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Extension of $C^{m,\omega}$-Smooth Functions by Linear Operators

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Abstract

Let $C^{m,\omega}(\mathbb{R}^n)$ be the space of functions on $\mathbb{R}^n$ whose m^{th} derivatives have modulus of continuity ω. For $E \subset \mathbb{R}^n$, let $C^{m,\omega}(E)$ be the space of all restrictions to E of functions in $C^{m,\omega}(\mathbb{R}^n)$. We show that there exists a bounded linear operator $T : C^{m,\omega}(E) \to C^{m,\omega}(\mathbb{R}^n)$ such that, for any $f \in C^{m,\omega}(E)$, we have $Tf = f$ on E.

0. Introduction

Let f be a real-valued function defined on a subset $E \subset \mathbb{R}^n$. Continuing from [10,...,14], we study the problem of extending f to a function F, defined on all of $\mathbb{R}^n$, and belonging to $C^m(\mathbb{R}^n)$ or $C^{m,\omega}(\mathbb{R}^n)$. (See also Whitney [23, 24, 25], Glaeser [16], Brudnyi-Shvartsman [3,. . . ,9, 18, 19, 20], and Bierstone-Milman-Pawłucki [1, 2]). Here, $C^{m,\omega}(\mathbb{R}^n)$ denotes the space of all C^m functions on $\mathbb{R}^n$ whose m^{th} derivatives have modulus of continuity ω. In this paper and [15], we show that an essentially optimal F can be found by applying a linear operator to f. We begin with a few basic definitions.

As usual, $C^m(\mathbb{R}^n)$ consists of all real-valued C^m functions F on $\mathbb{R}^n$, for which the norm

$$\| F \|_{C^m(\mathbb{R}^n)} = \max_{|\beta|\leq m} \sup_{x\in\mathbb{R}^n} |\partial^\beta F(x)|$$

is finite.

Similarly, for suitable functions $\omega : [0, 1] \to [0, 1]$, the space $C^{m,\omega}(\mathbb{R}^n)$ consists of all real-valued C^m functions F on $\mathbb{R}^n$, for which the norm

$$(1) \quad \| F \|_{C^{m,\omega}(\mathbb{R}^n)} = \max\left\{ \| F \|_{C^m(\mathbb{R}^n)}, \max_{|\beta|=m} \sup_{\substack{x,y\in\mathbb{R}^n \\ 0<|x-y|\leq 1}} \frac{|\partial^\beta F(x) - \partial^\beta F(y)|}{\omega(|x-y|)} \right\}$$

is finite.

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We require that ω be a "regular modulus of continuity", which means that it satisfies the following conditions:

$$\begin{aligned}
&\omega(0) = \lim_{t\to 0+} \omega(t) = 0\,;\\
&\omega(1) = 1;\\
&\omega(t) \ \text{ is increasing;}\\
&\omega(t)/t \ \text{ is decreasing.}
\end{aligned}$$

(We do not require that $\omega(t)$ be strictly increasing, or that $\omega(t)/t$ be strictly decreasing.) This is a very mild restriction on ω.

Now let E be an arbitrary subset of $\mathbb{R}^n$. We write $C^m(E)$ for the Banach space of all restrictions to E of functions $F \in C^m(E)$. The norm on $C^m(E)$ is given by

$$\| f \|_{C^m(E)} = \inf\left\{\| F \|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n) \text{ and } F = f \text{ on } E\right\}. \tag{2}$$

Similarly, we write $C^{m,\omega}(E)$ for the space of all restrictions to E of functions in $C^{m,\omega}(\mathbb{R}^n)$. The norm on $C^{m,\omega}(E)$ is given by

$$\| f \|_{C^{m,\omega}(E)} = \inf\left\{\| F \|_{C^{m,\omega}(\mathbb{R}^n)} : F \in C^{m,\omega}(\mathbb{R}^n) \text{ and } F = f \text{ on } E\right\}.$$

Theorem 1. *Given a non-empty set $E \subset \mathbb{R}^n$, and given $m \geq 1$, there exists a linear map*

$$T: C^m(E) \to C^m(\mathbb{R}^n),$$

with the following properties.

(A) *The norm of T is bounded by a constant depending only on m and n.*

(B) *For any $f \in C^m(E)$, we have $Tf = f$ on E.*

Theorem 2. *Given a non-empty set $E \subset \mathbb{R}^n$, a regular modulus of continuity ω, and an integer $m \geq 1$, there exists a linear map*

$$T: C^{m,\omega}(E) \to C^{m,\omega}(\mathbb{R}^n),$$

with the following properties.

(A) *The norm of T is bounded by a constant depending only on m and n.*

(B) *For any $f \in C^{m,\omega}(E)$, we have $Tf = f$ on E.*

This paper contains the proof of Theorem 2, together with a substantial generalization of Theorem 2 that will be needed for the proof of Theorem 1. The proof of Theorem 1 appears in [15].

To state our generalization of Theorem 2, we introduce some notation and definitions, most of which come from [12,14]. We fix $m, n \geq 1$ throughout this paper. We write $\mathcal{R}_x$ for the ring of m-jets of (real-valued) smooth functions at $x \in \mathbb{R}^n$. If $F \in C^m(\mathbb{R}^n)$, then we write $J_x(F)$ for the m-jet of F at x. We identify $J_x(F)$ with the Taylor polynomial

$$y \mapsto \sum_{|\beta| \leq m} \frac{1}{\beta!} \left(\partial^\beta F(x)\right) \cdot (y-x)^\beta .$$

Thus, as a vector space, $\mathcal{R}_x$ is identified with the vector space $\mathcal{P}$ of m^{th} degree polynomials on $\mathbb{R}^n$.

Now suppose we are given a point $x \in \mathbb{R}^n$, a subset $\sigma \subseteq \mathcal{R}_x$, and a positive real number A. We say that "σ is Whitney convex with Whitney constant A" if the following two conditions hold.

(I) The set σ is closed, convex, and symmetric (i.e., $P \in \sigma$ implies $-P \in \sigma$).

(II) Let $P \in \sigma$, $Q \in \mathcal{R}_x$, $\delta \in (0, 1]$. Suppose that P and Q satisfy

 (a) $|\partial^\beta P(x)| \leq \delta^{m-|\beta|}$ for $|\beta| \leq m$, and
 (b) $|\partial^\beta Q(x)| \leq \delta^{-|\beta|}$ for $|\beta| \leq m$.

Then $P \odot Q \in A\sigma$, where $\odot$ denotes multiplication in $\mathcal{R}_x$.

If ω is a regular modulus of continuity, then we say that "σ is Whitney ω-convex with Whitney constant A", provided (I) and (II) hold, with (II)(a) replaced by

$$|\partial^\beta P(x)| \leq \omega(\delta) \cdot \delta^{m-|\beta|} \text{ for } |\beta| \leq m .$$

Note that whenever σ is Whitney convex with Whitney constant A, it follows trivially that σ is also Whitney ω-convex with Whitney constant A.

The notion of Whitney convexity is not well understood, but there are interesting examples of Whitney convex sets. Moreover, Whitney convexity plays a crucial role in our solution [12] of "Whitney's extension problem", which is closely related to Theorem 1, and which we discuss later in this introduction.

Now let $E \subset \mathbb{R}^n$ be non-empty, and suppose that, for each $x \in E$, we are given a convex, symmetric subset $\sigma(x) \subseteq \mathcal{R}_x$.

We will define a space $C^m(E, \sigma(\cdot))$, generalizing $C^m(E)$.
This space consists of families of m-jets,

$$f = (f(x))_{x \in E}, \quad \text{with } f(x) \in \mathcal{R}_x \text{ for each } x \in E.$$

We say that f belongs to $C^m(E, \sigma(\cdot))$ if there exist a function $F \in C^m(\mathbb{R}^n)$ and a finite constant M, such that

(3) $\quad \| F \|_{C^m(\mathbb{R}^n)} \leq M$, and $J_x(F) \in f(x) + M\sigma(x)$ for all $x \in E$.

The seminorm $\| f \|_{C^m(E,\sigma(\cdot))}$ is defined as the infimum of all possible M in (3).

Similarly, let E, $\sigma(x)$ be as above, and suppose once more that $f = (f(x))_{x \in E}$, with $f(x) \in \mathcal{R}_x$ for each $x \in E$. Let ω be a regular modulus of continuity. We say that f belongs to $C^{m,\omega}(E, \sigma(\cdot))$ if there exist a function $F \in C^{m,\omega}(\mathbb{R}^n)$ and a finite constant M such that

$$\text{(4)} \qquad \| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq M \,, \text{ and } J_x(F) \in f(x) + M\sigma(x) \text{ for all } x \in E \,.$$

The seminorm $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))}$ is defined as the infimum of all possible M in (4).

Thus, $C^m(E, \sigma(\cdot))$ and $C^{m,\omega}(E, \sigma(\cdot))$ are vector spaces equipped with seminorms.

We are now in position to state our generalization of Theorem 2.

Theorem 3. *Let ω be a regular modulus of continuity, and let $E \subset \mathbb{R}^n$ be non-empty. For each $x \in E$, let $\sigma(x) \subseteq \mathcal{R}_x$ be Whitney ω-convex, with a Whitney constant A independent of x. Then there exists a linear map*

$$T : C^{m,\omega}(E, \sigma(\cdot)) \to C^{m,\omega}(\mathbb{R}^n)$$

with the following properties:

(A) *The norm of T is bounded by a constant depending only on m, n and A.*

(BBBB) *If $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq 1$, then $J_x(Tf) \in f(x) + A'\sigma(x)$ for all $x \in E$, with A' depending only on m, n and A.*

To recover Theorem 2 from Theorem 3, we simply take

$$\text{(5)} \qquad \sigma(x) = \{P \in \mathcal{R}_x : P = 0 \text{ at } x\} \text{ for each } x \in E \,.$$

One checks trivially that $\sigma(x)$ is Whitney ω-convex with Whitney constant 1. Theorem 3 for the case (5) easily implies Theorem 2. (To see this, we use a natural injection i from $C^{m,\omega}(E)$ into $C^{m,\omega}(E, \sigma(\cdot))$ with σ as in (5). The injection is given by $(if)(x) =$ [the constant polynomial $f(x)$] for $f \in C^{m,\omega}(E)$ and $x \in E$.)

An intermediate result between Theorems 2 and 3 may be obtained as follows. Let ω be a regular modulus of continuity, let $E \subset \mathbb{R}^n$ be non-empty, and let $\widehat{\sigma} : E \to [0, \infty)$.
We say that $F : E \to \mathbb{R}$ belongs to $C^{m,\omega}(E, \widehat{\sigma})$ if there exist a function $F \in C^{m,\omega}(\mathbb{R}^n)$ and a constant $M < \infty$ such that

$$\text{(6)} \qquad \| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq M \text{ and } |F(x) - f(x)| \leq M\widehat{\sigma}(x) \text{ for all } x \in E \,.$$

The norm $\| f \|_{C^{m,\omega}(E,\widehat{\sigma})}$ is defined as the infimum of all possible M in (6).

Taking

$$\text{(7)} \qquad \sigma(x) = \{P \in \mathcal{R}_x : |P(x)| \leq \widehat{\sigma}(x)\} \text{ for each } x \in E \,,$$

we again find that $\sigma(x)$ is Whitney ω-convex, with Whitney constant 1.

In this case, Theorem 3 specializes to the following result.

Theorem 4. *Let ω be a regular modulus of continuity, let $E \subset \mathbb{R}^n$ be non-empty, and let $\widehat{\sigma} : E \to [0, \infty)$. Then, there exists a linear map*

$$T : C^{m,\omega}(E, \widehat{\sigma}) \to C^{m,\omega}(\mathbb{R}^n)$$

with the following properties.

(A) *The norm of T is bounded by a constant depending only on m and n.*

(B) *If $\| f \|_{C^{m,\omega}(E,\widehat{\sigma})} \leq 1$, then $|Tf(x) - f(x)| \leq C\widehat{\sigma}(x)$ for all $x \in E$, with C depending only on m and n.*

Theorem 4 specializes to Theorem 2 by taking $\widehat{\sigma} \equiv 0$.
The case $\omega(t) = t$ of Theorem 4 was proven in [10].

We are interested in Theorem 3 in full generality, primarily because it easily implies the following result, which forms a first step in our proof of Theorem 1 in [15].

Theorem 5. *Let $E \subset \mathbb{R}^n$ be a non-empty finite set.*
For each $x \in E$, let $\sigma(x) \subseteq \mathcal{R}_x$ be Whitney convex, with Whitney constant A. Then there exists a linear map

$$T : C^m(E, \sigma(\cdot)) \to C^m(\mathbb{R}^n)$$

with the following properties.

(A) *The norm of T is bounded by a constant depending only on m, n and A.*

(B) *If $\| f \|_{C^m(E,\sigma(\cdot))} \leq 1$, then $J_x(Tf) \in f(x) + A'\sigma(x)$ for all $x \in E$, with A' depending only on m, n and A.*

To deduce Theorem 5 from Theorem 3, we invoke the following version of the classical

Whitney Extension Theorem. *Let ω be a regular modulus of continuity, and let $E \subset \mathbb{R}^n$ be closed and non-empty. Suppose we associate to each $x \in E$ a polynomial $P^x \in \mathcal{P}$. Assume that the P^x satisfy the estimates*

(a) $|\partial^\beta P^x(x)| \leq 1$ for $|\beta| \leq m$, $x \in E$; and

(b) $|\partial^\beta(P^x - P^y)(y)| \leq \omega(|x - y|) \cdot |x - y|^{m-|\beta|}$ for $|\beta| \leq m, |x - y| \leq 1$, $x, y \in E$.

Then there exists a function $F \in C^{m,\omega}(\mathbb{R}^n)$, with the following properties.

(A) $\| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C$, *with C depending only on m and n.*

(B) $J_x(F) = P^x$ *for all $x \in E$.*

Now let E and $\sigma(\cdot)$ be as in the hypotheses of Theorem 5. Since E is finite, the constant $\delta = \min_{x,y\in E, x\neq y} |x-y|$ is strictly positive. We define $\omega(t) = \min\{\delta^{-1}t, 1\}$ for $t \in [0,1]$. One checks trivially that ω is a regular modulus of continuity. Since each $\sigma(x)$ is Whitney convex with Whitney constant A, we know that $\sigma(x)$ is Whitney ω-convex with Whitney constant A. Moreover, we have $\omega(|x-y|) = 1$ for any two distinct points $x, y \in E$. Consequently, the Whitney Extension Theorem tells us that

$$C^{m,\omega}(E,\sigma(\cdot)) = C^m(E,\sigma(\cdot)), \text{ and that} \tag{8}$$

$$c_1 \, \| f \|_{C^m(E,\sigma(\cdot))} \leq \| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq C_1 \, \| f \|_{C^m(E,\sigma(\cdot))} \tag{9}$$

for all $f \in C^m(E,\sigma(\cdot))$, with c_1 and C_1 depending only on m, n.

In view of (8) and (9), Theorem 5 follows at once from Theorem 3, by taking ω as above.

Thus, Theorems 2,..., 5 all follow from Theorem 3. We give the proof of Theorem 3 in Sections 1,..., 5 below. See also Section 6, where we give refinements of Theorems 2,..., 5.

This paper is part of an effort by several authors, going back to Whitney [23, 24, 25], addressing the following questions.

Whitney Extension Problems. Suppose we are given a subset $E \subset \mathbb{R}^n$, and a Banach space X of functions on $\mathbb{R}^n$. (We might take $X = C^m(\mathbb{R}^n)$ or $X = C^{m,\omega}(\mathbb{R}^n)$.)
Let $X(E)$ denote the Banach space of all restrictions to E of functions in X.

Problem 1: How can we tell whether a given function on E belongs to $X(E)$?

Problem 2: Is there a bounded linear map $T : X(E) \to X$ such that $Tf\big|_E = f$ for all $f \in X(E)$?

Whitney [24] settled these questions for $X = C^m(\mathbb{R})$ in one dimension ($n = 1$) using finite differences; and he discovered the classical Whitney extension theorem.

G. Glaeser [16] settled the case $X = C^1(\mathbb{R}^n)$ in terms of a geometrical object called the "iterated paratangent space". Glaeser's work influenced all subsequent work on Whitney's problems.

A series of papers [3,...,9, 18, 19, 20] by Y. Brudnyi and P. Shvartsman conjectured solutions to Problems 1 and 2 for $X = C^{m,\omega}(\mathbb{R}^n)$ and related spaces. They proved their conjectures for the case $m = 1$ by the elegant method of "Lipschitz selection", which is of independent interest. Their work on Problem 1 involves restricting attention to an arbitrary subset of E with cardinality bounded by a constant $k^{\#}$ determined by the space X. See [9], which produces linear extension operators from $(m-1)$-jets into $C^{m,\omega}(\mathbb{R}^n)$.

This may be viewed as an instance of our Theorem 3. We refer the reader to [3,…, 9, 18, 19, 20] for additional results and conjectures.

The next progress on the Whitney problems was the work of E. Bierstone, P. Milman and W. Pawłucki [1]. They discovered an analogue of Glaeser's iterated paratangent space relevant to $C^m(\mathbb{R}^n)$. They conjectured a geometrical solution to Problem 1 for $X = C^m(\mathbb{R}^n)$ based on their paratangent space, and they showed that a version of their conjecture holds for subanalytic sets E.

My own papers [10,…,15] study Problems 1 and 2 above for $X = C^{m,\omega}(\mathbb{R}^n)$ and $X = C^m(\mathbb{R}^n)$, and broaden the discussion by introducing $\sigma(x)$ and $\widehat{\sigma}(x)$ as in Theorems 3 and 4 above. See also Bierstone-Milman-Pawłucki [2] in connection with [12].

Theorems 1 and 2, as stated here, solve Problem 2 for $X = C^m(\mathbb{R}^n)$ and for $X = C^{m,\omega}(\mathbb{R}^n)$.

