac{|f'(\zeta)|(1 - |\zeta|^2)}{|1 - z\bar\zeta|^2} \right.$$

(3.12)
$$\left. + \frac{|f''(\zeta)|(1 - |\zeta|^2)^2}{|1 - z\bar\zeta|^2} \right) dA(\zeta).$$

As a consequence,

$$(3.13) \quad \sum_{n=1}^{\infty} F_n(z) \leq \sum_{n=1}^{\infty} |F_n(z)| \leq \sum_{n=1}^{\infty} \sum_{j=1}^{4} |F_{j,n}(z)|$$

$$\leq C \, \mathbf{C}_a \int_{\mathbb{D}} \frac{|f(\zeta)| + |f'(\zeta)|(1 - |\zeta|^2) + |f''(\zeta)|(1 - |\zeta|^2)^2}{|1 - z\bar\zeta|^2} \, dA(\zeta).$$

The maximal Bergman projection is known to be bounded on L^p, i.e., there exists a constant $C > 0$ such that

$$(3.14) \qquad \left\| \int_{\mathbb{D}} \frac{|g(\zeta)|}{|1 - z\bar\zeta|^2} \, dA(\zeta) \right\|_p \leq C \|g\|_p$$

for all $g \in L^p$ (see [10, Corollary 3.13]). Moreover, by Theorem 4.28 of [10], the functions $|f'(\zeta)|(1 - |\zeta|^2)$ and $|f''(\zeta)|(1 - |\zeta|^2)^2$ belong to L^p. Using (3.13)

we thus find that the series (see (3.6))

$$\sum_{n=1}^{\infty} |F_n(z)|$$

can be pointwise bounded by an L^p function, and thus it converges for almost all z. Hence, the Toeplitz operator can be defined as explained in Remark 2.4, 1°. Moreover, by the same arguments,

$$(3.15) \qquad \Big\| \sum_{n=1}^{\infty} F_n(z) \Big\|_p \leq C\, \mathbf{C}_a \|f\|_p,$$

which proves the desired boundedness of T_a.   ∎

Corollary 3.2. *If $a \in BA^1$, then $T_a : A^p \to A^p$ is bounded, and*

$$\|T_a\| \leq C \|a\|_{BA^1}.$$

4. Proof of Theorem 2.6

We sketch the changes needed in Section 3. By definitions, we actually have

$$(4.1) \qquad \lim_{n \to \infty} \sup_{\zeta \in D_n} |I_n(\zeta)| = 0.$$

To prove Theorem 2.6 one takes an arbitrary sequence $(f_k)_{k=1}^{\infty} \subset A^p$ with $\|f_k\|_p \leq 1$ for all k such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. It is enough to prove that $\|T_a f_k\|_p \to 0$.

Let $\varepsilon > 0$ be arbitrary. Let the expressions $F_{j,n}(z)$ be as in Section 3, and let $F_{j,n,k}(z)$ be equal to $F_{j,n}(z)$ with f is replaced by f_k. Consider $F_{1,n,k}(z)$. Applying (4.1) to (3.7) and deducing as in (3.7)–(3.10), we obtain instead of (3.10) the estimate

$$(4.2) \qquad |F_{1,n,k}(z)| \leq C\varepsilon_n \int_{D_n} \frac{|f_k(\zeta)|}{|1 - z\bar{\zeta}|^2} \, dA(\zeta),$$

where we may assume that $\varepsilon_n \to 0$. Choose $N \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ for $n \geq N$.

Since the closure of the set $D^{(N)} := \cup_{n \leq N} D_n$ is compact, there exists a constant $C_N > 0$ such that

$$\sum_{n \leq N} \int_{D_n} \frac{1}{|1 - z\bar{\zeta}|^2} \, dA(\zeta) \leq C_N$$

for all $z \in \mathbb{D}$. Since the sequence (f_k) converges to 0 on compact subsets, we may choose $M \in \mathbb{N}$ such that

$$|f_k(\zeta)| \leq \frac{\varepsilon}{1 + C_N}$$

for all $k \geq M$ and $\zeta \in D^{(N)}$.

