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End-point estimates and multi-parameter paraproducts on higher dimensional tori

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Abstract

Analogues of multi-parameter multiplier operators on $\mathbb{R}^d$ are defined on the torus $\mathbb{T}^d$. It is shown that these operators satisfy the classical Coifman-Meyer theorem. In addition, $L(\log L)^n$ end-point estimates are proved

1. Introduction

This article is, in part, a continuation of [13, 14]. It is also derived from the author's dissertation, which can be found in full at [17].

Recall the multi-linear Coifman-Meyer [5] operator

$$\Lambda_m^{(1)}(f_1, \ldots, f_d)(x) = \int_{\mathbb{R}^d} m(t)\widehat{f_1}(t_1) \cdots \widehat{f_d}(t_d) e^{2\pi i x(t_1+\cdots+t_d)}\, dt,$$

for Schwartz functions f_j and where m satisfies a standard Marcinkiewicz-Mihlin-Hörmander type condition [12]. It is well known this operator maps $L^{p_1} \times \cdots \times L^{p_d} \to L^p$ for $1/p_1 + \cdots + 1/p_d = 1/p$ and $1 < p_j < \infty$. The case when $p \geq 1$ was originally shown by Coifman and Meyer. The general case $p > 1/d$ was settled later in [9, 11].

Led by natural questions in non-linear partial differential equations, extensions of this operator were considered by Muscalu et. al.: first the so-called bi-parameter multiplier [13], then multi-parameter multipliers [14]. In this setting, m is allowed to belong to a much wider class of multipliers which behave like the product of standard multipliers. Special cases of these multiplier operators had been previously considered by Christ and Journé [4, 10]. In [13, 14], it is shown that these multiplier operators satisfy the same $L^{p_1} \times \cdots \times L^{p_d} \to L^p$ property.

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However, in the single-parameter case of Coifman and Meyer, more is known. We have "end-point" estimates corresponding to the case when any or all of the p_j are equal to 1. Here, the result is $L^{p_1} \times \cdots \times L^{p_d} \to L^{p,\infty}$. In the multi-parameter setting, no such end-point estimates are known.

A natural candidate for such an estimate would involve $L \log L$ spaces, because of how they arise in interpolation results. Naively, an operator which maps $L^1 \to L^{1,\infty}$, and also satisfies some L^p result, is often thought to also satisfy some $L \log L$ to L^1 property. Indeed, we recall the result of Stein [16], which states Mf is locally integrable if and only if f is locally in $L \log L$; alternatively, C. Fefferman [6] showed the maximal double Hilbert transform maps $L \log L([0,1]^2)$ to $L^{1,\infty}([0,1]^2)$.

That $L \log L$ estimates can only be gained in the compact setting is a rather common obstacle. To avoid this, we instead consider analogues of multiplier operators defined on the torus $\mathbb{T}^d$. This also allows a departure from the classical definition of $L \log L$ spaces to a more iterative approach which blends perfectly with our methods. Ultimately, we show that the s-parameter multiplier operator $\Lambda_m^{(s)}$ in this setting satisfies the classical Coifman-Meyer theorem, along with the desired end-point estimate: for $p_j = 1$ each L^{p_j} is replaced by $L(\log L)^{s-1}$.

The organization is as follows. In the next section, characterizations of $L(\log L)^n$ are developed for any probability space, and several important results therein are proved. Section 3 details the connections between $L(\log L)^n$ spaces and the Hardy-Littlewood maximal operator. Section 4 deals with the notion of adapted families and a particular square function of Littlewood-Paley type. Section 5 introduces hybrid square-max operators. In Section 6, bi-parameter multiplier operators are handled, while section 7 is a non-rigorous survey of the proof for multi-parameter multipliers.

A remark on the notation used: we will write $A \lesssim B$ whenever $A \le C \cdot B$ with some universal constant C.

2. Zygmund spaces and $L(\log L)^n$

Let (X, ρ) be a probability space. For $f : (X, \rho) \to \mathbb{C}$, denote the decreasing rearrangement of f by f^*.

Definition. For $t > 0$ and $f : (X, \rho) \to \mathbb{C}$, let $f^{(*,1)}(t) = f^*(t)$ and for integers $n \geq 2$, set $f^{(*,n)}(t) = \frac{1}{t} \int_0^t f^{(*,n-1)}(s)\, ds$.

On a probability space, f^* is supported on $[0,1]$. It is advantageous to informally think of each $f^{(*,n)}$ as being defined only on $(0,1]$.

We can immediately verify the following properties: (1) $f^{(*,n)}$ is nonnegative, decreasing, and identically 0 if and only if $f = 0$ a.e.$[\rho]$; (2) $f^{(*,n)} \leq$

$f^{(*,n+1)}$; (3) $(\alpha f)^{(*,n)} = |\alpha| f^{(*,n)}$ (4) $|f| \le |g|$ a.e.$[\rho]$ implies $f^{(*,n)} \le g^{(*,n)}$ pointwise; (5) $|f_k| \uparrow |f|$ a.e.$[\rho]$ implies $f_k^{(*,n)} \uparrow f^{(*,n)}$ pointwise.

We would also like to show $(f+g)^{(*,n)}(t) \le f^{(*,n)}(t)+g^{(*,n)}(t)$ for all $t>0$ and $n \ge 2$; this property does not hold for $n=1$. By induction, it suffices to prove the result for $n=2$. However, this is an immediate consequence of the following technical result of Bennett and Sharpley [3]:

$$t f^{(*,2)}(t) = \int_0^t f^*(s)\,ds = \inf_{f=g+h} \big\{ \|g\|_1 + t\|h\|_\infty \big\}.$$

Definition. For $f : (X,\rho) \to \mathbb{C}$ and integers $n \ge 0$, define $\|f\|_{L(\log L)^n}$ by

$$\|f\|_{L(\log L)^n} = \int_0^1 f^{(*,n+1)}(t)\,dt.$$

Define the Zygmund space $L(\log L)^n(X)$ as the set of functions f with $\|f\|_{L(\log L)^n} < \infty$.

We note that $L(\log L)^0(X) = L^1(X)$, which is a useful notational short-cut. Clearly, $\|\cdot\|_{L(\log L)^n}$ is a norm with the additional properties that $|f| \le |g|$ a.e.$[\rho]$ implies $\|f\|_{L(\log L)^n} \le \|g\|_{L(\log L)^n}$ and $|f_k| \uparrow |f|$ a.e.$[\rho]$ implies $\|f_k\|_{L(\log L)^n} \uparrow \|f\|_{L(\log L)^n}$. Further, this definition of $L(\log L)^n$ coincides with the classical space.

Theorem 2.1. *$f \in L(\log L)^n(X)$ if and only if*

$$\int_X |f(x)| \big(\log^+ |f(x)|\big)^n \rho(dx) < \infty.$$

The proof is fairly technical but straightforward and is left to the reader. Using Hardy's inequality, it is also easy to establish the following.

Theorem 2.2. *For any $1 < p \le \infty$ and $n \ge 0$,*

$$L^p(X) \subseteq L(\log L)^{n+1}(X) \subseteq L(\log L)^n(X) \subseteq L^1(X),$$

with $\|f\|_1 \le \|f\|_{L(\log L)^n} \le \|f\|_{L(\log L)^{n+1}} \lesssim \|f\|_p$.

The principal reason for defining $L(\log L)^n$ as we have is the ease in which we gain interpolation results.