We refer the reader also to A. Brudnyi and Y. Brudnyi [4] for results on the analogue of Problem 2 for $X = \mathrm{Lip}(1)$, with $\mathbb{R}^n$ replaced by a more general metric space. See also N. Zobin [26, 27] for the solution of another problem, also going back to Whitney's work, that may be closely related to Problems 1 and 2.

We know very little about Problems 1 and 2 for function spaces other than C^m and $C^{m,\omega}$.

It is a pleasure to acknowledge the influence of E. Bierstone, Y. Brudnyi, P. Milman, W. Pawłucki, P. Shvartsman, and N. Zobin. I am grateful to Gerree Pecht for LaTeXing my paper to the highest standards.

1. Plan of the Proof

In this section, we explain our plan for the proof of Theorem 3. We recall the main result of [14], namely

Theorem 6. *Given $m, n \geq 1$, there exists $k^\#$, depending only on m and n, for which the following holds.*

Let ω be a regular modulus of continuity, let $E \subset \mathbb{R}^n$, and let $A > 0$.

For each $x \in E$, suppose we are given an m-jet $f(x) \in \mathcal{R}_x$, and a Whitney ω-convex subset $\sigma(x) \subseteq \mathcal{R}_x$ with Whitney constant A. Suppose that, given $S \subseteq E$ with cardinality at most $k^\#$, there exists $F^S \in C^{m,\omega}(\mathbb{R}^n)$, satisfying

$$\| F^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq 1 \quad \text{and} \quad J_x(F^S) - f(x) \in \sigma(x) \quad \text{for all } x \in S.$$

Then there exists $F \in C^{m,\omega}(\mathbb{R}^n)$, satisfying

$$\| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A', \quad \text{and} \quad J_x(F) - f(x) \in A'\sigma(x) \quad \text{for all } x \in E.$$

Here, A' depends only on m, n and on the Whitney constant A.

We will prove a modification of Theorem 6 in which the m-jet $f(x)$ and the function F depend on a parameter ξ. We take ξ to belong to a vector space Ξ, equipped with a seminorm $|\cdot|$.

We don't assume that our seminorm is a norm, or that Ξ is complete.

Our modification of Theorem 6 is as follows.

Theorem 7. *Given $m, n \geq 1$, there exists $k^{\#}$, depending only on m and n, for which the following holds.*

Let Ξ be a vector space with a seminorm $|\cdot|$. Let ω be a regular modulus of continuity, let $E \subset \mathbb{R}^n$, and let $A > 0$.

For each $x \in E$, suppose we are given a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$.

Also, for each $x \in E$, suppose we are given a Whitney ω-convex subset $\sigma(x) \subseteq \mathcal{R}_x$, with Whitney constant A.

Assume that, given $\xi \in \Xi$ with $|\xi| \leq 1$, and given $S \subseteq E$ with cardinality at most $k^{\#}$, there exists $F_\xi^S \in C^{m,\omega}(\mathbb{R}^n)$, satisfying

$$\text{(1)} \qquad \| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq 1\,, \quad \textit{and} \quad J_x(F_\xi^S) - f_\xi(x) \in \sigma(x) \quad \textit{for all } x \in S.$$

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, whenever $|\xi| \leq 1$, we have

$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A'\,, \quad \textit{and} \quad J_x(F_\xi) - f_\xi(x) \in A'\sigma(x) \quad \textit{for all } x \in E\,.$$

Here, A' depends only on m, n, and on the Whitney constant A.

Theorem 3 follows easily from Theorem 7. To see this, assume the hypotheses of Theorem 3, and take $\Xi = C^{m,\omega}(E, \sigma(\cdot))$, equipped with the seminorm $|\xi| = 2 \| f \|_{C^{m,\omega}(E,\sigma(\cdot))}$ for $\xi = f \in C^{m,\omega}(E, \sigma(\cdot))$. There is a tautological linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, for each $x \in E$. In fact, for $\xi = (f(x))_{x \in E} \in C^{m,\omega}(E, \sigma(\cdot))$, we just set $f_\xi(x) = f(x)$.

We check that hypothesis (1) of Theorem 7 holds here.

In fact, suppose $\xi = f = (f(x))_{x\in E} \in \Xi$, with $|\xi| \leq 1$.

Then $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq 1/2$. By definition, this implies that there exists $F \in C^{m,\omega}(\mathbb{R}^n)$, satisfying

$$\| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq 1, \quad \text{and} \quad J_x(F) - f(x) \in \sigma(x) \quad \text{for all } x \in E\,.$$

Consequently, given any subset $S \subseteq E$, (1) holds with $F_\xi^S = F$.

Thus, (1) holds here, as claimed.

Applying Theorem 7, we obtain a linear map $f \mapsto F_f$, from $C^{m,\omega}(E, \sigma(\cdot))$ into $C^{m,\omega}(\mathbb{R}^n)$, such that, for $f = (f(x))_{x \in E} \in C^{m,\omega}(\mathbb{R}^n)$ with $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq 1/2$, we have

$$\| F_f \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A', \quad \text{and} \quad J_x(F_f) - f(x) \in A'\sigma(x) \quad \text{for all } x \in E.$$

This immediately yields the conclusion of Theorem 3, with $2A'$ in place of A'. The reduction of Theorem 3 to Theorem 7 is complete. We turn our attention to the proof of Theorem 7.

Except at a few key points, we can simply carry along the proof of Theorem 6, and every relevant quantity will depend linearly on our parameter ξ. However, at a few key points, the proof of Theorem 6 makes non-linear constructions. Here, new arguments are needed. We proceed by adapting [10], where a transition like that from Theorem 6 to Theorem 7 was carried out in an easier case.

After a few elementary results on convex sets (given in Section 2 below), we prove in Section 3 the basic lemmas that preserve linear dependence on ξ, in the few crucial places where the arguments in [14] depart from it. The adaptations of [14] needed for Theorem 7 are then given in Section 4. At every point in Section 4 where one needs an idea, we apply a result from Section 3.

We will use freely the classical Whitney extension theorem for $C^{m,\omega}(\mathbb{R}^n)$, which we now state in the case of finite sets E.

Whitney's Extension Theorem for Finite Sets.

For a finite set $E \subset \mathbb{R}^n$*, let* $C(E)$ *denote the space of maps* $x \mapsto P^x$ *from* E *into* $\mathcal{P}$.

Then, given a finite set $E \subset \mathbb{R}^n$ *and a regular modulus of continuity* ω*, there exists a linear map* $T : C(E) \to C^{m,\omega}(\mathbb{R}^n)$*, with the following properties.*

(A) *Suppose* $F = T\vec{P}$*, with* $\vec{P} = (x \mapsto P^x) \in C(E)$.

Then $J_x(F) = P^x$ *for all* $x \in E$.

(B) *Suppose* $F = T\vec{P}$*, with* $\vec{P} = (x \mapsto P^x) \in C(E)$.

Assume that $\vec{P}$ *satisfies*

(i) $|\partial^\alpha P^x(x)| \leq 1$ *for* $|\alpha| \leq m$, $x \in E$; *and*

(ii) $|\partial^\alpha (P^x - P^y)(y)| \leq \omega(|x-y|) \cdot |x-y|^{m-|\alpha|}$ *for* $|\alpha| \leq m$, $x, y \in E$, $|x-y| \leq 1$.

Then $\| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C$*, with* C *depending only on* m *and* n.

A proof of Whitney's extension theorem as stated here (but without the restriction to finite sets) may be found in [16, 17, 21, 23].

2. Elementary Properties of Convex Sets

We start by recalling the Lemma of Fritz John.

Lemma 2.1. *Let $\sigma \subset \mathbb{R}^d$ be a compact, convex, symmetric set with non-empty interior. Then there exists a positive-definite quadratic form g on $\mathbb{R}^d$ such that*

$$\{x \in \mathbb{R}^d : g(x) \le c\} \subset \sigma \subset \{x \in \mathbb{R}^d : g(x) \le 1\},$$

with $c > 0$ depending only on the dimension d.

For a proof of Lemma 2.1, see e.g. [22].

We need to weaken the hypotheses of Fritz John's Lemma. To do so, we first prove the following.

Lemma 2.2. *Let σ be a closed, convex, symmetric subset of $\mathbb{R}^d$.*

Then we can write $\mathbb{R}^d$ as a direct sum of vector spaces $\mathbb{R}^d = I_1 \oplus I_2 \oplus I_3$, in terms of which we have $\sigma = \sigma_1 \oplus I_2 \oplus \{0\}$, with $\sigma_1 \subset I_1$ compact, convex, and symmetric, and having non-empty interior in I_1.

Proof. Set

$$I = \bigcap_{\lambda>0} \lambda\sigma \quad \text{and} \quad I^+ = \bigcup_{\lambda>0} \lambda\sigma.$$

One checks trivially that I and I^+ are vector subspaces of $\mathbb{R}^d$, with $I \subseteq I^+$. Hence, we may write $\mathbb{R}^d$ as a direct sum

(1) $\mathbb{R}^d = I_1 \oplus I_2 \oplus I_3$, with

(2) $I = I_2$ and $I^+ = I_1 \oplus I_2$.

Note that if $v \in \sigma$ and $w \in I$, then $v + w \in \sigma$. To see this, let $\tau \in (0, 1)$, and write

$$(1-\tau)v + w \;=\; (1-\tau)v \;+\; \tau(\tau^{-1}w) \;\in\; (1-\tau)\sigma \;+\; \tau\sigma \;=\; \sigma\,.$$

Letting $\tau \to 0+$, and recalling that σ is closed, we obtain $v + w \in \sigma$ as claimed.

We have also $\sigma \subseteq I^+$. These remarks show that, in terms of the direct sum decomposition (1), we have

$$\sigma = \sigma_1 \oplus I_2 \oplus \{0\},$$

with $\sigma_1 \subseteq I_1$ closed, convex, and symmetric.

It remains to show that σ_1 is compact and has non-empty interior in I_1.

By virtue of (2), any non-zero $x \in I_1 \oplus \{0\} \oplus \{0\}$ belongs to I^+ but not to I. Consequently,

(3) Given $x \in I_1 \smallsetminus \{0\}$, there exist $\lambda, \lambda' > 0$ with $x \in \lambda\sigma_1$ but $x \notin \lambda'\sigma_1$.

Let $e_1, \dots, e_m$ be a basis for I_1. By (3), there exist $\lambda_1, \dots, \lambda_m > 0$, with $e_i \in \lambda_i \sigma_1$ for $i = 1, \dots, m$. Consequently, if $|t_1|, |t_2|, \dots, |t_m| \leq 1$, then $t_1 e_1 + \cdots + t_m e_m$ belongs to $(\lambda_1 + \cdots + \lambda_m)\sigma_1$. It follows that σ_1 contains a neighborhood U of the origin in I_1.

Next, let S be the unit sphere in I_1, and suppose we are given $\widehat{x} \in S$. Then, for some $\widehat{\lambda} > 0$, we have $\widehat{x} \notin \widehat{\lambda}\sigma_1$, thanks to (3). If $x \in S$ is so close to $\widehat{x}$ that $\widehat{x} - x \in \frac{1}{2}\widehat{\lambda}U$, then we cannot have $x \in \frac{1}{2}\widehat{\lambda}\sigma$. (Otherwise, $\widehat{x} - x$ and x would both belong to $\frac{1}{2}\widehat{\lambda}\sigma_1$, hence $\widehat{x} \in \widehat{\lambda}\sigma_1$.) Hence, given $\widehat{x} \in S$, there exist a neighborhood $\widehat{U}$ of $\widehat{x}$ in S, and a positive number $\widehat{\lambda}$, such that $x \notin \widehat{\lambda}\sigma_1$ for all $x \in \widehat{U}$. By compactness of S, it follows that there exists $\tilde{\lambda} > 0$ such that no point of S belongs to $\tilde{\lambda}\sigma_1$. It follows that $\tilde{\lambda}\sigma_1$ is contained in the open unit ball in I_1. (If $x \in \tilde{\lambda}\sigma_1$ with $|x| \geq 1$, then $|x|^{-1}x \in \tilde{\lambda}\sigma_1 \cap S$.)

Thus, σ_1 is bounded. Since also σ_1 is closed, it is compact.

The proof of Lemma 2.2 is complete. ■

Combining Lemmas 2.1 and 2.2, we obtain at once the following result.

Lemma 2.3. *Let $\sigma \subseteq \mathbb{R}^d$ be closed, convex, and symmetric.*

Then there exist a vector subspace $I \subseteq \mathbb{R}^d$ and a positive semidefinite quadratic form g on I, such that

$$\{P \in I : g(P) \leq c\} \subseteq \sigma \subseteq \{P \in I : g(P) \leq 1\},$$

with $c > 0$ depending only on the dimension d.

Next, we prove a variant of Helly's theorem [22]. In [10], we proved the following result.

Lemma 2.4. *Let $(\sigma_\alpha)_{\alpha \in \mathcal{A}}$ be a finite collection of compact, convex, symmetric subsets of $\mathbb{R}^d$, with each σ_α having non-empty interior.*

Then there exist $\alpha_1, \alpha_2, \dots, \alpha_{d\cdot(d+1)} \in \mathcal{A}$, such that

$$\bigcap_{i=1}^{d\cdot(d+1)} \sigma_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha \in \mathcal{A}} \sigma_\alpha,$$

with C depending only on the dimension d.

We will need the following variant of the above result.

Lemma 2.5. *Let $(\sigma_\alpha)_{\alpha \in \mathcal{A}}$ be a finite collection of compact, convex, symmetric subsets of $\mathbb{R}^d$.*

Then there exist $\alpha_1, \alpha_2, \dots, \alpha_{(d+1)^2} \in \mathcal{A}$, such that

$$\bigcap_{i=1}^{(d+1)^2} \sigma_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha \in \mathcal{A}} \sigma_\alpha,$$

with C depending only on the dimension d.

Proof of Lemma 2.5. By Lemma 2.2, for each $\alpha \in \mathcal{A}$ there exist a vector space $I_\alpha \subseteq \mathbb{R}^d$ and a positive number ϵ_α, such that

(4) $\{x \in I_\alpha : |x| < \epsilon_\alpha\} \subseteq \sigma_\alpha \subseteq I_\alpha$.

Let $\widehat{I} = \bigcap_{\alpha\in\mathcal{A}} I_\alpha$, and let $\widehat{\sigma} = \bigcap_{\alpha\in\mathcal{A}} \sigma_\alpha$. Since $\mathcal{A}$ is finite, (4) shows that

(5) $\widehat{\sigma} \subseteq \widehat{I}$, and

(6) $\widehat{\sigma}$ contains a neighborhood of 0 in $\widehat{I}$.

For each $\alpha \in \mathcal{A}$, let $\widehat{\sigma}_\alpha = \sigma_\alpha \cap \widehat{I}$. Thus, each $\widehat{\sigma}_\alpha$ is a compact, convex, symmetric subset of $\widehat{I}$. Moreover, each $\widehat{\sigma}_\alpha$ has non-empty interior in $\widehat{I}$, thanks to (6). We have also

$$\widehat{\sigma} = \bigcap_{\alpha\in\mathcal{A}} \widehat{\sigma}_\alpha, \quad \text{by (5).}$$

Hence, we may apply Lemma 2.4, with $\widehat{I}$ in place of $\mathbb{R}^d$. Thus, there exist $\alpha_1, \ldots, \alpha_{d\cdot(d+1)} \in \mathcal{A}$, such that

$$\bigcap_{i=1}^{d\cdot(d+1)} \widehat{\sigma}_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha\in\mathcal{A}} \widehat{\sigma}_\alpha, \quad \text{with } C \text{ depending only on } d.$$

That is,

$$\text{(7)} \qquad \widehat{I} \cap \bigcap_{i=1}^{d\cdot(d+1)} \sigma_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha\in\mathcal{A}} \sigma_\alpha.$$

Next, we pick successively $\beta_1, \beta_2, \cdots \in \mathcal{A}$ by the following rule.

Once we have picked $\beta_1, \cdots, \beta_{i-1}$ (which is true vacuously when $i = 1$), we pick any $\beta_i \in \mathcal{A}$ such that $\dim(I_{\beta_1} \cap \cdots \cap I_{\beta_i}) < \dim(I_{\beta_1} \cap \cdots \cap I_{\beta_{i-1}})$. If there is no such $\beta_i \in \mathcal{A}$, then the process of picking β's stops with β_{i-1}. Since $0 \leq \dim(I_{\beta_1} \cap \cdots \cap I_{\beta_i}) \leq d - i + 1$ by induction on i, the process of picking $\beta_1, \beta_2 \cdots$ must end with some $\beta_s, s \leq d+1$.

Given any $\beta \in \mathcal{A}$, we cannot have $\dim(I_{\beta_1} \cap \cdots \cap I_{\beta_s} \cap I_\beta) < \dim(I_{\beta_1} \cap \cdots \cap I_{\beta_s})$, since the process of picking $\beta_1, \beta_2, \cdots$ stops with β_s. Consequently,

$$\text{(8)} \qquad \widehat{I} = \bigcap_{\beta\in\mathcal{A}} I_\beta = I_{\beta_1} \cap \cdots \cap I_{\beta_s}.$$

From (4) and (8), we see that $\sigma_{\beta_1} \cap \cdots \cap \sigma_{\beta_s} \subseteq \widehat{I}$, and therefore (7) implies

$$\text{(9)} \qquad \sigma_{\beta_1} \cap \cdots \cap \sigma_{\beta_s} \cap \bigcap_{i=1}^{d\cdot(d+1)} \sigma_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha\in\mathcal{A}} \sigma_\alpha,$$

with C depending only on the dimension d. Since $s \leq d+1$, the number of σ's being intersected on the left in (9) is at most $(d+1)+d\cdot(d+1) = (d+1)^2$.

The proof of Lemma 2.5 is complete. ■

Although I haven't found Lemmas 2.3 and 2.5 in the literature, these elementary results are very likely known, and in a sharper form than the versions stated here.

We will need also the following slight variant of Lemma 2.4.