We get for all $k \geq M$

$$\sum_{n=1}^{\infty} |F_{1,n,k}(z)| \leq \sum_{n \leq N} \int_{D_n} \frac{|f_k(\zeta)|}{|1 - z\bar{\zeta}|^2} \, dA(\zeta) + \sum_{n > N} \int_{D_n} \varepsilon_n \frac{|f_k(\zeta)|}{|1 - z\bar{\zeta}|^2} \, dA(\zeta)$$

$$\leq \sum_{n \leq N} \frac{\varepsilon}{1 + C_N} \int_{D_n} \frac{1}{|1 - z\bar{\zeta}|^2} \, dA(\zeta) + \varepsilon \int_{\mathbb{D}} \frac{|f_k(\zeta)|}{|1 - z\bar{\zeta}|^2} \, dA(\zeta)$$

$$\leq \varepsilon + \varepsilon \int_{\mathbb{D}} \frac{|f_k(\zeta)|}{|1 - z\bar{\zeta}|^2} \, dA(\zeta).$$

The other expressions $F_{j,n,k}(z)$, $j = 2, 3, 4$, are treated similarly, keeping in mind that also the derivatives of the functions f_k converge to 0 uniformly on compact subsets of $\mathbb{D}$. Arguing as in the proof of Theorem 2.3 we find that $\|T_a f_k\|_p \leq C\varepsilon$ for large enough k. ∎

Comparing the definition of the space VA^1 and the condition (2.11) immediately yields the following

Corollary 4.1. *If $a \in VA^1$, then $T_a : A^p \to A^p$ is compact.*

5. Proof of Theorem 2.8

Before proceeding with the proof, we recall for the convenience of the reader some Fredholm theory that will be needed.

Theorem 5.1. *Let T be a bounded linear operator on a Banach space X. Then T is Fredholm, if it has a regularizer, that is, if there exists a bounded linear operator $S : X \to X$ such that $ST - I$ and $TS - I$ are both compact.*

The index function Ind is constant on each connected component of the space of all Fredholm operators on X. In particular, if $S : X \to X$ is compact, then $\mathrm{Ind}(T + S) = \mathrm{Ind}\, T$.

For these facts, see, for example, [4].

We now proceed to prove Theorem 2.8. Since $a \in VMO_{\partial}^1$, it is enough to assume (2.13) only for ζ as in Remark 2.4. Also the symbol a has a decomposition

$$(5.1) \qquad\qquad a = a_1 + a_2$$

with $a_1 \in VO$ and $a_2 \in VA^1$. Because of Corollary 4.1, it is enough to show that T_{a_1} is Fredholm. Moreover, we may assume that a_1 satisfies (2.13) and (2.4); thus, by redefining the notation, we may assume that $a \in VO$.

Let $R > 0$ be so large that every $D \in \mathcal{D}$ is contained in a hyperbolic disc $B(z, R)$. By the VO–condition (with the same C as in (2.13)) we may assume that

$$(5.2) \qquad\qquad |a(z) - a(w)| \leq \frac{C}{2}$$

for all $|z|$ sufficiently large, for all $w \in B(z, R)$. We claim that $a(z) \neq 0$ for $|z|$ sufficiently large, say, $|z| \geq s$, where $0 < s < 1$. If this were not true, we would find a set $D \in \mathcal{D}$ with $d(D) \leq \delta$ such that $a(\zeta) = 0$ for some $\zeta \in D$ and such that (5.2) would hold for some $B(z, R)$ containing D. As a consequence, $|a(w)| \leq C/2$ for all $w \in D$, which implies a contradiction with (2.13). In the same way we find that a must be bounded. (However, the original a need not be bounded!)

We form a regularizer of T_a by defining first

$$(5.3) \qquad\qquad b(z) = \frac{1}{a(z)} \qquad \text{for } |z| \geq s$$

and $b(z) = 1$ for $|z| \leq s$. We have

$$T_a T_b = I - P(I - M_{ab}) - P M_a (I - P) M_b$$
$$= I - T_{1-ab} - P M_a H_b.$$

Here T_{1-ab} is obviously compact, since $1 - a(z)b(z) = 0$ for all $|z| \geq s$. For the compactness of H_b, note that b is in VO, which is generated by functions in $C(\overline{\mathbb{D}})$. Moreover, the operator $P M_a$ is bounded on $L^p \supset H_b(A^p)$, since a is a bounded function, so $T_a T_b = I + K$ for K compact. Similarly, $T_b T_a = I + K'$ for K' compact. Thus, T_a has a regularizer and is hence Fredholm (see Theorem 5.1).