Lemma 2.3. *Let T be a sublinear operator which maps $L^1(X) \to L^{1,\infty}(X)$ and $L^p(X) \to L^{q,\infty}(X)$, for some $1 < p,q < \infty$. Then, for $n \in \mathbb{N}$,*

$$(Tf)^{(*,n)}(t) \lesssim \left[\frac{1}{t} \int_0^{t^m} f^{(*,n)}(s)\,ds + t^{-1/q} \int_{t^m}^1 s^{1/p-1} f^{(*,n)}(s)\,ds \right],$$

where $m = (\frac{1}{q}-1)(\frac{1}{p}-1)^{-1}$.

Proof. We show this by induction. The $n = 1$ case is a technical result established in [3]. Assume it is true for $n-1$. Then,

$$\begin{aligned}(Tf)^{(*,n)}(t) &= \frac{1}{t}\int_0^t T^{(*,n-1)}(s)\,ds \\ &\lesssim \frac{1}{t}\int_0^t \frac{1}{s}\int_0^{s^m} f^{(*,n-1)}(u)\,du\,ds + \frac{1}{t}\int_0^t s^{-1/q}\int_{s^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\,du\,ds \\ &=: I + II.\end{aligned}$$

By the change of variables $r = s^m$,

$$I = \frac{1}{m}\frac{1}{t}\int_0^{t^m}\frac{1}{r}\int_0^r f^{(*,n-1)}(u)\,du\,dr = \frac{1}{m}\frac{1}{t}\int_0^{t^m} f^{(*,n)}(r)\,dr.$$

On the other hand, changing the order of integration gives

$$\begin{aligned}II &= \frac{1}{t}\int_0^{t^m} u^{1/p-1} f^{(*,n-1)}(u)\int_0^{u^{1/m}} s^{-1/q}\,ds\,du \\ &\quad + \frac{1}{t}\int_{t^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\int_0^t s^{-1/q}\,ds\,du \\ &= \frac{1}{1-1/q}\,\frac{1}{t}\int_0^{t^m} f^{(*,n-1)}(u)\,du + \frac{1}{1-1/q}\,t^{-1/q}\int_{t^m}^1 u^{1/p-1} f^{(*,n-1)}(u)\,du \\ &\le \frac{1}{1-1/q}\left[\frac{1}{t}\int_0^{t^m} f^{(*,n)}(u)\,du + t^{-1/q}\int_{t^m}^1 u^{1/p-1} f^{(*,n)}(u)\,du\right].\end{aligned}$$

■

Theorem 2.4. *Let T be a sublinear operator which maps $L^1(X) \to L^{1,\infty}(X)$ and $L^p(X) \to L^{q,\infty}(X)$, for some $1 < p, q < \infty$. Then, for all $n \in \mathbb{N}$, T also maps $L(\log L)^n(X) \to L(\log L)^{n-1}(X)$.*

Proof. Set $m = (\frac{1}{q} - 1)(\frac{1}{p} - 1)^{-1}$. Using Lemma 2.3 and the same change of variables and Fubini arguments,

$$\begin{aligned}\|Tf\|_{L(\log L)^{n-1}} &= \int_0^1 (Tf)^{(*,n)}(t)\,dt \\ &\lesssim \int_0^1 \frac{1}{t}\int_0^{t^m} f^{(*,n)}(s)\,ds\,dt + \int_0^1 t^{-1/q}\int_{t^m}^1 s^{1/p-1} f^{(*,n)}(s)\,ds\,dt \\ &= \frac{1}{m}\int_0^1 \frac{1}{u}\int_0^u f^{(*,n)}(s)\,ds\,du + \int_0^1 s^{1/p-1} f^{(*,n)}(s)\int_0^{s^{1/m}} t^{-1/q}\,dt\,ds \\ &= \frac{1}{m}\int_0^1 f^{(*,n+1)}(u)\,du + \frac{1}{1-1/q}\int_0^1 f^{(*,n)}(s)\,ds \lesssim \|f\|_{L(\log L)^n}.\end{aligned}$$

■

Corollary 2.5. *Let T be a sublinear operator. If for some $1 < p, r < \infty$*

$$\Big\| \Big(\sum_{k=1}^{\infty} |Tf_k|^r \Big)^{1/r} \Big\|_{1,\infty} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_{1} \quad \textit{and}$$

$$\Big\| \Big(\sum_{k=1}^{\infty} |Tf_k|^r \Big)^{1/r} \Big\|_{p} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_{p},$$

then for all $n \in \mathbb{N}$

$$\Big\| \Big(\sum_{k=1}^{\infty} |Tf_k|^r \Big)^{1/r} \Big\|_{L(\log L)^{n-1}} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_{L(\log L)^{n}}.$$

Proof. This only requires viewing the above theory through the wider scope of Banach space-valued functions $f : (X, \rho) \to (B, \|\cdot\|_B)$ (see [8]). If instead one defined the decreasing rearrangement f^* for Banach space-valued functions, in the natural way, and repeated the definitions and arguments of this section, everything would still hold. In particular, the previous theorem is valid; if T is sublinear operator mapping $L^1_B(X)$ to $L^{1,\infty}_B(X)$ and $L^p_B(X)$ to $L^{q,\infty}_B(X)$, then $T : L(\log L)^n_B(X) \to L(\log L)^{n-1}_B(X)$. But, simply by definition, $f^*(t) = (\|f\|_B)^*(t)$, where $(\|f\|_B)^*$ is understood as the decreasing rearrangement of the map $x \mapsto \|f(x)\|_B$. Thus,

$$\|f\|_{L(\log L)^n_B} = \big\| \|f\|_B \big\|_{L(\log L)^n}.$$

Let $B = \ell^r$ and $\overline{T}(f) = (Tf_1, Tf_2, \ldots)$, so that $\overline{T} : L^1_B(X) \to L^{1,\infty}_B(X)$ and $L^p_B(X) \to L^p_B(X)$. Thus, $\overline{T} : L(\log L)^n_B(X) \to L(\log L)^{n-1}_B(X)$, which is what was promised. ■

3. Connections to Hardy-Littlewood

Let us turn our attention to the probability space $(\mathbb{T}, m)$. Let Mf denote the standard Hardy-Littlewood maximal operator on $\mathbb{T}$. Of course, M maps $L^1(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$ and $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for all $1 < p < \infty$. So, by the interpolation results of the previous section, $M : L(\log L)^n(\mathbb{T}) \to L(\log L)^{n-1}(\mathbb{T})$. Further, from Fefferman and Stein [7], we know

$$\Big\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \Big\|_{1,\infty} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_{1} \quad \text{and}$$

$$\Big\| \Big(\sum_{k=1}^{\infty} |Mf_k|^r \Big)^{1/r} \Big\|_{p} \lesssim \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_{p},$$

for all $1 < p, r < \infty$, and therefore Corollary 2.5 applies. However, much more can be said.

Theorem 3.1. *$f^{(*,n+1)}(t) \sim (Mf)^{(*,n)}(t)$, where the underlying constants do not depend on f or t.*

It clearly suffices, by induction, to prove $f^{(*,2)}(t) \sim (Mf)^*(t)$. But, this is a well-known result; see [2, 3].