Lemma 2.6. *Let $(\sigma_\alpha)_{\alpha\in\mathcal{A}}$ be a collection of compact, convex symmetric subsets of $\mathbb{R}^d$. Assume that*

$$\bigcap_{\alpha\in\mathcal{A}} \sigma_\alpha \text{ has non-empty interior in } \mathbb{R}^d. \tag{10}$$

Then there exist $\alpha_1, \ldots, \alpha_{d\cdot(d+1)} \in \mathcal{A}$, such that

$$\bigcap_{i=1}^{d\cdot(d+1)} \sigma_{\alpha_i} \subseteq C \cdot \bigcap_{\alpha\in\mathcal{A}} \sigma_\alpha .$$

with C depending only on the dimension d.

Lemma 2.6 differs from Lemma 2.4 in that we now assume (10) in place of the finiteness of $\mathcal{A}$. Since the finiteness of $\mathcal{A}$ was used in the proof of Lemma 2.4 in [10] only to establish (10), that proof gives us Lemma 2.6 as well.

3. Linear Selection by Least Squares

The results in this section show that certain choices can be made to depend linearly on a parameter ξ in a vector space Ξ, by using least squares.

Lemma 3.1. *Suppose we are given a vector space Ξ equipped with a semi-norm $|\cdot|$, a constant $A > 0$, a point $x_0 \in \mathbb{R}^n$, a number $\delta \in (0,1]$, a regular modulus of continuity ω, a closed convex symmetric subset $\sigma_0 \subseteq \mathcal{R}_{x_0}$, and a linear map $\xi \mapsto f_{0,\xi}$ from Ξ into $\mathcal{R}_{x_0}$.*
Assume that, whenever $\xi \in \Xi$ with $|\xi| \le 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

(a) $|\partial^\beta F(x_0)| \le A\omega(\delta) \cdot \delta^{m-|\beta|}$ for $|\beta| \le m$; and

(b) $J_{x_0}(F) \in f_{0,\xi} + A\sigma_0$.

Then there exists a linear map $\xi \mapsto \widetilde{F}_\xi$, from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, whenever $\xi \in \Xi$ with $|\xi| \le 1$, the following hold.

(A) $|\partial^\beta \widetilde{F}_\xi(x)| \le CA\omega(\delta) \cdot \delta^{m-|\beta|}$ *for* $|\beta| \le m$, $x \in \mathbb{R}^n$.

(B) $|\partial^\beta \widetilde{F}_\xi(x') - \partial^\beta \widetilde{F}_\xi(x'')| \le CA\omega(|x'-x''|)$ *for* $|\beta| = m$, $|x'-x''| \le 1$ $x', x'' \in \mathbb{R}^n$.

(C) $J_{x_0}(\widetilde{F}_\xi) \in f_{0,\xi} + CA\sigma_0$.

Here, C depends only on m and n.

Proof. In the proof of Lemma 3.1, we call a constant "controlled" if it depends only on m, n; and we write c, C, C', etc. to denote controlled constants.

By Lemma 2.3, there exist a vector space $I_0 \subseteq \mathcal{R}_{x_0}$, and a positive semidefinite quadratic form g_0 on I_0, such that

$$\{P \in I_0 : g_0(P) \leq c\} \subseteq \sigma_0 \subseteq \{P \in I_0 : g_0(P) \leq 1\}. \tag{1}$$

Fix I_0 and g_0 as in (1). For $\xi \in \Xi$ and $P \in I_0$, define

$$\mathcal{Q}(\xi, P) = \sum_{|\alpha| \leq m} \left(\frac{\partial^\alpha (f_{0,\xi} + P)(x_0)}{A\, \omega(\delta) \cdot \delta^{m-|\alpha|}} \right)^2 + \frac{g_0(P)}{A^2}. \tag{2}$$

Note that

$$\mathcal{Q}(\xi, P) = \mathcal{Q}_0(\xi) + \mathcal{Q}_1(\xi, P) + \mathcal{Q}_2(P),$$

where $\mathcal{Q}_0(\xi)$ is a quadratic form in ξ; $\mathcal{Q}_1(\xi, P)$ is a bilinear form in ξ, P; and $\mathcal{Q}_2(P)$ is a positive-definite quadratic form in P.
Hence, for each fixed $\xi \in \Xi$, there is a unique minimizer $P_\xi \in I_0$ for the function $P \mapsto \mathcal{Q}(\xi, P)$ $(P \in I_0)$; moreover, P_ξ depends linearly on ξ.

Next, suppose $|\xi| \leq 1$. With F as in (a), (b) above, we set $\widehat{P} = J_{x_0}(F) - f_{0,\xi} \in A\, \sigma_0 \subseteq I_0$. From (a) we obtain

$$|\partial^\alpha (f_{0,\xi} + \widehat{P})(x_0)| \leq A\, \omega(\delta) \delta^{m-|\alpha|} \text{ for } |\alpha| \leq m. \tag{3}$$

From (b) and (1), we obtain

$$g_0(\widehat{P}) \leq A^2. \tag{4}$$

Putting (3) and (4) into (2), we see that $\mathcal{Q}(\xi, \widehat{P}) \leq C$, with $\widehat{P} \in I_0$. Since P_ξ minimizes $\mathcal{Q}(\xi, P)$ over all $P \in I_0$, it follows that $\mathcal{Q}(\xi, P_\xi) \leq C$. This means that

$$|\partial^\alpha (f_{0,\xi} + P_\xi)(x_0)| \leq CA\, \omega(\delta) \cdot \delta^{m-|\alpha|} \text{ for } |\alpha| \leq m, \ |\xi| \leq 1; \text{ and} \tag{5}$$

$$P_\xi \in I_0 \text{ and } g_0(P_\xi) \leq CA^2 \text{ for } |\xi| \leq 1. \tag{6}$$

Comparing (6) with (1), we find that

$$P_\xi \in CA\, \sigma_0 \text{ for } |\xi| \leq 1. \tag{7}$$

Next, we apply the classical Whitney Extension Theorem for finite sets (see Section 1), with $E = \{x_0\}$. Composing the linear map T from Whitney's extension theorem, with the linear map $\xi \mapsto f_{0,\xi} + P_\xi$, we obtain a linear map $\xi \mapsto F_\xi$, from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, with the following properties.

(8) $$\text{If } |\xi| \le 1\,, \text{ then } \| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le CA\,.$$

(9) $$\text{If } |\xi| \le 1\,, \text{ then } J_{x_0}(F_\xi) \in f_{0,\xi} + CA\,\sigma_0\,.$$

(10) $$\text{If } |\xi| \le 1\,, \text{ then } |\partial^\alpha F_\xi(x_0)| \le CA\,\omega(\delta) \cdot \delta^{m-|\alpha|} \text{ for } |\alpha| \le m\,.$$

In fact, (8) and (10) follow from (5); and (9) follows from (7).

Next, we fix a cutoff function $\theta \in C^{m+1}(\mathbb{R}^n)$, with

(11) $$\operatorname{supp}\theta \subset B(x_0, \tfrac{1}{2}\delta)\,;$$

(12) $$|\partial^\alpha\theta(x)| \le C\delta^{-|\alpha|} \text{ for } |\alpha| \le m+1\,,\ x \in \mathbb{R}^n; \text{ and}$$

(13) $$J_{x_0}(\theta) = 1\,.$$

We set $\widetilde{F}_\xi = \theta \cdot F_\xi$ for $\xi \in \Xi$.

Thus, $\xi \mapsto \widetilde{F}_\xi$ is a linear map from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, and we have

$$|\xi| \le 1 \text{ implies } J_{x_0}(\widetilde{F}_\xi) \in f_{0,\xi} + CA\,\sigma_0,$$

by (9) and (13). Thus, conclusion (C) of Lemma 3.1 holds for the linear map $\xi \mapsto \widetilde{F}_\xi$. We check that conclusions (A) and (B) hold as well. This will complete the proof of Lemma 3.1.

Let $\xi \in \Xi$ be given, with $|\xi| \le 1$. From (8) we have

$$|\partial^\beta F_\xi(x) - \partial^\beta F_\xi(x_0)| \le CA\,\omega(|x-x_0|) \le CA\,\omega(\delta) \text{ for } |\beta| = m,\ x \in B(x_0,\delta).$$

Together with (10), this yields

(14) $$|\partial^\beta F_\xi(x)| \le CA\,\omega(\delta) \text{ for } |\beta| = m\,,\ x \in B(x_0,\delta)\,.$$

From (14) and another application of (10), we find that

(15) $$|\partial^\beta F_\xi(x)| \le CA\,\omega(\delta) \cdot \delta^{m-|\beta|} \text{ for } |\beta| \le m\,,\ x \in B(x_0,\delta)\,.$$

Assertion (A) of Lemma 3.1 now follows from (11), (12), (15). We turn to assertion (B). Again, we suppose $|\xi| \le 1$.

If $|x'-x''| \ge \frac{1}{10}\delta$, then we have $\omega(|x'-x''|) \ge \frac{1}{10}\omega(\delta)$, since ω is a regular modulus of continuity. Hence, assertion (B) in this case follows from assertion (A), which we already know.

Also, if $|x'-x''| < \frac{1}{10}\delta$ and either x' or x'' lies outside $B(x_0,\delta)$, then both x' and x'' lie outside $B(x_0, \frac{1}{2}\delta)$. Hence, in this case, assertion (B) holds trivially, since $\partial^\beta\widetilde{F}_\xi(x') = \partial^\beta\widetilde{F}_\xi(x'') = 0$ by (11).

Thus, to prove assertion (B), we may assume that

$$x', x'' \in B(x_0, \delta) \text{ and } |x' - x''| < \frac{1}{10}\delta. \tag{16}$$

In this case, we argue as follows. For $|\beta| = m$, we have

$$\begin{aligned}\partial^\beta \widetilde{F}_\xi(x') - \partial^\beta \widetilde{F}_\xi(x'') &= \theta(x')\partial^\beta F_\xi(x') - \theta(x'')\partial^\beta F_\xi(x'') \\ &+ \sum_{\substack{\beta' + \beta'' = \beta \\ |\beta''| < m}} c(\beta', \beta'') \cdot \left[\partial^{\beta'}\theta(x') \cdot \partial^{\beta''}F_\xi(x') - \partial^{\beta'}\theta(x'') \cdot \partial^{\beta''}F_\xi(x'')\right].\end{aligned} \tag{17}$$

For $\beta' + \beta'' = \beta$, $|\beta''| < m$, we have

$$|\nabla\{(\partial^{\beta'}\theta) \cdot (\partial^{\beta''}F_\xi)\}| \leq CA\,\omega(\delta) \cdot \delta^{m-|\beta'|-|\beta''|-1} = CA\,\omega(\delta) \cdot \delta^{-1} \text{ on } B(x_0, \delta),$$

thanks to (12) and (15). Hence, for x', x'' as in (16), and for β', β'' in (17), we have

$$|\partial^{\beta'}\theta(x')\partial^{\beta''}F_\xi(x') - \partial^{\beta'}\theta(x'')\partial^{\beta''}F_\xi(x'')| \leq CA\,\omega(\delta)\,\delta^{-1} \cdot |x' - x''|.$$

Hence, in case (16), equation (17) yields

$$\begin{aligned}&|\partial^\beta \widetilde{F}_\xi(x') - \partial^\beta \widetilde{F}_\xi(x'')| \\ &\quad\leq |\theta(x')\partial^\beta F_\xi(x') - \theta(x'')\partial^\beta F_\xi(x'')| + CA\,\omega(\delta) \cdot \delta^{-1} \cdot |x' - x''| \\ &\quad\leq |\theta(x')| \cdot |\partial^\beta F_\xi(x') - \partial^\beta F_\xi(x'')| + |\theta(x') - \theta(x'')| \cdot |\partial^\beta F_\xi(x'')| \\ &\qquad + CA\,\omega(\delta) \cdot \delta^{-1} \cdot |x' - x''| \\ &\quad\leq CA\,\omega(|x' - x''|) + CA\,\omega(\delta) \cdot \delta^{-1} \cdot |x' - x''|,\end{aligned} \tag{18}$$

thanks to (8), (11), (15). In case (16), we have $\omega(\delta) \cdot \delta^{-1} \cdot |x' - x''| \leq \omega(|x' - x''|)$, since ω is a regular modulus of continuity. Hence, (18) shows that assertion (B) holds in case (16), completing the proof of Lemma 3.1 ∎

From the special case $\delta = 1$ of Lemma 3.1, we obtain at once the following result.

Corollary 3.1.1. *Suppose we are given a vector space Ξ with a seminorm $|\cdot|$, a positive constant A, a regular modulus of continuity ω, a point $x_0 \in \mathbb{R}^n$, a closed convex symmetric set $\sigma_0 \subseteq \mathcal{R}_{x_0}$, and a linear map $\xi \mapsto f_{0,\xi}$ from Ξ into $\mathcal{R}_{x_0}$.*
Assume that, whenever $|\xi| \leq 1$, there exists $F \in C^{m,\omega}(\mathbb{R}^n)$, with $\|F\|_{C^{m,\omega}(\mathbb{R}^n)} \leq A$ and $J_{x_0}(F) \in f_{0,\xi} + A\,\sigma_0$.

Then there exists a linear map $\xi \mapsto \tilde{f}_\xi$, from Ξ into $\mathcal{R}_{x_0}$, such that, whenever $|\xi| \leq 1$, we have

$$|\partial^\beta \tilde{f}_\xi(x_0)| \leq CA \textit{ for } |\beta| \leq m, \textit{ and } \quad \tilde{f}_\xi \in f_{0,\xi} + CA\,\sigma_0,$$

with C depending only on m, n.

For the next lemma, let $D = \dim \mathcal{P}$.

Lemma 3.2. *Suppose* $k^{\#} \geq (D+1)^{10} \cdot k_1^{\#}$, $k_1^{\#} \geq 1$, $A > 0$.

Let Ξ be a vector space, with a seminorm $|\cdot|$. Let ω be a regular modulus of continuity.

Suppose we are given a finite set $E \subset \mathbb{R}^n$; and for each point $x \in E$, suppose we are given a closed convex symmetric set $\sigma(x) \subseteq \mathcal{R}_x$ and a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$.

Assume that, given $\xi \in \Xi$ with $|\xi| \leq 1$, and given $S \subseteq E$ with cardinality at most $k^{\#}$, there exists $F_\xi^S \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A, \quad \textit{and} \quad J_x(F_\xi^S) \in f_\xi(x) + A\sigma(x) \textit{ for each } x \in S.$$

Let $y_0 \in \mathbb{R}^n$. Then there exists a linear map $\xi \mapsto P_\xi$, from Ξ into $\mathcal{R}_{y_0}$, with the following property:

Given $\xi \in \Xi$ with $|\xi| \leq 1$, and given $S \subseteq E$ with cardinality at most $k_1^{\#}$, there exists $F_\xi^S \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq CA\,,\ J_x(F_\xi^S) \in f_\xi(x) + CA\,\sigma(x) \textit{ for } x \in S\,,\ \textit{and } J_{y_0}(F_\xi^S) = P_\xi\,.$$

Here, C depends only on m, n and $k^{\#}$.

Note that the functions F_ξ^S in Lemma 3.2 needn't depend linearly on ξ.

Proof of Lemma 3.2. In this proof, we will call a constant "controlled" if it depends only on m, n, and $k^{\#}$; we write c, C, C', etc. to denote controlled constants.

The proof of Lemma 10.1 in [14] shows that, whenever $\xi \in \Xi$ with $|\xi| \leq 1$, there exists a polynomial $\widehat{P}_\xi$, with the following property:

(19) Given $S \subseteq E$ with cardinality at most $(D+1)^9 \cdot k_1^{\#}$, there exists $F_\xi^S \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A,\ J_x(F_\xi^S) \in f_\xi(x) + A\,\sigma(x) \text{ for } x \in S\,,\ \text{and } J_{y_0}(F_\xi^S) = \widehat{P}_\xi\,.$$

We do not assert that $\widehat{P}_\xi$ or F_ξ^S depend linearly on ξ.

For $S \subseteq E$, we introduce the set

(20) $$\sigma(S) = \{J_{y_0}(F) : F \in C^{m,\omega}(\mathbb{R}^n), \| F \|_{C^{m,\omega}(\mathbb{R}^n)} \leq 1, J_x(F) \in \sigma(x) \text{ for } x \in S\}.$$

Note that each $\sigma(S)$ is a compact, convex, symmetric subset of $\mathcal{P}$.
(To check that $\sigma(S)$ is compact, we recall that the closed unit ball of $C^{m,\omega}(\mathbb{R}^n)$ is compact in the topology of C^m convergence on compact sets, thanks to Ascoli's theorem.)

We set

$$\widehat{\sigma} = \cap\{\sigma(S) : S \subseteq E\,,\, \#(S) \,\le\, (D+1)^6 \cdot k_1^{\#}\} \subseteq \mathcal{P}\,, \tag{21}$$

where $\#(S)$ denotes the cardinality of S.

Since E is assumed to be finite, there are only finitely many $\sigma(S)$, each of which is a compact, convex, symmetric subset of $\mathcal{P}$. Hence, Lemma 2.5 applies, i.e., there exist $S_1, \dots, S_{(D+1)^2} \subseteq E$, with $\#(S_i) \le (D+1)^6 \cdot k_1^{\#}$, such that

$$\bigcap_{1 \le i \le (D+1)^2} \sigma(S_i) \subseteq C\widehat{\sigma}.$$

We define $\bar{S} = S_1 \cup \cdots \cup S_{(D+1)^2}$. Note that

$$\bar{S} \subseteq E\,, \#(\bar{S}) \,\le\, (D+1)^8 \cdot k_1^{\#} \,\le\, k^{\#}\,, \tag{22}$$

and

$$\sigma(\bar{S}) \subseteq \sigma(S_i) \quad \text{for } i = 1, \dots, (D+1)^2.$$

Consequently,

$$\sigma(\bar{S}) \subseteq C\widehat{\sigma},$$

i.e.,

$$\sigma(\bar{S}) \subseteq C\sigma(S) \text{ for any } S \subseteq E \text{ with } \#(S) \,\le\, (D+1)^6 \cdot k_1^{\#}\,. \tag{23}$$

Next, we apply Lemma 2.3 to $\sigma(x)$ for each $x \in \bar{S}$.