It remains to prove the index formula. Returning back to the decomposition (5.1), we have

$$\operatorname{Ind} T_a = \operatorname{Ind} T_{a_1},$$

which follows from Corollary 4.1 and Theorem 5.1. Note that the decomposition (5.1) can be obtained by setting $a_1 = \tilde{a}$ or $a_1 = \hat{a}_r$. Hence, it suffices to prove the index formula for T_a with $a \in VO$ and $|a(z)| \geq \epsilon$ whenever $s \leq |z| < 1$, where ϵ, s are some positive constants. (See the argument after (5.2).) Let $a_s(\zeta) = a(s\zeta)$ for $\zeta \in \mathbb{T}$ and let $\kappa = \operatorname{ind} a_s$. Let $f(z) = (\bar{z}/|z|)^\kappa a$ for $|z| \geq s$, and extend f to $\mathbb{D}$ so that $|f(z)| \geq \epsilon'$ for some $\epsilon' > 0$, which implies that $1/f \in VO$. If $g(z) = (\bar{z}/|z|)^\kappa$ for $|z| \geq s$ and $g(z) = (\bar{z}/s)^\kappa$ for $|z| < s$, then $ga - f = 0$ for $|z| \geq s$, and so $T_{ga} = T_f + K$ for K compact, and thus

$$\kappa + \operatorname{Ind} T_a = \operatorname{Ind} T_g + \operatorname{Ind} T_a = \operatorname{Ind} T_f.$$

We still need to show that $\operatorname{Ind} T_f = 0$. Let $h_t = t(1/|f|) + (1-t)$ for $t \in [0,1]$. We observe that the mapping $t \mapsto T_{h_t}$ is continuous with respect to the operator norm, since by Theorem 2.3

$$
\begin{aligned}
\|T_{h_t} - T_{h_s}\| &= \|T_{h_t - h_s}\| \\
&\leq C \sup_{D \in \mathcal{D}} \frac{1}{|D|} \int_D |h_t(\zeta) - h_s(\zeta)| \, dA(\zeta) \\
&\leq C \sup_{D \in \mathcal{D}} \frac{1}{|D|} \int_D \left(|t - s| \frac{1}{|f(\zeta)|} + |t - s| \right) dA(\zeta) \\
&= C \sup_{D \in \mathcal{D}} \frac{1}{|D|} \int_D \left(1 + \frac{1}{\epsilon'} \right) |t - s| \, dA(\zeta) \\
&\leq C' \, |t - s| \qquad \text{for } t, s \in [0,1].
\end{aligned}
$$

Hence, all T_{h_t} are in the same component of the class of Fredholm operators on A^p and therefore they all have the same index (Theorem 5.1); in particular, $\operatorname{Ind} T_{1/|f|} = \operatorname{Ind} T_{h_1} = \operatorname{Ind} T_{h_0} = 0$. Thus,

$$
\operatorname{Ind} T_{f/|f|} = \operatorname{Ind} T_f + \operatorname{Ind} T_{1/|f|} = \operatorname{Ind} T_f,
$$

and so we can assume $|f| = 1$ and write $f(z) = e^{i \arg f(z)}$. For any integer $m > 0$, we have $\exp(i(\arg f)/m) \in VO$ and

$$
\operatorname{Ind} T_f = \operatorname{Ind} T_{\exp(i(\arg f)/m)}^m = m \operatorname{Ind} T_{\exp(i(\arg f)/m)} \;;
$$

since the index is always an integer, the last expression cannot be bounded for all m unless $\operatorname{Ind} T_{\exp(i(\arg f)/m)}$ equals zero. So, $\operatorname{Ind} T_f = 0$. ∎

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