Corollary 3.2. *$f \in L(\log L)^{n+1}(\mathbb{T})$ if and only if $Mf \in L(\log L)^n(\mathbb{T})$, and, in particular, $\|f\|_{L(\log L)^{n+1}} \sim \|Mf\|_{L(\log L)^n}$.*

4. Adapted families

Definition. A smooth function $\varphi : \mathbb{T} \to \mathbb{C}$ is adapted to an interval I with constants $C_m > 0$, $m \in \mathbb{N}$, if

$$|\varphi(x)| \leq C_m \left(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x, I)}{|I|}\right)^{-m} \text{ for all } x \in \mathbb{T}, m \in \mathbb{N},$$

$$|\varphi'(x)| \leq C_m \frac{1}{|I|} \left(1 + \frac{\operatorname{dist}_{\mathbb{T}}(x, I)}{|I|}\right)^{-m} \text{ for all } x \in \mathbb{T}, m \in \mathbb{N}.$$

A family of smooth functions $\varphi_I : \mathbb{T} \to \mathbb{C}$, indexed by the dyadic intervals, is called an adapted family if each φ_I is adapted to I with the same universal constants. We say $\{\varphi_I\}_I$ is a 0-mean adapted family if it is an adapted family, with the additional property that $\int_{\mathbb{T}} \varphi_I \, dm = 0$ for all I.

For an adapted family φ_I, define $\phi_I = |I|^{-1/2}\varphi_I$, where $|I|$ denotes Lebesgue measure. Note $\|\phi_I\|_2 \lesssim 1$ for all I. Often, ϕ_I is called an L^2-normalized family. Per our notation, φ_I will always represent an adapted family, and ϕ_I will always represent the L^2-normalization.

Conceptually, we often think of functions which are adapted to an interval I as being "almost supported" in I. The following theorem, which is a variation of a result in [14], gives some rigid meaning to this.

Theorem 4.1. *Let $\varphi_I : \mathbb{T} \to \mathbb{C}$ be adapted to a dyadic interval I, with $|I| = 2^{-N}$. Then, we can write*

$$\varphi_I = \sum_{k=1}^{\infty} 2^{-10k} \varphi_I^k,$$

where each φ_I^k is adapted to I, uniformly in k, with $\operatorname{supp}(\varphi_I^k) \subseteq 2^k I$ for $1 \leq k \leq N$ and $\varphi_I^k = 0$ otherwise. Further, if φ_I has integral 0, each φ_I^k can be chosen to have integral 0.

To clarify the notation above, for an interval I and constant $\alpha > 0$, αI is the interval concentric with I so that $|\alpha I| = \alpha |I|$.

Given an adapted family φ_I, its normalization ϕ_I, and $f : \mathbb{T} \to \mathbb{C}$, we will be interested in "averages" of f with respect to the family. Let

$$M'f(x) = \sup_I \frac{1}{|I|^{1/2}} |\langle \phi_I, f\rangle| \chi_I(x).$$

where the supremum is over all dyadic intervals. For a 0-mean adapted family φ_I, define the Littlewood-Paley (discrete) square function by

$$Sf(x) = \Big(\sum_I \frac{|\langle \phi_I, f\rangle|^2}{|I|} \chi_I(x) \Big)^{1/2},$$

where the sum is over all dyadic intervals.

Using Theorem 4.1, it is easily shown that $M'f \lesssim Mf$, so that M' satisfies the same properties as M. It is known that $S : L^1 \to L^{1,\infty}$ and $L^p \to L^p$ for $1 < p < \infty$ (see [17] for a new approach). We will need to establish Fefferman-Stein inequalities for S as well, but the only the special case $r = 2$ will be necessary.

Theorem 4.2. *For $1 < p < \infty$ and any sequence $f_1, f_2, \dots$ of complex-valued functions on $\mathbb{T}$*

$$\Big\| \Big(\sum_{k=1}^\infty |Sf_k|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{k=1}^\infty |f_k|^2 \Big)^{1/2} \Big\|_p,$$

$$\Big\| \Big(\sum_{k=1}^\infty |Sf_k|^2 \Big)^{1/2} \Big\|_{1,\infty} \lesssim \Big\| \Big(\sum_{k=1}^\infty |f_k|^2 \Big)^{1/2} \Big\|_1.$$

Only considering the $r = 2$ allows us to use Rademacher functions and Khinchine's inequality to "linearize." For the weak-L^1 inequality, an alternate characterization called the Kolmogorov condition is helpful (see [8]). For full details, see [17].

5. Hybrid operators

The definitions of the hybrid operators MS, SM, and SS, their properties, and their relevance in our context are borrowed from [13].

We say a set $R \subset \mathbb{T}^2$ is a dyadic rectangle if there exist dyadic intervals I and J so that $R = I \times J$. Given two (possibly distinct) adapted families φ_I and φ_J, we will write $\varphi_R(x, y) = \varphi_I(x)\varphi_J(y)$. For $\varphi_R = \varphi_I \otimes \varphi_J$, set $\phi_R = |R|^{-1/2}\varphi_R = \phi_I \otimes \phi_J$.

For functions $f : \mathbb{T}^2 \to \mathbb{C}$, define

$$MMf(x,y) = \sup_R \frac{1}{|R|^{1/2}} |\langle \phi_R, f\rangle| \chi_R(x,y).$$

If $\{\varphi_R\}$ is a family such that $\int_{\mathbb{T}} \varphi_J \, dm = 0$ for all J, then define

$$MSf(x,y) = \sup_I \frac{1}{|I|^{1/2}} \left(\sum_J \frac{|\langle \phi_R, f\rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \chi_I(x),$$

Analogously, if $\int_{\mathbb{T}} \varphi_I \, dm = 0$ for all I, define

$$SMf(x,y) = \left(\sum_I \frac{\left(\sup_J \frac{1}{|J|^{1/2}} |\langle \phi_R, f\rangle| \chi_J(y) \right)^2}{|I|} \chi_I(x) \right)^{1/2}.$$

Finally, if $\int_{\mathbb{T}} \varphi_I \, dm = \int_{\mathbb{T}} \varphi_J \, dm = 0$, set

$$SSf(x,y) = \left(\sum_R \frac{|\langle \phi_R, f\rangle|^2}{|R|} \chi_R(x,y) \right)^{1/2}.$$

Theorem 5.1. *Each of MM, MS, SM, and SS maps $L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ for all $1 < p < \infty$, $L(\log L)^{n+2}(\mathbb{T}^2) \to L(\log L)^n(\mathbb{T}^2)$ for all $n \geq 0$, and $L \log L(\mathbb{T}^2) \to L^{1,\infty}(\mathbb{T}^2)$.*

Proof. Let M_S denote the strong maximal operator (that is, where the supremum is taken over all bi-parameter rectangles). Define the 1^{st} and 2^{nd} variables maximal operators M_1 and M_2 as follows. For $f : \mathbb{T}^2 \to \mathbb{C}$, let $M_1 f(x_1, x_2) = M(f(\cdot, x_2))(x_1)$ and $M_2 f(x_1, x_2) = M(f(x_1, \cdot))(x_2)$. It is clear that M_1, M_2 satisfy all the L^p properties and Fefferman-Stein inequalities that M does. Define M_1', M_2', S_1, S_2 similarly.