Thus, for each $x \in \bar{S}$, we may pick a subspace $I_x \subseteq \mathcal{P}$, and a positive semidefinite quadratic form g_x on I_x, such that

$$\{P \in I_x : g_x(P) \,\le\, c\} \subseteq \sigma(x) \subseteq \{P \in I_x : g_x(P) \,\le\, 1\} \text{ for each } x \in \bar{S}\,. \tag{24}$$

We have to argue slightly differently for the two cases $y_0 \in \bar{S}$ and $y_0 \notin \bar{S}$.

Therefore, we define

$$\check{S} \;=\; \bar{S} \cup \{y_0\}; \tag{25}$$

$$\check{I}_x \;=\; I_x \text{ for } x \in \bar{S}\,; \tag{26}$$

$$\check{I}_{y_0} \;=\; \mathcal{P} \text{ if } y_0 \notin \bar{S}\,; \tag{27}$$

$$\check{f}_\xi(x) \;=\; f_\xi(x) \text{ for } x \in \bar{S}\,,\, \xi \in \Xi; \tag{28}$$

$$\check{f}_\xi(y_0) \;=\; 0 \text{ if } y_0 \notin \bar{S}\,. \tag{29}$$

Thus, for each $x \in \check{S}$, $\check{I}_x$ is a subspace of $\mathcal{P}$, and $\xi \mapsto \check{f}_\xi(x)$ is a linear map from Ξ into $\mathcal{P}$.

Now, for $\xi \in \Xi$ and $\vec{P} = (P^x)_{x \in \check{S}} \in \bigoplus_{x \in \check{S}} \check{I}_x$, we define

$$(30) \quad \mathcal{Q}(\xi, \vec{P}) = \sum_{|\beta| \le m} \sum_{x \in \check{S}} (\partial^\beta [\check{f}_\xi(x) + P^x](x))^2 + \sum_{|\beta| \le m} \sum_{\substack{x,y \in \check{S} \\ 0 < |x-y| \le 1}} \left(\frac{\partial^\beta [\check{f}_\xi(x) + P^x - \check{f}_\xi(y) - P^y](y)}{\omega(|x-y|) \cdot |x-y|^{m-|\beta|}} \right)^2 + \sum_{x \in \bar{S}} g_x(P^x) .$$

Note that

$$\mathcal{Q}(\xi, \vec{P}) = \mathcal{Q}_0(\xi) + \mathcal{Q}_1(\xi, \vec{P}) + \mathcal{Q}_2(\vec{P}),$$

where $\mathcal{Q}_0(\xi)$ is a quadratic form in ξ; $\mathcal{Q}_1(\xi, \vec{P})$ is bilinear in $\xi, \vec{P}$; and $\mathcal{Q}_2(\vec{P})$ is a positive-definite quadratic form in $\vec{P}$. Hence, for fixed $\xi \in \Xi$, there is a unique minimizer

$$(31) \quad \vec{P}_\xi = (P^x_\xi)_{x \in \check{S}} \in \bigoplus_{x \in \check{S}} \check{I}_x$$

for the function $\vec{P} \mapsto \mathcal{Q}(\xi, \vec{P})$ $(\vec{P} \in \bigoplus_{x \in \check{S}} \check{I}_x)$; moreover

$$(32) \quad \vec{P}_\xi \text{ depends linearly on } \xi .$$

We define

$$(33) \quad P_\xi = P^{y_0}_\xi + \check{f}_\xi(y_0)$$

for $\xi \in \Xi$, with $P^{y_0}_\xi$ arising from (31).

From (31), (32), (33), we see that

$$(33a) \quad \xi \mapsto P_\xi \text{ is a linear map from } \Xi \text{ into } \mathcal{P} = \mathcal{R}_{y_0}.$$

We will show that this linear map has the property asserted in the statement of Lemma 3.2. This will complete the proof of the Lemma.

Let $\xi \in \Xi$ be given, with $|\xi| \le 1$.

From the hypotheses of Lemma 3.2, and from (22), we obtain a function $F^{\bar{S}}_\xi \in C^{m,\omega}(\mathbb{R}^n)$, with

$$(34) \quad \| F^{\bar{S}}_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le A , \text{ and } J_x(F^{\bar{S}}_\xi) \in f_\xi(x) + A\,\sigma(x) \text{ for } x \in \bar{S} .$$

For $x \in \check{S}$, we define

$$(35) \quad \check{P}^x = J_x(F^{\bar{S}}_\xi) - \check{f}_\xi(x) .$$

In particular, (34), (35) and (28) give $\check{P}^x \in A\sigma(x)$ for all $x \in \bar{S}$, hence

$$\check{P}^x \in I_x \text{ and } g_x(\check{P}^x) \leq A^2\,, \text{ for all } x \in \bar{S}. \tag{36}$$

Also,

$$\check{P}^{y_0} \in \check{I}_{y_0}\,. \tag{37}$$

In fact, (37) is immediate from (36) in case $y_0 \in \bar{S}$, and from (27) otherwise.

Thus,

$$\check{P} = (\check{P}^x)_{x\in\check{S}} \in \bigoplus_{x\in\check{S}} \check{I}_x\,. \tag{38}$$

In view of (34) and (35), we have

$$\left|\partial^\beta[\check{f}_\xi(x) + \check{P}^x](x)\right| \leq CA \text{ for } |\beta| \leq m,\, x \in \check{S}\,; \text{ and} \tag{39}$$

$$\left|\partial^\beta[\check{f}_\xi(x) + \check{P}^x - \check{f}_\xi(y) - \check{P}^y](y)\right| \leq CA\,\omega(|x-y|)\cdot|x-y|^{m-|\beta|} \tag{40}$$

$$\text{for } |\beta| \leq m\,, x, y \in \check{S},\, |x-y| \leq 1\,.$$

In view of (36), (39), (40) and the definition (30) of $\mathcal{Q}(\xi, \vec{\mathcal{P}})$, we have

$$\mathcal{Q}(\xi, \check{P}) \leq CA^2. \tag{41}$$

From (38), (41), and the minimizing property of $\vec{P}_\xi$, we have also

$$\mathcal{Q}(\xi, \vec{P}_\xi) \leq CA^2. \tag{42}$$

By definition of $\mathcal{Q}$, this shows that $\vec{P}_\xi = (P^x_\xi)_{x\in\check{S}} \in \bigoplus_{x\in\check{S}} \check{I}_x$ satisfies:

$$P^x_\xi \in I_x \text{ for } x \in \bar{S} \text{ (see (26))}\,; \tag{43}$$

$$\left|\partial^\beta[\check{f}_\xi(x) + P^x_\xi](x)\right| \leq CA \text{ for } |\beta| \leq m\,,\, x \in \check{S}\,; \tag{44}$$

$$\left|\partial^\beta[\check{f}_\xi(x) + P^x_\xi - \check{f}_\xi(y) - P^y_\xi](y)\right| \leq CA\,\omega(|x-y|)\cdot|x-y|^{m-|\beta|} \tag{45}$$

$$\text{for } |\beta| \leq m\,,\, |x-y| \leq 1\,, x, y \in \check{S}\,; \text{ and}$$

$$g_x(P^x_\xi) \leq CA^2 \text{ for } x \in \bar{S}\,. \tag{46}$$

From (43), (46) and (24), we obtain

$$P^x_\xi \in CA\,\sigma(x) \text{ for } x \in \bar{S}\,. \tag{47}$$

In view of (44), (45), and Whitney's extension theorem for finite sets (see Section 1), there exists a function $\check{F}_\xi \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| \check{F}_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq CA\,, \tag{48}$$

and

$$J_x(\check{F}_\xi) = \check{f}_\xi(x) + P_\xi^x \text{ for all } x \in \check{S}\,. \tag{49}$$

In particular,

$$J_{y_0}(\check{F}_\xi) = P_\xi \text{ (see (33))}\,, \tag{50}$$

and

$$J_x(\check{F}_\xi) = f_\xi(x) + P_\xi^x \in f_\xi(x) + CA\,\sigma(x) \text{ for } x \in \bar{S} \text{ (see (28) and (47))}\,. \tag{51}$$

On the other hand, (19) and (22) produce a function $F_\xi^{\bar{S}} \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| F_\xi^{\bar{S}} \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A\,, \tag{52}$$

$$J_x(F_\xi^{\bar{S}}) \in f_\xi(x) + A\,\sigma(x) \text{ for } x \in \bar{S}\,, \text{ and} \tag{53}$$

$$J_{y_0}(F_\xi^{\bar{S}}) = \widehat{P}_\xi\,. \tag{54}$$

From (48), (52) we obtain

$$\| \check{F}_\xi - F_\xi^{\bar{S}} \|_{C^{m,\omega}(\mathbb{R}^n)} \leq CA\,, \tag{55}$$

while (51), (53) yield

$$J_x(\check{F}_\xi - F_\xi^{\bar{S}}) \in CA\,\sigma(x) \text{ for } x \in \bar{S}\,, \tag{56}$$

and (50), (54) imply

$$J_{y_0}(\check{F}_\xi - F_\xi^{\bar{S}}) = P_\xi - \widehat{P}_\xi\,. \tag{57}$$

Comparing (55), (56), (57) with the definition (20) of $\sigma(S)$, we find that $P_\xi - \widehat{P}_\xi \in CA\,\sigma(\bar{S})$. Hence, (23) implies

$$P_\xi - \widehat{P}_\xi \in CA\,\sigma(S) \text{ for any } S \subseteq E \text{ with } \#(S) \leq (D+1)^6 \cdot k_1^\#\,. \tag{58}$$

Again recalling the definition of $\sigma(S)$, we conclude from (58) that, given $S \subseteq E$ with $\#(S) \leq (D+1)^6 \cdot k_1^\#$, there exists $\widetilde{F}_\xi^S \in C^{m,\omega}(\mathbb{R}^n)$, with

$$\| \widetilde{F}_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq CA, \quad J_x(\widetilde{F}_\xi^S) \in CA\,\sigma(x) \text{ for } x \in S\,, \quad \text{and } J_{y_0}(\widetilde{F}_\xi^S) = P_\xi - \widehat{P}_\xi\,. \tag{59}$$

Now, given $S \subseteq E$ with $\#(S) \leq k_1^\#$, let F_ξ^S be as in (19), and let $\widetilde{F}_\xi^S$ be as in (59). Then, from (19) and (59), we have

(60) $$\| F_\xi^S + \widetilde{F}_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq CA\,,$$

(61) $$J_x(F_\xi^S + \widetilde{F}_\xi^S) \in f_\xi(x) + CA\,\sigma(x) \text{ for } x \in S\,, \text{ and}$$

(62) $$J_{y_0}(F_\xi^S + \widetilde{F}_\xi^S) = P_\xi\,.$$

Thus, we can achieve (60), (61), (62) whenever $\xi \in \Xi$ with $|\xi| \leq 1$ and $S \subseteq E$ with $\#(S) \leq k_1^\#$.

Our results (33a) and (60), (61), (62) immediately imply the conclusions of Lemma 3.2.

The proof of the Lemma is complete. ■

In [15], we will use the following variant of Lemma 3.2 for infinite sets E. We write $\#(S)$ for the cardinality of a set S. Also, we adopt the convention that $|x-y|^{m-|\beta|} = 0$ in the degenerate case $x = y$, $|\beta| = m$.

Lemma 3.3. *Suppose* $k^\# \geq (D+1)^{10} \cdot k_1^\#$, $k_1^\# \geq 1$, $A > 0$, $\delta > 0$. *Let* Ξ *be a vector space, with a seminorm* $|\cdot|$. *Let* $E \subseteq \mathbb{R}^n$, *and let* $x_0 \in E$. *For each* $x \in E$, *suppose we are given a vector subspace* $I(x) \subseteq \mathcal{R}_x$, *and a linear map* $\xi \mapsto f_\xi(x)$ *from* Ξ *into* $\mathcal{R}_x$.

Assume that the following conditions are satisfied.

(a) *Given* $\xi \in \Xi$ *and* $S \subseteq E$, *with* $|\xi| \leq 1$ *and* $\#(S) \leq k^\#$, *there exists* $F_\xi^S \in C^m(\mathbb{R}^n)$, *with* $\| F_\xi^S \|_{C^m(\mathbb{R}^n)} \leq A$, *and* $J_x(F) \in f_\xi(x) + I(x)$ *for each* $x \in S$.

(b) *Suppose* $P_0 \in I(x_0)$, *with* $|\partial^\beta P_0(x_0)| < \delta$ *for* $|\beta| \leq m$. *Then, given* $x_1, \dots, x_{k^\#} \in E$, *there exist* $P_1 \in I(x_1), \dots, P_{k^\#} \in I(x_{k^\#})$, *with*

$|\partial^\beta P_i(x_i)| \leq 1$ *for* $|\beta| \leq m$, $0 \leq i \leq k^\#$; *and*

$|\partial^\beta (P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\beta|}$ *for* $|\beta| \leq m$, $0 \leq i, j \leq k^\#$.

Then there exists a linear map $\xi \mapsto \tilde{f}_\xi(x_0)$, *from* Ξ *into* $\mathcal{R}_{x_0}$, *with the following property*:

(c) *Given* $\xi \in \Xi$ *with* $|\xi| \leq 1$, *and given* $x_1, \dots, x_{k_1^\#} \in E$, *there exist polynomials* $P_0, \dots, P_{k_1^\#} \in \mathcal{P}$, *with*:

$P_0 = \tilde{f}_\xi(x_0)$;

$P_i \in f_\xi(x_i) + I(x_i)$ for $0 \leq i \leq k_1^\#$;

$|\partial^\beta P_i(x_i)| \leq CA$ for $|\beta| \leq m$, $0 \leq i \leq k_1^\#$; and

$|\partial^\beta (P_i - P_j)(x_j)| \leq CA\,|x_i - x_j|^{m-|\beta|}$ for $|\beta| \leq m$, $0 \leq i, j \leq k_1^\#$.

Here, C *depends only on* m, n, *and* $k^\#$.

Proof of Lemma 3.3. We follow the idea of the proof of Lemma 3.2. In this proof, we call a constant "controlled" if it is determined by $m, n, k^{\#}$; and we write c, C, C', etc. to denote controlled constants.

If $\dim I(x_0) = 0$, then Lemma 3.3 is trivial; we simply take $\tilde{f}_\xi(x_0) = f_\xi(x_0)$. From now on, we suppose $\dim I(x_0) > 0$.

We introduce the following convex sets.

For $S \subseteq E$ with $x_0 \in S$, we define $\sigma(S) \subseteq I(x_0)$ as follows.

$P_0 \in \sigma(S)$ if and only if there exists a family of polynomials $(P^x)_{x\in S}$, with $P^x \in I(x)$ for $x \in S$; $|\partial^\beta P^x(x)| \le 1$ for $|\beta| \le m$, $x \in S$; $|\partial^\beta(P^x - P^y)(y)| \le |x-y|^{m-|\beta|}$ for $|\beta| \le m$, $x, y \in S$; and $P^{x_0} = P_0$.

For $\xi \in \Xi$, $M \in (0,\infty)$, and $S \subseteq E$ with $x_0 \in S$, we define $\Gamma_\xi(S, M) \subseteq f_\xi(x_0) + I(x_0)$ as follows.

$P_0 \in \Gamma_\xi(S, M)$ if and only if there exists a family of polynomials $(P^x)_{x\in S}$, with $P^x \in f_\xi(x) + I(x)$ for $x \in S$; $|\partial^\beta P^x(x)| \le M$ for $|\beta| \le m$, $x \in S$; $|\partial^\beta(P^x - P^y)(y)| \le M\,|x-y|^{m-|\beta|}$ for $|\beta| \le m$, $x, y \in S$; and $P^{x_0} = P_0$.

For $k \ge 1$, we define

$$\sigma(k) = \cap\{\sigma(S) : S \subseteq E \ \text{ with } x_0 \in S \text{ and } \#(S) \le k\}.$$

For $k \ge 1, \xi \in \Xi$, $M \in (0,\infty)$, we define

$$\Gamma_\xi(k, M) = \cap\{\Gamma_\xi(S, M) : S \subseteq E \ \text{ with } x_0 \in S \text{ and } \#(S) \le k\}.$$

Note that $\sigma(S)$ and $\sigma(k)$ are compact, convex, symmetric subsets of $I(x_0)$, while $\Gamma_\xi(S, M)$ and $\Gamma_\xi(k, M)$ are compact, convex subsets of $f_\xi(x_0) + I(x_0)$.

We give a few basic properties of the above convex sets.

First of all, note that $\sigma(k)$ has non-empty interior in $I(x_0)$, for $1 \le k \le k^{\#}$, thanks to hypothesis (b). We take $k = (D+1)^6 \cdot k_1^{\#}$, and apply Lemma 2.6, with $I(x_0)$ in place of $\mathbb{R}^d$. Thus, there exist $S_1, \dots, S_{D\cdot(D+1)} \subseteq E$, with $\#(S_i) \le (D+1)^6 \cdot k_1^{\#}$ and $x_0 \in S_i$ for each i, such that

$$\sigma(S_1) \cap \cdots \cap \sigma(S_{D\cdot(D+1)}) \subseteq C \cdot \sigma((D+1)^6 \cdot k_1^{\#})\,.$$

We define $\bar{S} = S_1 \cup \cdots \cup S_{D\cdot(D+1)}$. Note that

$$\bar{S} \subseteq E,\ x_0 \in \bar{S},\ \#(\bar{S}) \le (D+1)^8 \cdot k_1^{\#}\,, \tag{63}$$

and $\sigma(\bar{S}) \subseteq \sigma(S_i)$ for each i. Consequently,

$$\sigma(\bar{S}) \subseteq C\sigma(S) \text{ for any } S \subseteq E \text{ with } \#(S) \le (D+1)^6 \cdot k_1^{\#} \text{ and } x_0 \in S\,. \tag{64}$$

We turn to the $\Gamma_\xi(S, M)$. Suppose $|\xi| \leq 1$, $S \subseteq E$, $\#(S) \leq k^\#$, and $x_0 \in S$. Then $\Gamma_\xi(S, CA)$ is non-empty, by hypothesis (a). (For $x \in S$, we take $P^x = J_x(F_\xi^S)$, with F_ξ^S as in (a).)

Next, suppose $|\xi| \leq 1$, and $S_1, \ldots, S_{D+1} \subseteq E$, with $\#(S_i) \leq (D+1)^8 \cdot k_1^\#$ and $x_0 \in S_i$, for each i. Then $S = S_1 \cup \cdots \cup S_{D+1} \subseteq E$, with $\#(S) \leq k^\#$ and $x_0 \in S$. Consequently, $\Gamma_\xi(S, CA)$ is non-empty. On the other hand, $\Gamma_\xi(S, CA) \subseteq \Gamma_\xi(S_i, CA)$ for each i. Therefore, $\Gamma_\xi(S_1, CA) \cap \cdots \cap \Gamma_\xi(S_{D+1}, CA)$ is non-empty.

Applying Helly's theorem on convex sets (see, e.g., [22]), we therefore obtain the following result.

$$\Gamma_\xi\left((D+1)^8 \cdot k_1^\#, CA\right) \text{ is non-empty, for } |\xi| \leq 1\,. \tag{65}$$

Now, for $\xi \in \Xi$ and $\vec{P} = (P^x)_{x \in \bar{S}} \in \bigoplus_{x \in \bar{S}} I(x)$, we define

$$\begin{aligned} Q(\xi, \vec{P}) = &\sum_{|\beta| \leq m} \sum_{x \in \bar{S}} \left(\partial^\beta [f_\xi(x) + P^x](x)\right)^2 \\ &+ \sum_{|\beta| \leq m} \sum_{\substack{x, y \in \bar{S} \\ x \neq y}} \left(\frac{\partial^\beta [f_\xi(x) + P^x - f_\xi(y) - P^y](y)}{|x - y|^{m - |\beta|}}\right)^2 . \end{aligned} \tag{66}$$

Here, $\bar{S}$ is as in (63), (64).

Note that $Q(\xi, \vec{P}) = Q_0(\xi) + Q_1(\xi, \vec{P}) + Q_2(\vec{P})$, where $Q_0(\xi)$ is a quadratic form in ξ; $Q_1(\xi, \vec{P})$ is a bilinear form in ξ and $\vec{P}$; and $Q_2(\vec{P})$ is a positive-definite quadratic form in $\vec{P}$. Hence, for fixed $\xi \in \Xi$, there is a unique minimizer

$$\vec{P}_\xi = (P_\xi^x)_{x \in \bar{S}} \in \bigoplus_{x \in \bar{S}} I(x) \tag{67}$$

for the function $\vec{P} \mapsto Q(\xi, \vec{P})$ $(\vec{P} \in \bigoplus_{x \in \bar{S}} I(x))$.

Moreover,

$$\vec{P}_\xi \text{ depends linearly on } \xi \in \Xi\,. \tag{68}$$

For $\xi \in \Xi$ we define

$$\tilde{f}_\xi(x_0) = f_\xi(x_0) + P_\xi^{x_0}\,, \tag{69}$$

with $P_\xi^{x_0}$ arising from (67).

In view of (68) and the assumed linearity of $\xi \mapsto f_\xi(x)$, we have

$$\xi \mapsto \tilde{f}_\xi(x_0) \text{ is a linear map from } \Xi \text{ into } \mathcal{R}_{x_0}\,. \tag{70}$$

We will show that $\tilde{f}_\xi(x_0)$ satisfies property (c). This will complete the proof of the lemma.

Let $F_\xi^{\bar{S}}$ be as in hypothesis (a) with $S = \bar{S}$ (see (63)), and set

$$\check{P}^x = J_x(F_\xi^{\bar{S}}) - f_\xi(x) \quad \text{for } x \in \bar{S}.$$

The defining properties of $F_\xi^{\bar{S}}$ tell us that

$$\check{P} = (\check{P}^x)_{x\in\bar{S}} \in \bigoplus_{x\in\bar{S}} I(x)\,; \tag{71}$$

$$|\partial^\beta[f_\xi(x) + \check{P}^x]\,(x)| \le CA \text{ for } |\beta| \le m\,,\ x \in \bar{S}\,; \text{ and} \tag{72}$$

$$|\partial^\beta[f_\xi(x) + \check{P}^x - f_\xi(y) - \check{P}^y]\,(y)| \le CA\,|x-y|^{m-|\beta|} \quad \text{for } |\beta| \le m,\ x,y \in \bar{S}\,. \tag{73}$$

Together with the definition of $\mathcal{Q}$, our results (71), (72), (73) show that

$$\mathcal{Q}(\xi, \check{P}) \le CA^2,$$

and therefore

$$\mathcal{Q}(\xi, \vec{P}_\xi) \le CA^2 \tag{74}$$

by the minimizing property of $\vec{P}_\xi$.

From (74) and the definition of $\mathcal{Q}$, we learn that

$$P_\xi^x \in I(x) \text{ for } x \in \bar{S}\,; \tag{75}$$

$$|\partial^\beta[f_\xi(x) + P_\xi^x]\,(x)| \le CA \text{ for } |\beta| \le m\,,\ x \in \bar{S}\,; \text{ and} \tag{76}$$

$$|\partial^\beta[f_\xi(x) + P_\xi^x - f_\xi(y) - P_\xi^y]\,(y)| \le CA\,|x-y|^{m-|\beta|} \quad \text{for } |\beta| \le m\,, x,y \in \bar{S}\,. \tag{77}$$

On the other hand, fix

$$\widehat{P}_\xi \in \Gamma_\xi((D+1)^8 \cdot k_1^\#\,,\, CA) \tag{78}$$

(See (65), and note that $\widehat{P}_\xi$ need not depend linearly on ξ.) In view of (63), we have

$$\widehat{P}_\xi \in \Gamma_\xi(\bar{S}\,, CA).$$

Consequently, there exists a family of polynomials $(\widehat{P}_x)_{x\in\bar{S}}$, with the following properties.

$$\widehat{P}^x \in I(x) \text{ for } x \in \bar{S}\,. \tag{79}$$

$$|\partial^\beta[f_\xi(x) + \widehat{P}^x](x)| \le CA \text{ for } |\beta| \le m\,,\ x \in \bar{S}\,. \tag{80}$$

$$|\partial^\beta[f_\xi(x) + \widehat{P}^x - f_\xi(y) - \widehat{P}^y](y)\,| \le CA\,|x-y|^{m-|\beta|} \quad \text{for } |\beta| \le m\,, x,y \in \bar{S}\,. \tag{81}$$

$$f_\xi(x_0) + \widehat{P}^{x_0} = \widehat{P}_\xi\,. \tag{82}$$

Comparing (75),...,(77) with (79),...,(81); and comparing (69) with (82), we learn the following.

$$[P^x_\xi - \widehat{P}^x] \in I(x) \quad \text{for } x \in \bar{S}.$$
$$|\partial^\beta [P^x_\xi - \widehat{P}^x](x)| \leq CA \quad \text{for } |\beta| \leq m,\ x \in \bar{S}.$$
$$|\partial^\beta \{[P^x_\xi - \widehat{P}^x] - [P^y_\xi - \widehat{P}^y]\}(y)| \leq CA\,|x-y|^{m-|\beta|} \quad \text{for } |\beta| \leq m,\ x, y \in \bar{S}.$$
$$[P^{x_0}_\xi - \widehat{P}^{x_0}] = \tilde{f}_\xi(x_0) - \widehat{P}_\xi.$$

These properties, and the definition of $\sigma(\bar{S})$, show that $\tilde{f}_\xi(x_0) - \widehat{P}_\xi \in CA\sigma(\bar{S})$. Consequently, (64) tells us that

(83) $\tilde{f}_\xi(x_0) - \widehat{P}_\xi \in CA\sigma(S)$
for any $S \subseteq E$ with $\#(S) \leq (D+1)^6 \cdot k_1^\#$ and $x_0 \in S$.

On the other hand, (78) gives

(84) $\widehat{P}_\xi \in \Gamma_\xi(S, CA)$ for any $S \subseteq E$ with $\#(S) \leq (D+1)^6 \cdot k_1^\#$ and $x_0 \in S$.

From (83), (84), and the definitions of $\sigma(S)$ and $\Gamma_\xi(S, CA)$, we conclude that

$$\tilde{f}_\xi(x_0) \in \Gamma_\xi(S, CA).$$

Thus, we have proven the following result.

(85) $\tilde{f}_\xi(x_0) \in \Gamma_\xi(S, CA)$
for $|\xi| \leq 1$, $S \subseteq E$ with $\#(S) \leq (D+1)^6 \cdot k_1^\#$ and $x_0 \in S$.

This result trivially implies the desired property (c) of $\tilde{f}_\xi(x_0)$. In fact, given $\xi \in \Xi$ with $|\xi| \leq 1$, and given $x_1, \ldots, x_{k_1^\#} \in E$, we set $S = \{x_0, \ldots, x_{k_1^\#}\}$, and apply (85).
By definition of $\Gamma_\xi(S, CA)$, there exists a family of polynomials $(P^x)_{x \in S}$, with

$$P^{x_0} = \tilde{f}_\xi(x_0);$$
$$P^x \in f_\xi(x) + I(x) \quad \text{for all } x \in S;$$
$$|\partial^\beta P^x(x)| \leq CA \quad \text{for } |\beta| \leq m,\ x \in S; \text{ and}$$
$$|\partial^\beta (P^x - P^y)(y)| \leq CA\,|x-y|^{m-|\beta|} \quad \text{for } |\beta| \leq m,\ x, y \in S.$$

Setting $P_i = P^{x_i}$ for $i = 0, \ldots, k_1^\#$, we obtain all the properties asserted in (c).

The proof of Lemma 3.3 is complete. ∎

4. Adapting Previous Results

In this section, we show how to adapt [14], using the results of Section 3 above, to prove a local version of our present Theorem 7, in the case of (arbitrarily large) finite subsets $E \subset \mathbb{R}^n$. We assume from here on that the reader is completely familiar with [14].

Sections 2 and 3 of [14] require no changes. In Section 4 of [14], the statements of the two main lemmas should be changed to the following.

Weak Main Lemma for $\mathcal{A}$: *There exists* $k^\#$*, depending only on* m *and* n*, for which the following holds.*

Suppose we are given a vector space Ξ *with a seminorm* $|\cdot|$*; constants* C, a_0*; a regular modulus of continuity* ω*; a finite set* $E \subset \mathbb{R}^n$*; a point* $y^0 \in \mathbb{R}^n$*; and a family of polynomials* $P_\alpha \in \mathcal{P}$*, indexed by* $\alpha \in \mathcal{A}$*. Suppose also that, for each* $x \in E$*, we are given a linear map* $\xi \mapsto f_\xi(x)$ *from* Ξ *into* $\mathcal{R}_x$*, and a subset* $\sigma(x) \subset \mathcal{R}_x$*.*

Assume that the following conditions are satisfied.

(WL0) *For each* $x \in E$*, the set* $\sigma(x)$ *is Whitney* ω*-convex, with Whitney constant* C*.*

(WL1) $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ *for all* $\beta, \alpha \in \mathcal{A}$*.*

(WL2) $|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \le a_0$ *for all* $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$*.*

(WL3) *Given* $\alpha \in \mathcal{A}$ *and* $S \subset E$ *with* $\#(S) \le k^\#$*, there exists* $\varphi^S_\alpha \in C^{m,\omega}_{\ell oc}(\mathbb{R}^n)$*, with*

- (a) $|\partial^\beta \varphi^S_\alpha(x) - \partial^\beta \varphi^S_\alpha(y)| \le a_0 \cdot \omega(|x-y|)$ *for* $|\beta| = m,\ x, y \in \mathbb{R}^n, |x-y| \le 1$;
- (b) $J_x(\varphi^S_\alpha) \in C\,\sigma(x)$ *for all* $x \in S$; *and*
- (c) $J_{y^0}(\varphi^S_\alpha) = P_\alpha$.

(WL4) *Given* $\xi \in \Xi$ *with* $|\xi| \le 1$*, and given* $S \subseteq E$ *with* $\#(S) \le k^\#$*, there exists* $F^S_\xi \in C^{m,\omega}(\mathbb{R}^n)$*, with*

- (a) $\| F^S_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C$; *and*
- (b) $J_x(F^S_\xi) \in f_\xi(x) + C\,\sigma(x)$ *for all* $x \in S$.

(WL5) a_0 *is less than a small enough positive constant determined by* C, m, n*.*

Then there exists a linear map $\xi \mapsto F_\xi$*, from* Ξ *into* $C^{m,\omega}(\mathbb{R}^n)$*, such that, for any* $\xi \in \Xi$ *with* $|\xi| \le 1$*, we have*

(WL6) $\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C'$, *and*

(WL7) $J_x(F_\xi) \in f_\xi(x) + C'\,\sigma(x)$ *for all* $x \in E \cap B(y^0, c')$.

Here, C' *and* c' *in* (WL6,7) *depend only on* C, m, n*.*

Strong Main Lemma for $\mathcal{A}$: *There exists* $k^\#$, *depending only on* m *and* n, *for which the following holds.*

Suppose we are given a vector space Ξ *with a seminorm* $|\cdot|$; *constants* $C, \bar{a}_0$; *a regular modulus of continuity* ω; *a finite set* $E \subset \mathbb{R}^n$; *a point* $y^0 \in \mathbb{R}^n$; *and a family of polynomials* $P_\alpha \in \mathcal{P}$, *indexed by* $\alpha \in \mathcal{A}$. *Suppose also that, for each* $x \in E$, *we are given a linear map* $\xi \mapsto f_\xi(x)$ *from* Ξ *into* $\mathcal{R}_x$, *and a subset* $\sigma(x) \subseteq \mathcal{R}_x$.

Assume that the following conditions are satisfied.

(SL0) *For each* $x \in E$, *the set* $\sigma(x)$ *is Whitney* ω*-convex, with Whitney constant* C.

(SL1) $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ *for all* $\beta, \alpha \in \mathcal{A}$.

(SL2) $|\partial^\beta P_\alpha(y^0)| \leq C$ *for all* $\beta \in \mathcal{M}$, $\alpha \in \mathcal{A}$ *with* $\beta \geq \alpha$.

(SL3) *Given* $\alpha \in \mathcal{A}$ *and* $S \subseteq E$ *with* $\#(S) \leq k^\#$, *there exists*

$$\varphi_\alpha^S \in C^{m,\omega}_{\ell oc}(\mathbb{R}^n),$$

with

(a) $$|\partial^\beta \varphi_\alpha^S(x) - \partial^\beta \varphi_\alpha^S(y)| \leq \bar{a}_0 \omega(|x-y|) + C|x-y| \quad \text{for } |\beta| = m\,, x, y \in \mathbb{R}^n\,, |x-y| \leq 1\,;$$

(b) $J_x(\varphi_\alpha^S) \in C\sigma(x)$ *for all* $x \in S$; *and*

(c) $J_{y^0}(\varphi_\alpha^S) = P_\alpha$.

(SL4) *Given* $\xi \in \Xi$ *with* $|\xi| \leq 1$, *and given* $S \subseteq E$ *with* $\#(S) \leq k^\#$, *there exists*

$$F_\xi^S \in C^{m,\omega}(\mathbb{R}^n),$$

with

(a) $\| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C$, *and*

(b) $J_x(F_\xi^S) \in f_\xi(x) + C\sigma(x)$ *for all* $x \in S$.

(SL5) $\bar{a}_0$ *is less than a small enough positive constant determined by* C, m, n.

Then there exists a linear map $\xi \mapsto F_\xi$, *from* Ξ *into* $C^{m,\omega}(\mathbb{R}^n)$, *such that, for any* $\xi \in \Xi$ *with* $|\xi| \leq 1$, *we have*

(SL6) $\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C'$, *and*

(SL7) $J_x(F_\xi) \in f_\xi(x) + C'\sigma(x)$ *for all* $x \in E \cap B(y^0, c')$.

Here, C' *and* c' *in* (SL6, 7) *depend only on* C, m, n.

In Section 5 of [14], Lemmas 5.1, 5.2, 5.3 are to be left unchanged. The "Local Theorem" should be changed to read as follows.

Local Theorem: *There exists $k^{\#}$, depending only on m and n, for which the following holds.*

Suppose we are given a vector space Ξ with a seminorm $|\cdot|$; a regular modulus of continuity ω; a finite set $E \subset \mathbb{R}^n$; and, for each $x \in E$, a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, and a subset $\sigma(x) \subseteq \mathcal{R}_x$.

Assume that the following conditions are satisfied.

(I) *For each $x \in E$, the set $\sigma(x)$ is Whitney ω-convex, with Whitney constant C.*

(II) *Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $S \subseteq E$ with $\#(S) \le k^{\#}$, there exists*

$$F_\xi^S \in C^{m,\omega}(\mathbb{R}^n),$$

with

$$\| F_\xi^S \|_{C^{m,\omega}(\mathbb{R}^n)} \le C, \text{ and } J_x(F_\xi^S) \in f_\xi(x) + C\sigma(x) \text{ for each } x \in S.$$

Let $y^0 \in \mathbb{R}^n$ be given. Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, given any $\xi \in \Xi$ with $|\xi| \le 1$, we have

$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C', \text{ and } J_x(F_\xi) \in f_\xi(x) + C'\sigma(x) \text{ for each } x \in E \cap B(y^0, c').$$

Here, C' and c' depend only on C, m, n in (I) *and* (II).

Note that in the two Main Lemmas and the Local Theorem, we do not assume that F_ξ^S depends linearly on ξ, but we assert that F_ξ depends linearly on ξ.

In Section 6 of [14], the proof of Lemma 5.1 may be left unchanged, except that, in the discussion of (14), (15), (16) in that section, F^S should be replaced by F_ξ^S, and f should be replaced by f_ξ, for a given $\xi \in \Xi$ with $|\xi| \le 1$.