Using Theorem 4.1 as before, $MMf \lesssim M_S f$. But, $M_S f \leq M_1 \circ M_2 f$, so that

$$\begin{aligned}
\|MMf\|_p &\lesssim \|M_1 \circ M_2 f\|_p \lesssim \|M_2 f\|_p \lesssim \|f\|_p, \\
\|MMf\|_{L(\log L)^n} &\lesssim \|M_1 \circ M_2 f\|_{L(\log L)^n} \lesssim \|M_2 f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}}, \\
\|MMf\|_{1,\infty} &\lesssim \|M_1 \circ M_2 f\|_{1,\infty} \lesssim \|M_2 f\|_1 \lesssim \|f\|_{L \log L}.
\end{aligned}$$

We abuse notation slightly and write $\langle f, \phi_I \rangle$ to mean $\int_{\mathbb{T}} \overline{\phi}_I(x) f(x,y) \, dx$, a function of the variable y. Thus, $\langle \phi_R, f\rangle = \langle \phi_J, \langle f, \phi_I \rangle \rangle$ makes sense. Also, we can consider the two variable function $\langle f, \phi_I\rangle \chi_I$. In this manner,

$$SMf(x,y) = \Big(\sum_I \frac{\Big(\sup_J \frac{1}{|J|^{1/2}} |\langle \phi_R, f\rangle| \chi_J(y)\Big)^2}{|I|} \chi_I(x)\Big)^{1/2}$$

$$= \Big(\sum_I \Big(\sup_J \frac{1}{|J|^{1/2}} \big|\langle \phi_J, \frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I(x)\rangle\big| \chi_J(y)\Big)^2\Big)^{1/2}$$

$$= \Big(\sum_I M_2'\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I\Big)(x,y)^2\Big)^{1/2}.$$

By the Fefferman-Stein inequalities on M' (or M_2'),

$$\|SMf\|_p = \Big\|\Big(\sum_I M_2'\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I\Big)^2\Big)^{1/2}\Big\|_p$$

$$\lesssim \Big\|\Big(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|} \chi_I\Big)^{1/2}\Big\|_p = \|S_1 f\|_p \lesssim \|f\|_p,$$

and

$$\|SMf\|_{L(\log L)^n} = \Big\|\Big(\sum_I M_2'\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I\Big)^2\Big)^{1/2}\Big\|_{L(\log L)^n}$$

$$\lesssim \Big\|\Big(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|} \chi_I\Big)^{1/2}\Big\|_{L(\log L)^{n+1}}$$

$$= \|S_1 f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}},$$

and

$$\|SMf\|_{1,\infty} = \Big\|\Big(\sum_I M_2'\Big(\frac{\langle f, \phi_I\rangle}{|I|^{1/2}} \chi_I\Big)^2\Big)^{1/2}\Big\|_{1,\infty}$$

$$\lesssim \Big\|\Big(\sum_I \frac{|\langle f, \phi_I\rangle|^2}{|I|} \chi_I\Big)^{1/2}\Big\|_1 = \|S_1 f\|_1 \lesssim \|f\|_{L \log L}.$$

On the other hand,

$$MSf(x,y) = \sup_I \frac{1}{|I|^{1/2}} \Big(\sum_J \frac{|\langle \phi_R, f\rangle|^2}{|J|} \chi_J(y)\Big)^{1/2} \chi_I(x)$$

$$\le \Big(\sum_J \frac{\Big(\sup_I \frac{1}{|I|^{1/2}} |\langle \phi_R, f\rangle| \chi_I(x)\Big)^2}{|J|} \chi_J(y)\Big)^{1/2}.$$

This is essentially SM with the roles of I and J reversed. The same arguments as above can now be applied.

Finally,

$$\begin{aligned} SSf(x,y) &= \Big(\sum_R \frac{|\langle \phi_R, f\rangle|^2}{|R|}\chi_R(x,y)\Big)^{1/2} \\ &= \Big[\sum_I \sum_J \frac{1}{|J|}\big|\big\langle \phi_J, \frac{\langle f,\phi_I\rangle}{|I|^{1/2}}\chi_I(x)\big\rangle\big|^2 \chi_J(y)\Big]^{1/2} \\ &= \Big[\sum_I S_2\Big(\frac{\langle f,\phi_I\rangle}{|I|^{1/2}}\chi_I\Big)(x,y)^2\Big]^{1/2}, \end{aligned}$$

so that the same proof works. ■

6. Bi-parameter multipliers

Given a vector $\vec{t} = (t_1,\ldots,t_{2d}) \in \mathbb{R}^{2d}$, denote $\rho_1(\vec{t}) = (t_1, t_3, \ldots, t_{2d-1})$ and $\rho_2(\vec{t}) = (t_2, t_4, \ldots, t_{2d})$, which are both vectors in $\mathbb{R}^d$. For multi-indices of nonnegative integers α, we set $|\rho_1(\alpha)| = \alpha_1 + \alpha_3 + \cdots + \alpha_{2d-1}$, and similarly for $|\rho_2(\alpha)|$. Conversely, for $1 \le j \le d$, let $\vec{t_j} = (t_{2j-1}, t_{2j}) \in \mathbb{R}^2$, so that $\vec{t} = (\vec{t_1},\ldots,\vec{t_d})$.

Definition. Let $m : \mathbb{R}^{2d} \to \mathbb{C}$ be smooth away the origin and uniformly bounded. We say m is a bi-parameter multiplier if

$$|\partial^\alpha m(\vec{t})| \lesssim \|\rho_1(\vec{t})\|^{-|\rho_1(\alpha)|}\|\rho_2(\vec{t})\|^{-|\rho_2(\alpha)|}$$

for all vectors α with $|\alpha| \le 2d(d+3)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

Given such a multiplier m on $\mathbb{R}^{2d}$ and L^1 functions $f_1,\ldots,f_d : \mathbb{T}^2 \to \mathbb{C}$, we define the associated multiplier operator $\Lambda_m^{(2)}(f_1,\ldots,f_d) : \mathbb{T}^2 \to \mathbb{C}$ as

$$\Lambda_m^{(2)}(f_1,\ldots,f_d)(\vec{x}) = \sum_{\vec{t}\in\mathbb{Z}^{2d}} m(\vec{t})\widehat{f_1}(\vec{t_1})\cdots\widehat{f_d}(\vec{t_d})e^{2\pi i\vec{x}\cdot(\vec{t_1}+\cdots+\vec{t_d})}.$$

Consider the following theorem.

Theorem 6.1. *For any bi-parameter multiplier m on $\mathbb{R}^{2d}$, it follows that $\Lambda_m^{(2)} : L^{p_1}\times\cdots\times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $1/p_1+\cdots+1/p_d = 1/p$. If any or all of the p_j are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L\log L$. In particular, $\Lambda_m^{(2)} : L\log L\times\cdots\times L\log L \to L^{1/d,\infty}$.*

We focus only the bi-linear $d = 2$ case, but this makes no substantiative difference in the proof. Note that in this case, the bi-parameter multiplier condition can be stated

$$|\partial^{(\alpha,\beta)} m(\vec{s}, \vec{t})| \lesssim \|(s_1, t_1)\|^{-\alpha_1-\beta_1} \|(s_2, t_2)\|^{-\alpha_2-\beta_2}$$

for all two-dimensional indices α, β with $|\alpha|, |\beta| \leq 10$.

It is by now a well established fact (see [14, 15, 17]) that the study of multiplier operators of various sorts can be reduced to the study of finitely many discrete paraproducts. For $f, g : \mathbb{T}^2 \to \mathbb{C}$, the bi-parameter bi-linear paraproducts are defined by

$$T^{a,b}(f, g)(x, y) = \sum_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f\rangle \langle \phi_R^2, g\rangle \phi_R^3(x, y),$$

for $a, b = 1, 2, 3$, where ϕ_R^1, ϕ_R^2, and ϕ_R^3 are each the tensor product of two normalized adapted families, as in the previous secton. The sum is over all dyadic rectangles R. Further, if $\phi_R^i = \phi_I^i \otimes \phi_J^i$, then $\int_{\mathbb{T}} \phi_I^i \, dx = 0$ for $i \neq a$ and $\int_{\mathbb{T}} \phi_J^i \, dx = 0$ for $i \neq b$.