In Section 7 of [14], Lemma 7.1 and its proof may be left unchanged, except for the paragraph including (30), (31), (32). In that paragraph, we replace "we obtain $F \in C^{m,\omega}(\mathbb{R}^n)$" by "we obtain a linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$"; and we replace F, f by F_ξ, f_ξ in (30), (31), (32), for a given $\xi \in \Xi$ with $|\xi| \le 1$.

In Section 8 of [14], we make the following changes.

In the statement of Lemma 8.1, the phrase "we are given an m-jet $f(x) \in \mathcal{R}_x$" should be changed to "we are given a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$,". Also, hypothesis (G4) of Lemma 8.1 should be replaced by the following.

(G4) Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $S \subseteq E$ with $\#(S) \le k^{\#}_{\mathrm{old}}$, there exists
$$F^S_\xi \in C^{m,\omega}(\mathbb{R}^n),$$
with

(a) $\| \partial^\beta F^S_\xi \|_{C^0(\mathbb{R}^n)} \le A \cdot \omega(\delta_Q) \cdot \delta_Q^{m-|\beta|}$ for $|\beta| \le m$;

(b) $|\partial^\beta F^S_\xi(x') - \partial^\beta F^S_\xi(x'')| \le A \cdot \omega(|x' - x''|)$ for $|\beta| = m$, $x', x'' \in \mathbb{R}^n$, $|x' - x''| \le \delta_Q$;

(c) $J_x(F^S_\xi) \in f_\xi(x) + A \cdot \sigma(x)$ for all $x \in S$.

Moreover, the conclusion of Lemma 8.1 should be replaced by the following.
Then there exists a linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, for any $\xi \in \Xi$ with $|\xi| \le 1$, we have

(G5) $\| \partial^\beta F_\xi \|_{C^0(\mathbb{R}^n)} \le A' \cdot \omega(\delta_Q) \cdot \delta^{m-|\beta|}$ for $|\beta| \le m$;

(G6) $|\partial^\beta F_\xi(x') - \partial^\beta F_\xi(x'')| \le A' \cdot \omega(|x' - x''|)$ for $|\beta| = m$, $x', x'' \in \mathbb{R}^n$, $|x' - x''| \le \delta_Q$;

(G7) $J_x(F_\xi) \in f_\xi(x) + A' \cdot \sigma(x)$ for all $x \in E \cap Q^*$.

Here, A' is determined by A, m, n.

Again, note that we do not assume that F^S_ξ depends linearly on ξ, but we assert that F_ξ depends linearly on ξ.

In the proof of Lemma 8.1 in [14], we must insert subscript ξ's on F's and f's, as in our discussion of the changes to be made in Sections 6 and 7. More seriously, the paragraph containing (37) and (38) requires changes, because $\tilde{f}(\underline{x})$ there may depend non-linearly on ξ. We change that paragraph to the following.

Applying (G4), we obtain, for any $\xi \in \Xi$ with $|\xi| \le 1$, a function $F_\xi \in C^{m,\omega}(\mathbb{R}^n)$, with
$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le A, \quad \text{and} \quad J_{\underline{x}}(F_\xi) \in f_\xi(\underline{x}) + A\sigma(\underline{x}).$$

Hence, we may apply Corollary 3.1.1 (from Section 3 of this paper, not from [14]), with $f_{0,\xi} = f_\xi(\underline{x})$, and with $\sigma_0 = \sigma(\underline{x})$.

Thus, there exists a linear map $\xi \mapsto \tilde{f}_\xi(\underline{x})$, from Ξ into $\mathcal{R}_{\underline{x}}$, such that, whenever $|\xi| \le 1$, we have

(37) $|\partial^\beta[\tilde{f}_\xi(\underline{x})]\,(\underline{x})| \le CA$ for $|\beta| \le m$, and $\tilde{f}_\xi(\underline{x}) \in f_\xi(\underline{x}) + CA\sigma(\underline{x})$.

In view of (37) and (G4), we have the following property of $\tilde{f}_\xi$.

(38) Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $S \subseteq E$ with $\#(S) \le k^{\#}$, there exists
$$F^S_\xi \in C^{m,\omega}(\mathbb{R}^n),$$
with

(a) $\| \partial^\beta F^S_\xi \|_{C^0(\mathbb{R}^n)} \le A$ for $|\beta| \le m$;

(b) $|\partial^\beta F^S_\xi(x') - \partial^\beta F^S_\xi(x'')| \le A \cdot \omega(|x' - x''|)$ for $|\beta| = m$, $x', x'' \in \mathbb{R}^n$, $|x' - x''| \le 1$;

(c) $J_x(F^S_\xi) \in \tilde{f}_\xi(x) + CA \cdot \sigma(x)$ for all $x \in S$.

Note that, in the old (38) in Section 8 of [14], part (c) reads
$$J_x(F^S) \in \tilde{f}(x) + 2A \cdot \sigma(x) \quad \text{for all } x \in S.$$
Our new (38) has CA in place of $2A$.
Hence, in Claim (39) and its proof (in Section 8 of [14]), we must replace $2A$ by CA.

From this point on, the arguments in Section 8 of [14] go through with only minor changes of the sort discussed for Sections 6 and 7.

In Section 9 of [14], the first few paragraphs should be changed to read as follows.

In this section, we give the set-up for the proof of Lemma 5.2 in the monotonic case. We fix $m, n \ge 1$ and $\mathcal{A} \subseteq \mathcal{M}$.

We let $k^{\#}$ be a large enough integer, determined by m and n, to be picked later. We suppose we are given the following data:

- A vector space Ξ with a seminorm $|\cdot|$.
- Constants $C_0, a_1, a_2 > 0$.
- A regular modulus of continuity ω.
- A finite set $E \subset \mathbb{R}^n$.
- For each $x \in E$, a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, and a set $\sigma(x) \subset \mathcal{R}_x$.
- A point $y^0 \in \mathbb{R}^n$.
- A family of polynomials $P_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$.

We fix Ξ, $C_0, a_1, a_2, \omega, E, \xi \mapsto f_\xi(\cdot)$, $\sigma(\cdot)$, y^0, $(P_\alpha)_{\alpha \in \mathcal{A}}$ until the end of Section 16.

We make the following assumptions.

Also, in Section 9 of [14], we replace (SU8) by the following.

(SU8) Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $S \subseteq E$ with $\#(S) \le k^{\#}$, there exists $F^S_\xi \in C^{m,\omega}(\mathbb{R}^n)$, with

(a) $\| F^S_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C_0$; and

(b) $J_x(F^S_\xi) \in f_\xi(x) + C_0\sigma(x)$ for all $x \in S$.

Moreover, we replace Lemma 9.1 in [14] by the following.

Lemma 9.1. *Assume* (SU0),...,(SU8).

Then there exists a linear mapping $\xi \mapsto F_\xi$, *from* Ξ *into* $C^{m,\omega}(\mathbb{R}^n)$, *such that, for any* $\xi \in \Xi$ *with* $|\xi| \le 1$, *we have*

(a) $\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le A$, *and*

(b) $J_x(F_\xi) \in f_\xi(x) + A\sigma(x)$ *for all* $x \in E \cap B(y^0, a)$;

here, A *and* a *are determined by* a_1, a_2, m, n, C_0.

Lemma 9.2 and its proof require no changes.

In Section 10 of [14], we make the following changes:
In place of (1) and (2) in that section, we make the following definitions.
For $M > 0, S \subset E, y \in \mathbb{R}^n$, $\xi \in \Xi$, we define

$$(1) \qquad \mathcal{K}_\xi(y, S, M) = \{J_y(F) : F \in C^{m,\omega}(\mathbb{R}^n), \| F \|_{C^{m,\omega}(\mathbb{R}^n)} \le M \text{ and } J_x(F) \in f_\xi(x) + M\sigma(x) \text{ for all } x \in S\}.$$

For $M > 0$, $k \ge 1$, $y \in \mathbb{R}^n$, $\xi \in \Xi$, we define

$$(2) \qquad \mathcal{K}_\xi(y, k, M) = \cap\{\mathcal{K}_\xi(y, S, M) : S \subset E, \#(S) \le k\}.$$

Wherever we referred to $\mathcal{K}_f$ in Section 10 of [14], we refer now to $\mathcal{K}_\xi$, with $\xi \in \Xi$ assumed to satisfy $|\xi| \le 1$.
Also, in place of $\mathcal{K}^{\#}_f(y, k, M)$ from Section 10 of [14], we define

$$\mathcal{K}^{\#}_\xi(y, k, M) = \{P \in \mathcal{K}_\xi(y, k, M) : \partial^\beta P(y) = 0 \quad \text{for all } \beta \in \mathcal{A}\}.$$

With these changes, Lemmas 10.1 through 10.5 and their proofs go through just as in [14]. We shall also require another lemma, not essentially contained in [14]. That result is as follows.

Lemma 10.6. *Suppose* $k^{\#} \ge (D+1)^{10} \cdot k^{\#}_1$ *and* $k^{\#}_1 \ge 1$. *Then, for a large enough controlled constant* C_*, *the following holds.*

Given $y \in B(y^0, a_1)$, *there exists a linear map* $\xi \mapsto P^y_\xi$, *from* Ξ *into* $\mathcal{P}$, *such that, for any* $\xi \in \Xi$ *with* $|\xi| \le 1$, *we have*

$$P^y_\xi \in \mathcal{K}^{\#}_\xi(y, k^{\#}_1, C_*).$$

Sketch of Proof of Lemma 10.6. Suppose $k^{\#} \geq (D+1)^{10} \cdot k_1^{\#}$ and $k_1^{\#} \geq 1$.

Let $y \in B(y^0, a)$ be given. In view of hypothesis (SU8) and Lemma 3.2 (from this paper, not [14]), there exists a linear map $\xi \mapsto P_\xi$ from Ξ into $\mathcal{P}$, with the following property.

Given $\xi \in \Xi$ with $|\xi| \leq 1$, we have $P_\xi \in \mathcal{K}_\xi(y, k_1^{\#}, C)$.

We can now repeat the proof of Lemma 10.5 in [14], using the above P_ξ in place of the polynomial $P \in \mathcal{K}_f(y, k_1^{\#}, C)$ from that proof. In particular, the polynomial called $\tilde{P}$ in the proof of Lemma 10.5 in [14] now depends linearly on $\xi \in \Xi$.

From the proof of Lemma 10.5, we obtain $\tilde{P} \in \mathcal{K}_\xi^{\#}(y, k_1^{\#}, C_*)$ for $|\xi| \leq 1$. This concludes our sketch of the proof of Lemma 10.6. ■

Sections 11, 12 and 13 in [14] require no changes here.

Section 14 in [14] requires only the following trivial changes:
We replace f, $\mathcal{K}_f$, $\mathcal{K}_f^{\#}$ by $f_\xi, \mathcal{K}_\xi, \mathcal{K}_\xi^{\#}$ respectively.
We add to each of the lemmas in Section 14 the additional hypotheses $\xi \in \Xi$, $|\xi| \leq 1$.

Once these trivial changes are made, the proofs of Lemmas 14.1,...,14.5 go through unchanged.

Section 15 in [14] requires no changes here.

In Section 16 of [14], we make the following changes.
We replace (2) in that section by

$$(2) \qquad k^{\#} = (D+1)^{30} \cdot k_{\text{old}}^{\#}.$$

We replace the remarks immediately after (2) by the following.

For each ν, Lemma 10.6 gives us a linear map $\xi \mapsto P_{\nu,\xi}$, from Ξ into $\mathcal{P}$, such that

$$(3) \qquad P_{\nu,\xi} \in \mathcal{K}_\xi^{\#}(y_\nu, (D+1)^{20} \cdot k_{\text{old}}^{\#}, C) \text{ for } |\xi| \leq 1.$$

Applying Lemmas 14.3 and 14.5, we see that, whenever $\xi \in \Xi$ with $|\xi| \leq 1$, we have

$$(4) \qquad |\partial^\beta(P_{\mu,\xi} - P_{\nu,\xi})(y_\mu)| \leq C' \cdot (a_1)^{-(m+1)} \cdot a_2^{-1}\, \omega(\delta_\nu) \cdot \delta_\nu^{m-|\beta|} \quad \text{for } |\beta| \leq m, \text{ if } Q_\mu, Q_\nu \text{ abut;}$$

and

$$(5) \qquad |\partial^\beta(P_{\mu,\xi} - P_{\nu,\xi})(y_\mu)| \leq C' \cdot (a_1)^{-(m+1)} \cdot a_2^{-1} \cdot \omega(|y_\mu - y_\nu|) \cdot |y_\mu - y_\nu|^{m-|\beta|} \quad \text{for } |\beta| \leq m,\ \mu \neq \nu.$$

In Lemma 16.1 in [14], we replace "Fix ν" by "Fix ν, ξ, with $\xi \in \Xi$ and $|\xi| \leq 1$"; and replace $\widehat{F}_\nu^S$, $f(x)$, P_ν by $F_{\nu,\xi}^S$, $f_\xi(x)$, $P_{\nu,\xi}$, respectively. The proof of Lemma 16.1 goes through without further changes.

In the statement of Lemma 16.2 in [14], after the phrase "for the following data:", we insert the bullet

- The vector space Ξ with seminorm $|\cdot|$.

Also in the statement of that lemma, we replace

- The map $x \mapsto f(x) - P_\nu \in \mathcal{R}_x$ for $x \in E \cap Q^*_\nu$

by

- The map $\xi \mapsto f_\xi(x) - P_{\nu,\xi} \in \mathcal{R}_x$ $(\xi \in \Xi)$ for $x \in E \cap Q^*_\nu$.

In the proof of that lemma, we replace (31) by the following.

(31) Given $\xi \in \Xi$ with $|\xi| \leq 1$, and given $S \subset E \cap Q^*_\nu$ with $\#(S) \leq k^{\#}_{\mathsf{old}}$, there exists $F^S_\xi \in C^{m,\omega}(\mathbb{R}^n)$, with

(a) $\| \partial^\beta F^S_\xi \|_{C^0(\mathbb{R}^n)} \leq (a_1)^{-(m+2)} \cdot \omega(\delta_\nu) \cdot \delta_\nu^{m-|\beta|}$ for $|\beta| \leq m$;

(b) $|\partial^\beta F^S_\xi(x') - \partial^\beta F^S_\xi(x'')| \leq (a_1)^{-(m+2)} \cdot \omega(|x'-x''|)$
for $|\beta| = m$, $x', x'' \in \mathbb{R}^n$, $|x'-x''| \leq \delta_\nu$; and

(c) $J_x(F^S_\xi) \in (f_\xi(x) - P_{\nu,\xi}) + (a_1)^{-(m+2)}\sigma(x)$ for all $x \in S$.

The statement and proof of Lemma 16.3 in [14] should be replaced by the following.

Lemma 16.3. *For each* ν $(1 \leq \nu \leq \mu_{\max})$, *there exists a linear map* $\xi \mapsto F_{\nu,\xi}$ *from* Ξ *into* $C^{m,\omega}(\mathbb{R}^n)$, *such that for any* $\xi \in \Xi$ *with* $|\xi| \leq 1$, *we have*

$$\| \partial^\beta F_{\nu,\xi} \|_{C^0(\mathbb{R}^n)} \leq A'\omega(\delta_\nu) \cdot \delta_\nu^{m-|\beta|} \text{ for } |\beta| \leq m; \tag{32}$$

$$|\partial^\beta F_{\nu,\xi}(x') - \partial^\beta F_{\nu,\xi}(x'')| \leq A'\omega(|x'-x''|) \tag{33}$$
$$\text{for } |\beta| = m,\ x', x'' \in \mathbb{R}^n,\ |x'-x''| \leq \delta_\nu; \text{ and}$$

$$J_x(F_{\nu,\xi}) \in (f_\xi(x) - P_{\nu,\xi}) + A'\sigma(x) \text{ for all } x \in E \cap Q^*_\nu. \tag{34}$$

Here A' *depends only on* a_1, m, n *and the constant* C_0 *in* (SU0,...,8).

Proof. Fix ν. Either Q_ν is OK, or $E \cap Q^*_\nu$ contains at most one point.

If Q_ν is OK, then the conclusion of Lemma 16.3 is immediate from Lemmas 16.2 and 8.1.

If instead there is exactly one point in $E \cap Q^*_\nu$, then the conclusion of Lemma 16.3 is immediate from Lemma 16.1 with $S = E \cap Q^*_\nu$, together with Lemma 3.1 (from this paper, not [14]), where we take:

- x_0 to be the single element of $E \cap Q^*_\nu$;
- $\sigma = \sigma(x_0)$;

- $f_{0,\xi} = f_\xi(x_0)$;
- $\delta = \delta_\nu$;
- $A =$ the controlled constant C' from the conclusions of Lemma 16.1.

Finally, if $E \cap Q^*_\nu$ is empty, then we may simply set $F_{\nu,\xi} \equiv 0$, and (32), (33), (34) hold trivially.

The proof of Lemma 16.3 is complete ■

Immediately after the proof of Lemma 16.3 in [14], we replace "For each ν, we fix F_ν as in Lemma 16.3" by "For each ν, we fix a linear map $\xi \mapsto F_{\nu,\xi}$ as in Lemma 16.3".

Also, we replace (35) there by

$$|\partial^\beta F_{\nu,\xi}(x') - \partial^\beta F_{\nu,\xi}(x'')| \leq A'\omega(|x'-x''|) \quad \text{for } |\xi| \leq 1,\ |\beta| = m,\ x', x'' \in Q^*_\nu. \tag{35}$$

The proof of (35) in [14], with a trivial change in notation, establishes our present (35).

We replace (43) in Section 16 of [14] by

$$\left|\partial^\beta P_{\nu,\xi}(y_\nu)\right| \leq C \text{ for } |\xi| \leq 1,\ |\beta| \leq m,\ \text{all } \nu. \tag{43}$$

Immediately following (43), when we verify conditions (PLS1,...,8), we replace P_ν by $P_{\nu,\xi}$, where $\xi \in \Xi$ is assumed to satisfy $|\xi| \leq 1$.

In place of (44) in Section 16 of [14], we write the following:
For $\xi \in \Xi$, we define

$$\tilde{F}_\xi = \sum_{1 \leq \nu \leq \mu_{\max}} \theta_\nu \cdot [P_{\nu,\xi} + F_{\nu,\xi}] \text{ on } Q^0. \tag{44}$$

Note that $\xi \mapsto \tilde{F}_\xi$ is a linear map from Ξ to C^m functions on Q^0.

Fix $\xi \in \Xi$ with $|\xi| \leq 1$. We will write $\tilde{F}$ for $\tilde{F}_\xi$, and P_ν for $P_{\nu,\xi}$.

In Section 16 of [14], we replace the discussion after (61) by the following:

In view of (45), (46), (61), we have proven the following.

(62) $\xi \mapsto \tilde{F}_\xi$ is a linear map from Ξ to C^m functions on Q^0, such that, for any $\xi \in \Xi$ with $|\xi| \leq 1$, we have

(a) $|\partial^\beta \tilde{F}_\xi(x)| \leq A'$ for $|\beta| \leq m$, $x \in Q^0$;

(b) $|\partial^\beta \tilde{F}_\xi(x') - \partial^\beta \tilde{F}_\xi(x'')| \leq A'\omega(|x'-x''|)$ for $|\beta| = m$, $x', x'' \in Q^0$; and

(c) $J_x(\tilde{F}_\xi) \in f_\xi(x) + A'\sigma(x)$ for all $x \in E \cap Q^0$.

Unfortunately, $\tilde{F}_\xi(x)$ is defined only for $x \in Q^0$. To remedy this, we multiply $\tilde{F}_\xi$ by a cutoff function. We recall (see (11.1), (11.3)) that Q^0 is centered at y^0 and has diameter $ca_1 \leq \delta_{Q^0} \leq a_1$.

We introduce a cutoff function θ on $\mathbb{R}^n$, with

$$(63)\qquad \|\theta\|_{C^{m+1}(\mathbb{R}^n)} \le A'\,,\ \theta = 1 \text{ on } B(y^0, c'a_1)\,,\ \operatorname{supp}\theta \subset Q^0\,.$$

We then define a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, by setting $F_\xi = \theta \cdot \tilde{F}_\xi$ on $\mathbb{R}^n$.

From (62) and (63), we conclude that $\xi \mapsto F_\xi$ is a linear map from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, and that for $|\xi| \le 1$, we have

$$(64)\qquad \|F_\xi\|_{C^{m,\omega}(\mathbb{R}^n)} \le A'\,, \text{ and}$$

$$(65)\qquad J_x(F_\xi) \in f_\xi(x) + A'\sigma(x) \text{ for all } x \in E \cap B(y^0, c'a_1)\,.$$

Since the constants A' and $c'a_1$ in (64), (65) are determined by m, n, C_0, a_1, a_2 in (SU0,...,8), our results (64), (65) immediately imply the conclusions of Lemma 9.1 for the linear map $\xi \mapsto F_\xi$.

The proof of Lemma 9.1 is complete.

In view of Lemma 9.2, the proof of Lemma 5.2 is also complete. ■

This completes our discussion of Section 16 in [14].

Section 17 in [14] requires no change here.

In section 18 in [14], we make the following changes.
At the start of the section, the paragraph beginning "Also, suppose..." should be changed to the following.
Also, suppose we are given a vector space Ξ with a seminorm $|\cdot|$, and suppose that, for each $x \in E$, we are given a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, and a subset $\sigma(x) \subseteq \mathcal{R}_x$. Assume that these data satisfy conditions (SL0,...,5). We must show that there exists a linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, satisfying (SL6,7) with a constant C' determined by C, m, n.
We replace (7) in Section 18 of [14] by

$$(7)\qquad \bar{f}_\xi(\bar{x}) = (f_\xi(\tau(\bar{x}))) \circ \tau \in \mathcal{R}_{\bar{x}} \text{ for } \bar{x} \in \bar{E}\,,$$

and we note that $\xi \mapsto \bar{f}_\xi(\bar{x})$ is a linear map from Ξ into $\mathcal{R}_{\bar{x}}$.

The discussion of (24) in Section 18 of [14], starting with "Similarly, let $\bar{S} \subset \bar{E}$ be given", should be replaced by the following.

Similarly, let $\xi \in \Xi$ with $|\xi| \le 1$, and let $\bar{S} \subseteq \bar{E}$ with $\#(\bar{S}) \le k^\#$. Again, we set $S = \tau(\bar{S})$, and we apply (SL4). Let F^S_ξ be as in (SL4), and define

$$(24)\qquad \bar{F}^{\bar{S}}_\xi = F^S_\xi \circ \tau\,.$$

Thus, $\bar{F}^{\bar{S}}_\xi \in C^{m,\omega}(\mathbb{R}^n)$, since $F^S_\xi \in C^{m,\omega}(\mathbb{R}^n)$.
Fix $\xi \in \Xi$ with $|\xi| \le 1$, and set $\bar{F}^{\bar{S}} = \bar{F}^{\bar{S}}_\xi$.

We replace (29) in Section 18 of [14] by the following.

(29) Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $\bar{S} \subseteq \bar{E}$ with $\#(\bar{S}) \le k^{\#}$, there exists $\bar{F}_\xi^{\bar{S}} \in C^{m,\omega}(\mathbb{R}^n)$, with $\| \bar{F}_\xi^{\bar{S}} \|_{C^{m,\omega}(\mathbb{R}^n)} \le C$, and $J_{\bar{x}}(\bar{F}_\xi^{\bar{S}}) \in \bar{f}_\xi(\bar{x}) + CA^{-1}\bar{\sigma}(\bar{x})$ for all $\bar{x} \in \bar{S}$.

Similarly, we replace (51) in that section by the following.

(51) Given $\xi \in \Xi$ with $|\xi| \le 1$, and given $\bar{S} \subseteq \bar{E}$ with $\#(\bar{S}) \le k^{\#}$, there exists $\bar{F}_\xi^{\bar{S}} \in C^{m,\omega}(\mathbb{R}^n)$, with

(a) $\| \bar{F}_\xi^{\bar{S}} \|_{C^{m,\omega}(\mathbb{R}^n)} \le C_1$; and

(b) $J_{\bar{x}}(\bar{F}_\xi^{\bar{S}}) \in \bar{f}_\xi(\bar{x}) + CA^{-1}\bar{\sigma}(\bar{x})$ for all $\bar{x} \in \bar{S}$.

A couple of paragraphs later, when we specify the data that are to satisfy the hypotheses of the Weak Main Lemma for $\bar{\mathcal{A}}$, we add the bullet

- The vector space Ξ with seminorm $|\cdot|$,

and we change the bullet

- The m-jet $\bar{f}(\bar{x})$ associated to each $\bar{x} \in \bar{E}$

to

- The linear map $\xi \mapsto \bar{f}_\xi(\bar{x})$ from Ξ into $\mathcal{R}_{\bar{x}}$, associated to each $\bar{x} \in \bar{E}$.

Next, the discussion of (56), (57), (58) should be changed to the following.

There exists a linear map $\xi \mapsto \bar{F}_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, for any $\xi \in \Xi$ with $|\xi| \le 1$, we have

$$\| \bar{F}_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C' \text{ and} \tag{56}$$

$$J_{\bar{x}}(\bar{F}_\xi) \in \bar{f}_\xi(\bar{x}) + C'\bar{\sigma}(\bar{x}) \text{ for all } \bar{x} \in \bar{E} \cap B(\bar{y}^0, c')\,. \tag{57}$$

We fix $\xi \mapsto \bar{F}_\xi$ as above, and define

$$F_\xi = \bar{F}_\xi \circ \tau^{-1} \text{ for } \xi \in \Xi\,. \tag{58}$$

Thus, $\xi \mapsto F_\xi$ is a linear map from Ξ into $C^{m,\omega}(\mathbb{R}^n)$.

Fix $\xi \in \Xi$, with $|\xi| \le 1$, and write $F, \bar{F}, f$ for $F_\xi, \bar{F}_\xi, f_\xi$ respectively.

Thus, $F \in C^{m,\omega}(\mathbb{R}^n)$. We estimate its norm.

Finally, the sentence containing (71) in Section 18 of [14] should be changed to the following.

Therefore, (63) and (70) show that the linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$ satisfies the following property.

(71) For $\xi \in \Xi$ with $|\xi| \le 1$, we have

$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C'\,, \text{ and } J_x(F_\xi) \in f_\xi(x) + C'\sigma(x) \text{ for all } x \in E \cap B(y^0, c')\,,$$

with C' and c' determined by C, m, n in (SL0,...,5).

With the changes indicated above, the arguments in [14] prove Lemmas 5.1, 5.2, 5.3, and therefore establish the Weak Main Lemma and the Strong Main Lemma (in the form given here) for every $\mathcal{A}$. Consequently, we have proven the Local Theorem stated earlier in this section.

5. Passage to the Banach Limit

The Local Theorem proven in the preceding section gives a local version of Theorem 7 for the case of finite sets $E \subset \mathbb{R}^n$. In this section, we remove the restriction to finite E, by passing to a Banach limit as in [10]. We then pass from a local to a global result by a partition of unity, completing the proof of Theorem 7.

We start by recalling the standard notion of a Banach limit, in the particular form used in [10].

Let $E \subset \mathbb{R}^n$ be given, and let $\mathcal{D}$ denote the collection of all finite subsets of E. We introduce the Banach space $C^0(\mathcal{D})$, which consists of all bounded families of real numbers $\vec{\zeta} = (\zeta_{E_1})_{E_1 \in \mathcal{D}}$ indexed by elements of $\mathcal{D}$.

The norm in $C^0(\mathcal{D})$ is given by $\| \vec{\zeta} \|_{C^0(\mathcal{D})} = \sup_{E_1 \in \mathcal{D}} |\zeta_{E_1}|$.

A standard application of the Hahn-Banach theorem yields a linear functional

$$\ell_{\mathcal{D}} : C^0(\mathcal{D}) \to \mathbb{R},$$

with the following properties.

(1) $|\ell_{\mathcal{D}}(\vec{\zeta})| \leq \| \vec{\zeta} \|_{C^0(\mathcal{D})}$ for all $\vec{\zeta} \in C^0(\mathcal{D})$.

(2) Suppose $E_0 \in \mathcal{D}$, $\lambda \in \mathbb{R}$, and $\vec{\zeta} = (\zeta_{E_1})_{E_1 \in \mathcal{D}} \in C^0(\mathcal{D})$, with $\zeta_{E_1} \geq \lambda$ whenever $E_1 \supseteq E_0$. Then $\ell_{\mathcal{D}}(\vec{\zeta}) \geq \lambda$.

We fix $\ell_{\mathcal{D}}$ as above, and call it the "Banach limit."

Next, we start removing the finiteness assumption from the Local Theorem of the previous section. We fix $m, n \geq 1$, and take $k^\#$ as in the Local Theorem. Let Ξ, $|\cdot|$, $\omega, E, A, \xi \mapsto f_\xi(x)$ $(x \in E)$ and $\sigma(x)(x \in E)$ be as in the hypotheses of Theorem 7, for the $k^\#$ just given. We do not assume that E is finite.

We will call a constant "controlled" if it depends only on A, m, n in the hypotheses of Theorem 7; and we write c, C, C', etc. to denote controlled constants.

Let $y^0 \in \mathbb{R}^n$ be given. Then, for each $E_1 \in \mathcal{D}$, the hypotheses of the Local Theorem hold, with E_1 in place of E, and with a controlled constant C independent of E_1. Hence, applying the Local Theorem, we obtain for each $E_1 \in \mathcal{D}$ a linear map $\xi \mapsto F_\xi^{E_1}$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, with the following properties.

(3) For $|\xi| \leq 1$ and $E_1 \in \mathcal{D}$, we have $\| F_\xi^{E_1} \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C_1$.

(4) For $|\xi| \leq 1$, $E_1 \in \mathcal{D}$, $x \in E_1 \cap B(y^0, c_1)$, we have $J_x(F_\xi^{E_1}) \in f_\xi(x) + C_1\sigma(x)$.

We fix constants c_1 and C_1 as in (3) and (4). For $|\beta| \leq m$, $\xi \in \Xi$, $x \in \mathbb{R}^n$, $E_1 \in \mathcal{D}$, define

(5) $\zeta_{E_1}^{\beta,\xi}(x) = \partial^\beta F_\xi^{E_1}(x)$;

and for $|\beta| \leq m$, $\xi \in \Xi$, $x \in \mathbb{R}^n$, define

(6) $\vec{\zeta}^{\beta,\xi}(x) = (\zeta_{E_1}^{\beta,\xi}(x))_{E_1 \in \mathcal{D}}$.

In view of (3), we have

(7) $\vec{\zeta}^{\beta,\xi}(x) \in C^0(\mathcal{D})$ for $|\beta| \leq m$, $\xi \in \Xi$, $x \in \mathbb{R}^n$; and

(8) $\| \vec{\zeta}^{\beta,\xi}(x) \|_{C^0(\mathcal{D})} \leq C_1$ for $|\beta| \leq m$, $|\xi| \leq 1$, $x \in \mathbb{R}^n$.

Note also that

(9) $\xi \mapsto \vec{\zeta}^{\beta,\xi}(x)$ is a linear map from Ξ into $C^0(\mathcal{D})$, for each fixed β, x ($|\beta| \leq m$, $x \in \mathbb{R}^n$),

as we see at once from (5), (6), (7).

For $|\beta| \leq m$, $x \in \mathbb{R}^n$, $\xi \in \Xi$, we now define

(10) $F_{\beta,\xi}(x) = \ell_{\mathcal{D}}(\vec{\zeta}^{\beta,\xi}(x)) \in \mathbb{R}$,

where $\ell_{\mathcal{D}}$ is the Banach limit. This makes sense, thanks to (7).

From (8), (9), (10), we see that

(11) $\xi \mapsto F_{\beta,\xi}(x)$ is a linear map from Ξ to $\mathbb{R}$, for each fixed $x \in \mathbb{R}^n$, $|\beta| \leq m$; and

(12) $|F_{\beta,\xi}(x)| \leq C_1$ for $|\xi| \leq 1$, $|\beta| \leq m$, $x \in \mathbb{R}^n$.

We define

(13) $F_\xi(x) = F_{0,\xi}(x)$ for $\xi \in \Xi$, $x \in \mathbb{R}^n$,

where 0 denotes the zero multi-index. We will show that

(14) $F_\xi \in C^{m,\omega}(\mathbb{R}^n)$

and that

(15) $\partial^\beta F_\xi = F_{\beta,\xi}$ for $\xi \in \Xi$, $|\beta| \leq m$.

Moreover, we will show that

(16) $\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C$ for $|\xi| \leq 1$.

To prove (14), (15), (16), we fix a multi-index β, with $|\beta| \le m-1$. For $1 \le j \le n$, we write $\beta[j]$ for the sum of β and the j^{th} unit multi-index. From (3) and Taylor's theorem, we have

$$\left|\partial^\beta F_\xi^{E_1}(x+y) - \left[\partial^\beta F_\xi^{E_1}(x) + \sum_{j=1}^n y_j \partial^{\beta[j]} F_\xi^{E_1}(x)\right]\right| \le C\omega(|y|) \cdot |y|$$

for $E_1 \in \mathcal{D}$, $|\xi| \le 1$, $x, y \in \mathbb{R}^n$, $|y| \le 1$, $y = (y_1, \dots, y_n)$.
Comparing this with (4) and (5), we find that

$$\text{(17)} \quad \left\|\vec{\zeta}^{\beta,\xi}(x+y) - \left[\vec{\zeta}^{\beta,\xi}(x) + \sum_{j=1}^n y_j\, \vec{\zeta}^{\beta[j],\xi}(x)\right]\right\|_{C^0(\mathcal{D})} \le C\omega(|y|) \cdot |y|$$

for $|\xi| \le 1$, $x, y \in \mathbb{R}^n$, $|y| \le 1$, $y = (y_1, \dots, y_n)$.

Applying $\ell_{\mathcal{D}}$, and recalling (1) and (10), we conclude that

$$\text{(18)} \quad \left|F_{\beta,\xi}(x+y) - \left[F_{\beta,\xi}(x) + \sum_{j=1}^n y_j F_{\beta[j],\xi}(x)\right]\right| \le C\omega(|y|) \cdot |y|$$

for ξ, x, y as in (17). Since $\omega(t) \to 0$ as $t \to 0$, (18) implies

$$\text{(19)} \quad \left[\begin{array}{l} F_{\beta,\xi} \text{ is differentiable for } |\beta| \le m-1,\ |\xi| \le 1;\ \text{and moreover} \\ \frac{\partial}{\partial x_j} F_{\beta,\xi}(x) = F_{\beta[j],\xi}(x) \text{ for such } \beta, \xi, \text{ and for } j = 1, \dots, n. \end{array}\right]$$

Since $\xi \mapsto F_{\beta,\xi}$ is linear for $|\beta| \le m$, we may drop the assumption $|\xi| \le 1$ from (19).

Next, we return to (3), and conclude that, for $|\beta| \le m$ and $|\xi| \le 1$, we have

$$|\partial^\beta F_\xi^{E_1}(x) - \partial^\beta F_\xi^{E_1}(y)| \le C_1\, \omega(|x-y|) \text{ for } |x-y| \le 1,\ E_1 \in \mathcal{D}.$$

In view of (5), (6), this means that

$$\|\, \vec{\zeta}^{\beta,\xi}(x) - \vec{\zeta}^{\beta,\xi}(y) \,\|_{C^0(\mathcal{D})} \le C_1\, \omega(|x-y|) \text{ for } |x-y| \le 1.$$

Applying $\ell_{\mathcal{D}}$, and recalling (1) and (10), we find that

$$\text{(20)} \quad |F_{\beta,\xi}(x) - F_{\beta,\xi}(y)| \le C_1\, \omega(|x-y|) \text{ for } |x-y| \le 1,\ |\xi| \le 1,\ |\beta| \le m.$$

This shows in particular that $F_{\beta,\xi}$ is a continuous function on $\mathbb{R}^n$ for $|\beta| \le m$, $|\xi| \le 1$. Again, we may drop the assumption $|\xi| \le 1$, since $\xi \mapsto F_{\beta,\xi}$ is linear. Thus, for any $\xi \in \Xi$, $|\beta| \le m$, we see from (20) that

(21) $F_{\beta,\xi}$ is a continuous function with modulus of continuity $O(\omega(t))$.