In order to establish Theorem 6.1, we need only prove each paraproduct satisfies the same bounds. First, the following lemma is a well-known characterization of weak-L^p. A proof is given in [1].

Lemma 6.2. *Fix $0 < p < \infty$ and $f : \mathbb{T}^d \to \mathbb{C}$. Suppose that for every measurable set $|E| > 0$ in $\mathbb{T}^d$, we can choose a subset $E' \subseteq E$ with $|E'| > |E|/2$ and $|\langle f, \chi_{E'}\rangle| \leq A|E|^{1-1/p}$. Then, $\|f\|_{p,\infty} \lesssim A$. Conversely, if $\|f\|_{p,\infty} \leq A$, then for any measurable set $|E| > 0$ there exists $E' \subseteq E$ with $|E'| > |E|/2$ and $|\langle f, \chi_{E'}\rangle| \lesssim A|E|^{1-1/p}$.*

Theorem 6.3. *$T^{a,b} : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p$. If p_1 or p_2 or both are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L \log L$.*

Proof. We will assume $a = 1$ and $b = 2$, as the other cases will follow similarly.

First, suppose $p > 1$. Then, necessarily $p_1, p_2 > 1$ and $1 < p' < \infty$. Note, $1/p_1 + 1/p_2 + 1/p' = 1$. Fix $h \in L^{p'}(\mathbb{T})$ with $\|h\|_{p'} \leq 1$. Then,

$$\begin{aligned} |\langle T^{1,2}(f, g), h\rangle| &\leq \sum_R \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f\rangle| |\langle \phi_R^2, g\rangle| |\langle \phi_R^3, h\rangle| \\ &= \int_{\mathbb{T}^2} \sum_R \frac{|\langle \phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h\rangle|}{|R|^{1/2}} \chi_R(x, y) \, dx \, dy. \end{aligned}$$

Concentrating on the integrand,

$$
\begin{aligned}
\sum_R \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^2,g\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3,h\rangle|}{|R|^{1/2}}\chi_R(x,y) &= \\
&= \sum_I\sum_J \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^2,g\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3,h\rangle|}{|R|^{1/2}}\chi_R(x,y) \\
&\le \sum_I \Bigg[\bigg(\frac{1}{|I|^{1/2}}\chi_I(x)\sup_J \frac{|\langle\phi_R^2,g\rangle|}{|J|^{1/2}}\chi_J(y)\bigg) \\
&\qquad \cdot\bigg(\sum_J \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3,h\rangle|}{|R|^{1/2}}\chi_R(x,y)\bigg)\Bigg].
\end{aligned}
$$

Applying Hölder's inequality, the last term is bounded by

$$
SM(g)(x,y)\bigg(\sum_I\Big(\sum_J \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3,h\rangle|}{|R|^{1/2}}\chi_R(x,y)\Big)^2\bigg)^{1/2}.
$$

Applying Hölder to the inner sum,

$$
\begin{aligned}
&\bigg(\sum_I\Big(\sum_J \frac{|\langle\phi_R^1,f\rangle|}{|R|^{1/2}}\frac{|\langle\phi_R^3,h\rangle|}{|R|^{1/2}}\chi_R(x,y)\Big)^2\bigg)^{1/2} \le \\
&\le \bigg(\sum_I\Big(\sum_J \frac{|\langle\phi_R^1,f\rangle|^2}{|R|}\chi_R(x,y)\Big)\Big(\sum_J \frac{|\langle\phi_R^3,h\rangle|^2}{|R|}\chi_R(x,y)\Big)\bigg)^{1/2} \\
&\le \bigg(\sup_I \frac{1}{|I|}\chi_I(x)\sum_J \frac{|\langle\phi_R^1,f\rangle|^2}{|J|}\chi_J(y)\bigg)^{1/2}\bigg(\sum_I\sum_J \frac{|\langle\phi_R^3,h\rangle|^2}{|R|}\chi_R(x,y)\bigg)^{1/2} \\
&= MS(f)(x,y)SS(h)(x,y).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\langle T^{1,2}(f,g),h\rangle| &\le \int_{\mathbb{T}^2} MSf(x,y)SMg(x,y)SSh(x,y)\,dx\,dy \\
&\le \|MSf\|_{p_1}\|SMg\|_{p_2}\|SSh\|_{p'} \lesssim \|f\|_{p_1}\|g\|_{p_2}.
\end{aligned}
$$

As h in the unit ball of $L^{p'}$ is arbitrary, we have $\|T^{1,2}(f,g)\|_p \lesssim \|f\|_{p_1}\|g\|_{p_2}$.

Now assume $1/2 \le p \le 1$. By interpolation, it is sufficient to show $T^{1,2} : L^{p_1}\times L^{p_2} \to L^{p,\infty}$ for all $1 \le p_1, p_2 < \infty$. Fix $\|f\|_{p_1} = 1$ if $p_1 > 1$ or $\|f\|_{L\log L} = 1$ if $p_1 = 1$. Similarly for g and p_2. Let $E \subseteq \mathbb{T}^2$ with $|E| > 0$. By Lemma 6.2, we will be done if we can find $E' \subseteq E$, $|E'| > |E|/2$ so that $|\langle T^{1,2}(f,g),\chi_{E'}\rangle| \lesssim 1 \le |E|^{1-1/p}$.

For $\vec{k} \in \mathbb{N}^2$ and $R = I \times J$ a dyadic interval, denote $2^{\vec{k}}R = 2^{k_1}I \times 2^{k_2}J$, and $|\vec{k}| = k_1 + k_2$. Use Theorem 4.1 to write

$$\phi_R^3 = \sum_{\vec{k} \in \mathbb{N}^2} 2^{-10|\vec{k}|} \phi_R^{3,\vec{k}}$$

where each $\phi_R^{3,\vec{k}}$ is the normalization of the tensor product of two 0-mean adapted families which are uniformally adapted to I, J respectively. Further, $\operatorname{supp}(\phi_R^{3,\vec{k}}) \subseteq 2^{\vec{k}}R$ for $\vec{k}$ small enough, while $\phi_I^{3,\vec{k}}$ is identically 0 otherwise. Now

$$\langle T^{1,2}(f,g), \chi_{E'} \rangle = \sum_{\vec{k} \in \mathbb{N}^2} 2^{-10|\vec{k}|} \sum_R \frac{1}{|R|^{1/2}} \langle \phi_R^1, f \rangle \langle \phi_R^2, g \rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle.$$

Hence, it suffices to show $|\sum |R|^{-1/2} \langle \phi_R^1, f \rangle \langle \phi_R^2, g \rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle| \lesssim 2^{4|\vec{k}|}$, so long as the underlying constants are independent of $\vec{k}$.

Let $SS^{\vec{k}}$ be the double square operator with $\phi_R^{3,\vec{k}}$. For each $\vec{k} \in \mathbb{N}^2$, define

$$\begin{aligned}
\Omega_{-3|\vec{k}|} &= \{MSf > C2^{3|\vec{k}|}\} \cup \{SMg > C2^{3|\vec{k}|}\}, \\
\widetilde{\Omega}_{\vec{k}} &= \{M_S(\chi_{\Omega_{-3|\vec{k}|}}) > 1/100\}, \\
\widetilde{\widetilde{\Omega}}_{\vec{k}} &= \{M_S(\chi_{\widetilde{\Omega}_{\vec{k}}}) > 2^{-|\vec{k}|-1}\}.
\end{aligned}$$

and

$$\Omega = \bigcup_{\vec{k} \in \mathbb{N}^2} \widetilde{\widetilde{\Omega}}_{\vec{k}}.$$

Observe, C can be chosen independent of f and g so that $|\Omega| < |E|/2$. Set $E' = E - \Omega = E \cap \Omega^c$. Then, $E' \subseteq E$ and $|E'| > |E|/2$.

Fix $\vec{k} \in \mathbb{N}^2$, and set $Z_{\vec{k}} = \{MSf = 0\} \cup \{SMg = 0\} \cup \{SS^{\vec{k}}\chi_{E'} = 0\}$. Let $\mathcal{D}$ be any finite collection of dyadic rectangles. Consider three subcollections. Set $\mathcal{D}_1 = \{R \in \mathcal{D} : R \cap Z_{\vec{k}} \neq \emptyset\}$. For the remaining rectangles, let $\mathcal{D}_2 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \subseteq \widetilde{\Omega}_{\vec{k}}\}$ and $\mathcal{D}_3 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \cap \widetilde{\Omega}_{\vec{k}}^c \neq \emptyset\}$.

If $R \in \mathcal{D}_1$, then there is some $(x,y) \in R \cap Z_{\vec{k}}$. Namely, $MSf(x,y) = 0$, $SMg(x,y) = 0$, or $SS^{\vec{k}}(\chi_{E'})(x,y) = 0$. If it is the first, $\langle \phi_R^1, f \rangle = 0$. If it is the second, then $\langle \phi_R^2, g \rangle = 0$, and if it is the third, $\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle = 0$. As this holds for all $R \in \mathcal{D}_1$, we have

$$\sum_{R \in \mathcal{D}_1} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle| = 0.$$

Now suppose $R \in \mathcal{D}_2$, namely $R \subseteq \widetilde{\Omega}_{\vec{k}}$. For some $\vec{k}$, $\phi_R^{3,\vec{k}}$ is identically 0 and $\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle = 0$. For all others, $\phi_I^{3,\vec{k}}$ is supported in $2^{\vec{k}}R$. Let $(x,y) \in 2^{\vec{k}}R$, and observe

$$M_S(\chi_{\widetilde{\Omega}_{\vec{k}}})(x,y) \geq \frac{1}{|2^{\vec{k}}R|} \int_{2^{\vec{k}}R} \chi_{\widetilde{\Omega}_{\vec{k}}}\ dm \geq \frac{1}{2^{|\vec{k}|}} \frac{1}{|R|} \int_R \chi_{\widetilde{\Omega}_{\vec{k}}}\ dm = 2^{-|\vec{k}|} > 2^{-|\vec{k}|-1}.$$

That is, $2^{\vec{k}}R \subseteq \widetilde{\widetilde{\Omega}}_{\vec{k}} \subseteq \Omega$, a set disjoint from E'. Thus, $\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle = 0$. As this holds for all $R \in \mathcal{D}_2$, we have

$$\sum_{R \in \mathcal{D}_2} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^{3,\vec{k}}, \chi_{E'} \rangle| = 0.$$

Finally, we concentrate on $\mathcal{D}_3$. Define $\Omega_{-3|\vec{k}|+1}$ and $\Pi_{-3|\vec{k}|+1}$ by

$$\begin{aligned}
\Omega_{-3|\vec{k}|+1} &= \{MSf > C2^{3|\vec{k}|-1}\}, \\
\Pi_{-3|\vec{k}|+1} &= \{I \in \mathcal{D}_3 : |I \cap \Omega_{-3|\vec{k}|+1}| > |R|/100\}.
\end{aligned}$$

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\begin{aligned}
\Omega_n &= \{MSf > C2^{-n}\}, \\
\Pi_n &= \{R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|+1}^{n-1} \Pi_j : |R \cap \Omega_n| > |R|/100\}.
\end{aligned}$$

As every $R \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $MSf > 0$ on R, it is clear that each $R \in \mathcal{D}_3$ will be in one of these collections.

Set $\Omega'_{-3|\vec{k}|} = \Omega_{-3|\vec{k}|}$ for symmetry. Define $\Omega'_{-3|\vec{k}|+1}$ and $\Pi'_{-3|\vec{k}|+1}$ by

$$\begin{aligned}
\Omega'_{-3|\vec{k}|+1} &= \{SMg > C2^{3|\vec{k}|-1}\}, \\
\Pi'_{-3|\vec{k}|+1} &= \{R \in \mathcal{D}_3 : |R \cap \Omega'_{-3|\vec{k}|+1}| > |R|/100\}.
\end{aligned}$$

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\begin{aligned}
\Omega'_n &= \{SMg > C2^{-n}\}, \\
\Pi'_n &= \Big\{R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|+1}^{n-1} \Pi'_j : |R \cap \Omega'_n| > |R|/100\Big\}.
\end{aligned}$$

Again, all $R \in \mathcal{D}_3$ must be in one of these collections.

Choose an integer N big enough so that $\Omega''_{-N} = \{SS^{\vec{k}}(\chi_{E'}) > 2^N\}$ has very small measure. In particular, we take N big enough so that $|R \cap \Omega''_{-N}| < |R|/100$ for all $R \in \mathcal{D}_3$, which is possible since $\mathcal{D}_3$ is a finite collection. Define

$$\Omega''_{-N+1} = \{SS^{\vec{k}}(\chi_{E'}) > 2^{N-1}\},$$
$$\Pi''_{-N+1} = \{R \in \mathcal{D}_3 : |R \cap \Omega''_{-N+1}| > |R|/100\},$$

and

$$\Omega''_n = \{SS^{\vec{k}}(\chi_{E'}) > 2^{-n}\},$$
$$\Pi''_n = \Big\{R \in \mathcal{D}_3 - \bigcup_{j=-N+1}^{n-1} \Pi''_j : |R \cap \Omega''_n| > |R|/100\Big\},$$

Again, all $R \in \mathcal{D}_3$ must be in one of these collections.

Consider $R \in \mathcal{D}_3$, so that $R \cap \widetilde{\Omega}^c_{\vec{k}} \neq \emptyset$. Then, there is some $(x, y) \in R \cap \widetilde{\Omega}^c_{\vec{k}}$ which implies $|R \cap \Omega_{-3|\vec{k}|}|/|R| \le M_S(\chi_{\Omega_{-3|\vec{k}|}})(x, y) \le 1/100$. Write $\Pi_{n_1,n_2,n_3} = \Pi_{n_1} \cap \Pi'_{n_2} \cap \Pi''_{n_3}$. So,

$$\sum_{R\in\mathcal{D}_3} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle|$$
$$= \sum_{n_1,n_2>-3|\vec{k}|,\, n_3>-N} \Big[\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f\rangle||\langle \phi^2_R, g\rangle||\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle| \Big]$$
$$= \sum_{n_1,n_2>-3|\vec{k}|,\, n_3>-N} \Big[\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle \phi^1_R, f\rangle|}{|R|^{1/2}} \frac{|\langle \phi^2_R, g\rangle|}{|R|^{1/2}} \frac{|\langle \phi^{3,\vec{k}}_R, \chi_{E'}\rangle|}{|R|^{1/2}} |R| \Big].$$

Suppose $R \in \Pi_{n_1,n_2,n_3}$. If $n_1 > -3|\vec{k}| + 1$, then $R \in \Pi_{n_1}$, which in particular says $R \notin \Pi_{n_1-1}$. So, $|R \cap \Omega_{n_1-1}| \le |R|/100$. If $n_1 = -3|\vec{k}| + 1$, then we still have $|R \cap \Omega_{-3|\vec{k}|}| \le |R|/100$, as $R \in \mathcal{D}_3$. Similarly, If $n_2 > -3k + 1$, then $R \in \Pi'_{n_2}$, which in particular says $R \notin \Pi'_{n_2-1}$. So, $|R \cap \Omega'_{n_2-1}| \le |R|/100$. If $n_2 = -3|\vec{k}| + 1$, then we still have $|R \cap \Omega'_{-3|\vec{k}|}| = |R \cap \Omega_{-3|\vec{k}|}| \le |R|/100$, as $R \in \mathcal{D}_3$. Finally, if $n_3 > -N + 1$, then $R \notin \Pi''_{n_3-1}$ and $|R \cap \Omega''_{n_3-1}| \le |R|/100$. If $n_3 = -N + 1$, then $|R \cap \Omega''_{-N}| \le |R|/100$ by the choice of N. So, $|R \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1}| \ge \frac{97}{100}|R|$. Let $\Omega_{n_1,n_2,n_3} = \bigcup\{R : R \in \Pi_{n_1,n_2,n_3}\}$. Then,

$$|R \cap \Omega^c_{n_1-1} \cap \Omega'^c_{n_2-1} \cap \Omega''^c_{n_3-1} \cap \Omega_{n_1,n_2,n_3}| \ge \frac{97}{100}|R|$$

for all $R \in \Pi_{n_1,n_2,n_3}$. Further,

$$\begin{aligned}
&\sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle\phi_R^2, g\rangle}{|R|^{1/2}} \frac{|\langle\phi_R^{3,\vec{k}}, \chi_{E'}\rangle|}{|R|^{1/2}} |R| \\
&\lesssim \sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle\phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle\phi_R^{3,\vec{k}}, \chi_{E'}\rangle|}{|R|^{1/2}} \\
&\qquad \times |R \cap \Omega_{n_1-1}^c \cap \Omega_{n_2-1}'^c \cap \Omega_{n_3-1}''^c \cap \Omega_{n_1,n_2,n_3}| \\
&= \int_{\Omega_{n_1-1}^c \cap \Omega_{n_2-1}'^c \cap \Omega_{n_3-1}''^c \cap \Omega_{n_1,n_2,n_3}} \chi_R(x,y) \\
&\qquad \times \sum_{R\in\Pi_{n_1,n_2,n_3}} \frac{|\langle\phi_R^1, f\rangle|}{|R|^{1/2}} \frac{|\langle\phi_R^2, g\rangle|}{|R|^{1/2}} \frac{|\langle\phi_R^{3,\vec{k}}, \chi_{E'}\rangle|}{|R|^{1/2}} \, dx\, dy \\
&\leq \int_{\Omega_{n_1-1}^c \cap \Omega_{n_2-1}'^c \cap \Omega_{n_3-1}''^c \cap \Omega_{n_1,n_2,n_3}} MSf(x,y) SMg(x,y) SS^{\vec{k}}(\chi_{E'})(x,y)\, dx\, dy \\
&\lesssim C^2 2^{-n_1} 2^{-n_2} 2^{-n_3} |\Omega_{n_1,n_2,n_3}|.
\end{aligned}$$

Note,

$$\begin{aligned}
|\Omega_{n_1,n_2,n_3}| &\leq |\bigcup\{R : R \in \Pi_{n_1}\}| \leq |\{M_S(\chi_{\Omega_{n_1}}) > 1/100\}| \\
&\lesssim |\Omega_{n_1}| = |\{MSf > C2^{-n_1}\}| \lesssim C^{-p_1} 2^{p_1 n_1}.
\end{aligned}$$

Repeating the argument,

$$|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega'_{n_2}| = |\{SMg > C2^{-n_2}\}| \lesssim C^{-p_2} 2^{p_2 n_2}, \quad \text{and}$$

$$|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega''_{n_3}| = |\{SS^{\vec{k}}(\chi_{E'}) > 2^{-n_3}\}| \lesssim 2^{\alpha n_3}$$

for any $\alpha \geq 1$. Thus, $|\Omega_{n_1,n_2,n_3}| \lesssim C^{-p_1-p_2} 2^{\theta_1 p_1 n_1} 2^{\theta_2 p_2 n_2} 2^{\theta_3 \alpha n_3}$ for any $\theta_1 + \theta_2 + \theta_3 = 1$, $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$. Hence,

$$\begin{aligned}
&\sum_{R\in\mathcal{D}_3} \frac{1}{|R|^{1/2}} |\langle\phi_R^1, f\rangle||\langle\phi_R^2, g\rangle||\langle\phi_R^{3,\vec{k}}, \chi_{E'}\rangle| \\
&\qquad \lesssim \sum_{n_1,n_2 > -3|\vec{k}|,\ n_3 > 0} 2^{(\theta_1 p_1 - 1)n_1} 2^{(\theta_2 p_2 - 1)n_2} 2^{(\theta_3 \alpha - 1)n_3} \\
&\qquad\qquad + \sum_{n_1,n_2 > -3|\vec{k}|,\ -N < n_3 \leq 0} 2^{(\theta_1 p_1 - 1)n_1} 2^{(\theta_2 p_2 - 1)n_2} 2^{(\theta_3 \alpha - 1)n_3} \\
&\qquad =: A + B.
\end{aligned}$$

For the first term, take $\theta_1 = 1/(2p_1)$, $\theta_2 = 1/(2p_2)$, $\theta_3 = 1 - 1/(2p)$, and $\alpha = 1$. For the second term, take $\theta_1 = 1/(3p_1)$, $\theta_2 = 1/(3p_2)$, $\theta_3 = 1 - 1/(3p) > 0$, and $\alpha = 2/\theta_3$ to see

$$A = \sum_{n_1,n_2>-3|\vec{k}|,\ n_3>0} 2^{-n_1/2}2^{-n_2/2}2^{-n_3/2p} \lesssim 2^{3|\vec{k}|}2^{1/2p} \le 2^{3|\vec{k}|+1},$$

$$B = \sum_{n_1,n_2>-3|\vec{k}|,\ -N<n_3\le 0} 2^{-2n_1/3}2^{-2n_2/3}2^{n_3} \le \sum_{n_1,n_2>-3|\vec{k}|, n_3\le 0} 2^{-2n_1/3}2^{-2n_2/3}2^{n_3} \lesssim 2^{4|\vec{k}|}.$$

Note, there is no dependence on the number N, which depends on $\mathcal{D}$, or C, which depends on E.

Combining the estimates for $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, we see

$$\sum_{R\in\mathcal{D}} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f\rangle||\langle \phi_R^2, g\rangle||\langle \phi_R^{3,\vec{k}}, \chi_{E'}\rangle| \lesssim 2^{4|\vec{k}|},$$

where the constant has no dependence on the collection $\mathcal{D}$. Hence, as $\mathcal{D}$ is arbitrary, we have

$$\Big| \sum_{R} \frac{1}{|R|^{1/2}} \langle \phi_R^1, f\rangle \langle \phi_R^2, g\rangle \langle \phi_R^{3,\vec{k}}, \chi_{E'}\rangle \Big| \\ \le \sum_{R} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f\rangle||\langle \phi_R^2, g\rangle||\langle \phi_R^{3,\vec{k}}, \chi_{E'}\rangle| \lesssim 2^{4|\vec{k}|},$$

which completes the proof. ■

It should now be clear that proving the above for $(a,b) \neq (1,2)$ follows by permuting the roles of MM, MS, SM, and SS. For instance, if $(a,b) = (1,1)$, then we consider MMf, SSg, and $SS^{\vec{k}}\chi_{E'}$.

7. Multi-parameter multipliers

Finally, we would like to consider multipliers, and their corresponding operators, which are multi-parameter. That is, m acts as if the product of s standard multipliers.

For a vector $\vec{t} \in \mathbb{R}^{sd}$ and $1 \le j \le s$, let $\rho_j(\vec{t}) = (t_j, t_{j+s}, \ldots, t_{j+s(d-1)}) \in \mathbb{R}^d$. Conversely, for $1 \le j \le d$, let $\vec{t_j} = (t_{s(j-1)+1}, \ldots, t_{js}) \in \mathbb{R}^s$ so that $\vec{t} = (\vec{t_1}, \ldots, \vec{t_d})$.

Let $m : \mathbb{R}^{sd} \to \mathbb{C}$ be smooth away from the origin and uniformly bounded. We say m is an s-parameter multiplier if

$$|\partial^\alpha m(\vec{t})| \lesssim \prod_{j=1}^{s} \|\rho_j(\vec{t})\|^{-|\rho_j(\alpha)|}$$

for all indices $|\alpha| \le sd(d+3)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

Given such a multiplier m on $\mathbb{R}^{sd}$ and L^1 functions $f_1, \dots, f_d : \mathbb{T}^s \to \mathbb{C}$, we define the associated multiplier operator $\Lambda_m^{(s)}(f_1, \dots, f_d) : \mathbb{T}^s \to \mathbb{C}$ as

$$\Lambda_m^{(s)}(f_1, \dots, f_d)(\vec{x}) = \sum_{\vec{t} \in \mathbb{Z}^{sd}} m(\vec{t}) \widehat{f}_1(\vec{t}_1) \cdots \widehat{f}_d(\vec{t}_d) e^{2\pi i \vec{x} \cdot (\vec{t}_1 + \cdots + \vec{t}_d)}.$$

The familiar L^p estimates of still hold with minor modifications.

Theorem 7.1. *For any s-parameter multiplier m on $\mathbb{R}^{sd}$, it follows that $\Lambda_m^{(s)} : L^{p_1} \times \cdots \times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $1/p_1 + \cdots + 1/p_d = 1/p$. If any or all of the p_j are equal to 1, this still holds with L^p replaced by $L^{p,\infty}$ and L^{p_j} replaced by $L(\log L)^{s-1}$. In particular, $\Lambda_m^{(s)} : L(\log L)^{s-1} \times \cdots \times L(\log L)^{s-1} \to L^{1/d,\infty}$.*

In view of these results, we now have a good perception of the heuristics. Away from $p_j = 1$, each of these operators act the same. However, it is these endpoint cases which are the most interesting. Each time we go up a parameter, we "gain a log" at the endpoint.

Just as in the bi-parameter case, we can reduce to paraproducts. We say $Q \subset \mathbb{T}^s$ is a dyadic rectangle if $Q = I_1 \times \cdots \times I_s$ for dyadic intervals I_j. Let $\varphi_Q : \mathbb{T}^s \to \mathbb{C}$ be the s-fold tensor product of adapted families. The appropriate (bi-linear) paraproducts in this setting are

$$T_\epsilon^{a_1, \dots, a_s}(f, g)(\vec{x}) = \sum_Q \frac{1}{|Q|^{1/2}} \langle \phi_Q^1, f \rangle \langle \phi_Q^2, g \rangle \phi_Q^3(\vec{x})$$

where the sum is over all dyadic rectangles Q. Each a_j ranges over $1, 2, 3$. If $\phi_Q^i = \phi_{I_1}^i \otimes \cdots \otimes \phi_{I_s}^i$, then $\int_{\mathbb{T}} \phi_{I_j}^i \, dx = 0$ whenever $i \neq a_j$.

To complete the proof on s-parameter multiplier operators, it suffices to show the associated paraproducts satisfy the same bounds. The same stopping time argument works equally well in all dimensions, given the correct s-fold hybrid operators. Therefore, we will understand the paraproducts if we can show each s-fold hybrid operator maps $L^p \to L^p$ for $1 < p < \infty$ and $L(\log L)^{s-1} \to L^{1,\infty}$.

For illustrative purposes, we show this for one specific operator when $s = 3$. For $f : \mathbb{T}^3 \to \mathbb{C}$ define

$$SSMf(x, y, z) = \left(\sum_{I_1} \sum_{I_2} \frac{\left(\sup_{I_3} \frac{1}{|I_3|^{1/2}} |\langle \phi_Q, f \rangle| \chi_{I_3}(z) \right)^2}{|I_1||I_2|} \chi_{I_1}(x) \chi_{I_2}(y) \right)^{1/2}.$$

Using the same notational conveniences as before,

$$SSMf = \left(\sum_{I_1} \sum_{I_2} M_3' \Big(\frac{\langle f, \phi_{I_1} \otimes \phi_{I_2} \rangle}{|I_1|^{1/2} |I_2|^{1/2}} \chi_{I_1} \chi_{I_2} \Big)^2 \right)^{1/2}.$$

So,

$$\begin{aligned}
\|SSMf\|_p &= \Big\|\Big(\sum_{I_1}\sum_{I_2} M_3'\Big(\frac{\langle f,\phi_{I_1}\otimes\phi_{I_2}\rangle}{|I_1|^{1/2}|I_2|^{1/2}}\chi_{I_1}\chi_{I_2}\Big)^2\Big)^{1/2}\Big\|_p \\
&\lesssim \Big\|\Big(\sum_{I_1}\sum_{I_2}\frac{|\langle f,\phi_{I_1}\otimes\phi_{I_2}\rangle|^2}{|I_1||I_2|}\chi_{I_1}\chi_{I_2}\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\sum_{I_1} S_2\Big(\frac{\langle f,\phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_{I_1}\frac{|\langle f,\phi_{I_1}\rangle|^2}{|I_1|}\chi_{I_1}\Big)^{1/2}\Big\|_p \\
&= \|S_1 f\|_p \lesssim \|f\|_p,
\end{aligned}$$

and

$$\begin{aligned}
\|SSMf\|_{1,\infty} &= \Big\|\Big(\sum_{I_1}\sum_{I_2} M_3'\Big(\frac{\langle f,\phi_{I_1}\otimes\phi_{I_2}\rangle}{|I_1|^{1/2}|I_2|^{1/2}}\chi_{I_1}\chi_{I_2}\Big)^2\Big)^{1/2}\Big\|_{1,\infty} \\
&\lesssim \Big\|\Big(\sum_{I_1} S_2\Big(\frac{\langle f,\phi_{I_1}\rangle}{|I_1|^{1/2}}\chi_{I_1}\Big)^2\Big)^{1/2}\Big\|_1 \lesssim \|S_1 f\|_{L\log L} \lesssim \|f\|_{L(\log L)^2}.
\end{aligned}$$

The recipe for arbitrary s-fold hybrid operators should now be clear. Each such operator is pointwise smaller than one of the form $SS...SMM...M$. In this case, the $M...MM$ part is bounded by $M_j \circ M_{j+1} \circ \cdots \circ M_s$. Repeated iterations of Fefferman-Stein eliminate these M_j, while the remaining $SS...S$ part can be dealt with as usual.

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