From (19) and (21), we see that (14) and (15) hold.
Moreover, (16) follows from (12), (15), (20).
This completes the proof of (14), (15), (16).

Next, we prove that

(22) $\quad J_x(F_\xi) \in f_\xi(x) + C_1\sigma(x)$ for $|\xi| \le 1$, $x \in E \cap B(y^0, c_1)$,

with C_1, c_1 as in (4). To see this, fix $x_0 \in E \cap B(y^0, c_1)$ and $\xi \in \Xi$ with $|\xi| \le 1$.

Then $f_\xi(x_0) + C_1\sigma(x_0)$ is a closed, convex subset of $\mathcal{R}_{x_0}$. Hence, it is an intersection of closed half-spaces. A closed half-space in $\mathcal{R}_{x_0}$ has the form

$$\Big\{ J_{x_0}(F) : \sum_{|\beta| \le m} a_\beta \partial^\beta F(x_0) \ge \lambda \Big\}$$

for coefficients $a_\beta \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Consequently, there exists a collection Ω, consisting of pairs $((a_\beta)_{|\beta|\le m}, \lambda)$, with each $a_\beta \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, with the following property:

(23) Let $F \in C^m(\mathbb{R}^n)$. Then $J_{x_0}(F)$ belongs to $f_\xi(x_0) + C_1\sigma(x_0)$ if and only if we have

$$\sum_{|\beta| \le m} a_\beta \, \partial^\beta F(x_0) \ \ge \lambda \text{ for all } ((a_\beta)_{|\beta|\le m}, \lambda) \in \Omega.$$

Now suppose we are given $E_1 \in \mathcal{D}$, with E_1 containing x_0. Then (4) gives

$$J_{x_0}\,(F_\xi^{E_1}) \ \in \ f_\xi(x_0) \ + \ C_1\,\sigma(x_0),$$

hence

$$\sum_{|\beta| \le m} a_\beta \, \partial^\beta F_\xi^{E_1}(x_0) \ \ge \ \lambda \quad \text{for } ((a_\beta)_{|\beta|\le m}, \lambda) \ \in \ \Omega, \text{ by (23).}$$

In view of (5), this means that

(24) $\quad \displaystyle\sum_{|\beta| \le m} a_\beta \zeta_{E_1}^{\beta,\xi}(x_0) \ \ge \ \lambda$ for $((a_\beta)_{|\beta|\le m}, \lambda) \in \Omega$, $E_1 \in \mathcal{D}$ containing x_0.

Fix $((a_\beta)_{|\beta|\le m}, \lambda) \in \Omega$, and set

$$\vec{\zeta} \ = \sum_{|\beta| \le m} a_\beta \, \vec{\zeta}^{\beta,\xi}(x_0) \ \in \ C^0(\mathcal{D})$$

(see (7)). From (24) and (5), (6), we see that

$$\vec{\zeta} \ = \ (\zeta_{E_1})_{E_1 \in \mathcal{D}}, \quad \text{with } \zeta_{E_1} \ge \lambda \text{ whenever } E_1 \text{ contains } x_0.$$

Taking $E_0 = \{x_0\} \in \mathcal{D}$, and applying (2), we learn that $\ell_{\mathcal{D}}(\vec{\zeta}) \ge \lambda$. That is,

$$\sum_{|\beta| \le m} a_\beta \, \ell_{\mathcal{D}}(\vec{\zeta}^{\beta,\xi}(x_0)) \ \ge \ \lambda.$$

Recalling (10) and (15), we obtain

$$\sum_{|\beta|\le m} a_\beta\, \partial^\beta F_\xi(x_0) \ge \lambda.$$

This holds for any $((a_\beta)_{|\beta|\le m}, \lambda) \in \Omega$. Therefore, (23) gives $J_{x_0}(F_\xi) \in f_\xi(x_0) + C_1\sigma(x_0)$, completing the proof of (22).

In view of (11), (13), (14), (16), (22), we have proven the following result.

Local Theorem 7. *Assume the hypotheses of Theorem 7. Then, given $y^0 \in \mathbb{R}^n$, there exists a linear map $\xi \mapsto F_\xi^{y^0}$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$ such that, whenever $\xi \in \Xi$ with $|\xi| \le 1$, we have*

$$\| F_\xi^{y^0} \|_{C^{m,\omega}(\mathbb{R}^n)} \le C$$

and

$$J_x(F_\xi^{y^0}) \in f_\xi(x) + C\sigma(x) \quad \textit{for all } x \in E \cap B(y^0, c_1).$$

Here, C and c_1 depend only on A, m, n in the hypotheses of Theorem 7.

Finally, we pass from a local to a global result, to complete the proof of Theorem 7. To do so, we assume the hypotheses of Theorem 7, and fix a partition of unity

(25) $$1 = \sum_\nu \theta_\nu \text{ on } \mathbb{R}^n,$$

with

(26) $$\| \theta_\nu \|_{C^{m+1}(\mathbb{R}^n)} \le C,$$

and

(27) $\operatorname{supp}\theta_\nu \subset B(y_\nu, \tfrac{1}{2}c_1)$ (with c_1 as in the Local Theorem 7).

Here, $y_\nu \in \mathbb{R}^n$ are points with the following property.

(28) Any given ball of radius 1 in $\mathbb{R}^n$ meets at most C of the balls $B(y_\nu, c_1)$.

Applying the Local Theorem 7, we obtain for each ν, a linear map $\xi \mapsto F_{\nu,\xi}$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, such that, for any $\xi \in \Xi$ with $|\xi| \le 1$, we have

(29) $$\| F_{\nu,\xi} \|_{C^{m,\omega}(\mathbb{R}^n)} \le C$$

and

(30) $$J_x(F_{\nu,\xi}) \in f_\xi(x) + C\sigma(x) \text{ for all } x \in E \cap B(y_\nu, c_1).$$

We define a linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$, by setting

(31) $$F_\xi = \sum_\nu \theta_\nu \cdot F_{\nu,\xi}.$$

For $|\xi| \le 1$, we have

$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \le C \,, \tag{32}$$

thanks to (26),..., (29).

Moreover, suppose $x \in E$, $\xi \in \Xi$ are given, with $|\xi| \le 1$. By (25), we can find some μ for which $x \in \operatorname{supp} \theta_\mu$. In particular, we have

$$x \in E \cap B\Big(y_\mu, \frac{1}{2} c_1\Big) \,. \tag{33}$$

In view of (31) and (25), we have

$$J_x(F_\xi) = J_x(F_{\mu,\xi}) + \sum_\nu J_x(\theta_\nu) \odot J_x(F_{\nu,\xi} - F_{\mu,\xi}) \,, \tag{34}$$

where $\odot$ denotes multiplication in $\mathcal{R}_x$.

In (34), we may assume that the sum is taken only over those ν for which $x \in B(y_\nu, c_1)$. (In fact, $J_x(\theta_\nu) = 0$ for all other ν, by (27).)

Let ν be given, with $x \in B(y_\nu, c_1)$. Then (29) and (30), applied to μ and ν show that

$$J_x(F_{\mu,\xi}) \in f_\xi(x) + C\sigma(x) \,; \tag{35}$$

$$J_x(F_{\nu,\xi}) \in f_\xi(x) + C\sigma(x); \tag{36}$$

and

$$|\partial^\beta F_{\mu,\xi}(x)| \,, |\partial^\beta F_{\nu,\xi}(x)| \le C \text{ for } |\beta| \le m \,. \tag{37}$$

These remarks imply

$$J_x(F_{\nu,\xi} - F_{\mu,\xi}) \in C\sigma(x) \tag{38}$$

and

$$|\partial^\beta (F_{\nu,\xi} - F_{\mu,\xi})(x)\,| \le C \text{ for } |\beta| \le m \,. \tag{39}$$

From (38), (39), (26), and the Whitney ω-convexity hypothesis of Theorem 7, we conclude that

$$J_x(\theta_\nu) \odot J_x(F_{\nu,\xi} - F_{\mu,\xi}) \in C\sigma(x) \,. \tag{40}$$

This holds whenever $B(y_\nu, c_1)$ contains x. There are at most C such ν, thanks to (28). Consequently, (40) yields

$$\sum_{B(y_\nu, c_1) \ni x} J_x(\theta_\nu) \odot J_x(F_{\nu,\xi} - F_{\mu,\xi}) \in C\sigma(x) \,.$$

Together with (34), (35), this in turn yields

$$J_x(F_\xi) \in f_\xi(x) + C\sigma(x) \,. \tag{41}$$

We have proven (41) for any $x \in E$ and any $\xi \in \Xi$ with $|\xi| \le 1$.

Since C depends only on A, m, n in the hypotheses of Theorem 7, our results (32) and (41) are precisely the conclusions of Theorem 7, for the linear map $\xi \mapsto F_\xi$ from Ξ into $C^{m,\omega}(\mathbb{R}^n)$. The proof of Theorem 7 is complete. ■

6. Further Results

What kind of linear maps are needed in Theorems 1,. . . , 5? To shed light on this, we introduce the notion of an operator of bounded "depth".

We start by recalling the basic vector spaces arising in Theorems 1,. . . , 5, namely

$$C^m(E),\ C^{m,\omega}(E),\ C^m(E, \sigma(\cdot)),\ C^{m,\omega}(E, \sigma(\cdot)),\ C^{m,\omega}(E, \widehat{\sigma}).$$

We denote any of these spaces by $X(E)$. Note that, whenever $S \subset E$, there is a natural restriction map $f \mapsto f|_E$ from $X(E)$ into $X(S)$.

Next, suppose $E \subset \mathbb{R}^n$, and let $B \subset \mathbb{R}^n$ be a ball. We say that B "avoids E" if the distance from B to E exceeds the radius of B.

Now, let $T : X(E) \to C^m(\mathbb{R}^n)$ be a linear map, and let k be a positive integer. Then we say that T has "depth k" if it satisfies the following two conditions.

(1) Given $x \in E$, there exists $S_x \subset E$ of cardinality at most k, such that, when f varies in $X(E)$, the jet $J_x(Tf)$ is uniquely determined by $f|_{S_x}$.

(2) Let $B \subset \mathbb{R}^n$ be any ball that avoids E. Then there exists $S_B \subset E$ of cardinality at most k, such that, when f varies in $X(E)$, the function $Tf|_B$ is uniquely determined by $f|_{S_B}$.

In terms of these definitions, we can state a refinement of Theorems 2,..., 5.

We have also proven an analogous refinement of Theorem 1, which we discuss in a later paper. Our refinement of Theorems 2,..., 5 is as follows.

Theorem 8. *Suppose $E \subset \mathbb{R}^n$ is finite. Then, in Theorems 2,..., 5, we can take the linear map T to have depth k, where k depends only on m and n.*

To prove Theorem 8, we give a refinement of Theorem 7. We need a few more definitions. In the setting of Theorem 7, we fix an arbitrary set $\widehat{\Xi}$ of (not necessarily bounded) linear functionals on Ξ.

Suppose $T : \Xi \to V$ is a linear map from Ξ to a finite dimensional vector space V. Then we call T "k-admissible" if there exist k functionals $\ell_1, \dots, \ell_k \in \widehat{\Xi}$ and a linear map $\tilde{T} : \mathbb{R}^k \to V$ such that

$$T\xi = \tilde{T}(\ell_1(\xi), \dots, \ell_k(\xi)) \quad \text{for all } \xi \in \Xi.$$

Also, suppose $T : \Xi \to C^m(\mathbb{R}^n)$ is a linear map. Then we call T "k-admissible" if, for each $x \in \mathbb{R}^n$, the linear map $\xi \mapsto J_x(T\xi)$, from Ξ into $\mathcal{R}_x$, is k-admissible.

We can now state our refinement of Theorem 7.

Theorem 9. *Given* $m, n \geq 1$*, there exists* $k^{\#}$*, depending only on* m *and* n*, for which the following holds.*

Let Ξ *be a vector space with a seminorm* $|\cdot|$*, let* $\widehat{\Xi}$ *be a set of linear functionals on* Ξ*, and let* $\widehat{k}$ *be a positive integer .*

Let ω *be a regular modulus of continuity, let* $E \subset \mathbb{R}^n$ *be a finite set, and let* $A > 0$*.*

For each $x \in E$*, suppose we are given a* $\widehat{k}$*-admissible linear map* $\xi \mapsto f_\xi(x)$ *from* Ξ *into* $\mathcal{R}_x$*.*

Also, for each $x \in E$*, suppose we are given a Whitney* ω*-convex subset* $\sigma(x) \subset \mathcal{R}_x$*, with Whitney constant* A*.*

Assume that, given $\xi \in \Xi$ *with* $|\xi| \leq 1$*, and given* $S \subset E$ *with cardinality at most* $k^{\#}$*, there exists* $F^S_\xi \in C^{m,\omega}(\mathbb{R}^n)$*, satisfying*

$$\| F^S_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq 1, \ \textit{and}\ J_x(F^S_\xi) \in f_\xi(x) + \sigma(x) \ \textit{for all}\ x \in S\,.$$

Then there exists a linear map $\xi \mapsto F_\xi$ *from* Ξ *into* $C^{m,\omega}(\mathbb{R}^n)$*, with the following properties.*

(I) *For any* $\xi \in \Xi$ *with* $|\xi| \leq 1$*, we have*

$$\| F_\xi \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A', \ \textit{and}\ \ J_x(F_\xi) \in f_\xi(x) + A'\sigma(x) \ \textit{for all}\ x \in E.$$

Here, A' *depends only on* m, n*, and the Whitney constant* A*.*

(II) *The map* $\xi \mapsto F_\xi$ *is* k^**-admissible, where* k^* *depends only on* $\widehat{k}, m, n$*.*

The proof of Theorem 9 is a straightforward adaptation of that of Theorem 7, without the Banach limit.
(We needn't introduce the Banach limit, since we assume E finite. If we needed the Banach limit here, then we would lose k^*-admissibility.)

We use Theorem 9 to prove Theorem 8, just as we use Theorem 7 to prove Theorem 3. We sketch the argument here. The heart of the matter is to prove the refinement of Theorem 3 indicated in Theorem 8. As in the proof of Theorem 3 in Section 1, we take

$$\Xi = C^{m,\omega}(E, \sigma(\cdot)) \ \text{ and } \ |\xi| = 2 \| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \ \text{ for } \ \xi = f \in C^{m,\omega}(E, \sigma(\cdot));$$

and we use the tautological map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, given by $\xi = (f(x))_{x \in E} \mapsto f_\xi(x_0) = f(x_0)$ for $x_0 \in E$.

We define $\widehat{\Xi}$ to consist of all the functionals on Ξ of the form

$$\xi \mapsto \ell(f_\xi(x)) \text{ for } x \in E \text{ and } \ell \text{ a linear functional on } \mathcal{R}_x.$$

As in Section 1, we find that the hypotheses of Theorem 9 hold for our Ξ, $|\cdot|$, $f_\xi(x)$, $\widehat{\Xi}$. The only new point to be checked is that $\xi \mapsto f_\xi(x)$ is $\widehat{k}$-admissible for each $x \in E$. This holds, with $\widehat{k} = \dim \mathcal{P}$, thanks to our choice of $\widehat{\Xi}$. Consequently, Theorem 9 produces a linear map $f \mapsto \tilde{F}_f$ from $C^{m,\omega}(E, \sigma(\cdot)) \to C^{m,\omega}(\mathbb{R}^n)$, with the following properties.

(3) Given $f = (f(x))_{x \in E} \in C^{m,\omega}(E, \sigma(\cdot))$ with $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq 1$, we have

$$\| \tilde{F}_f \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A', \text{ and } J_x(\tilde{F}_f) \in f(x) + A'\sigma(x) \text{ for all } x \in E,$$

with A' depending only on A, m, n.

(4) For each $x \in \mathbb{R}^n$ there exist $x_1, \ldots, x_{k^*} \in E$ such that, as $f = (f(x))_{x \in E}$ varies in $C^{m,\omega}(E, \sigma(\cdot))$, the m-jet $J_x(\tilde{F}_f)$ depends only on $f(x_1), \ldots, f(x_{k^*})$.

Here, k^* depends only on m and n.

Our result (4) is not as strong as the desired conditions (1), (2) that define an operator of depth k^*. However, the proof of the classical Whitney extension theorem, applied to the family of m-jets $(J_x(\tilde{F}_f))_{x \in E}$, produces a function $F_f \in C^{m,\omega}(\mathbb{R}^n)$, depending linearly on f, with the following properties.

(5) $J_x(F_f) = J_x(\tilde{F}_f)$ for all $x \in E$.

(6) $\| F_f \|_{C^{m,\omega}(\mathbb{R}^n)} \leq C \| \tilde{F}_f \|_{C^{m,\omega}(\mathbb{R}^n)}$ with C depending only on m, n.

(7) Let $B \subset \mathbb{R}^n$ be any ball that avoids E. Then $F_f|_B$ is determined by the m-jets of $\tilde{F}_f$ at points $x_1, \ldots, x_{\underline{k}} \in E$, with $\underline{k}$ depending only on m, n.

From (3), (5), (6) we see that the linear map $f \mapsto F_f$ satisfies

(8) Suppose $f = (f(x))_{x \in E} \in C^{m,\omega}(E, \sigma(\cdot))$, with $\| f \|_{C^{m,\omega}(E,\sigma(\cdot))} \leq 1$. Then

$$\| F_f \|_{C^{m,\omega}(\mathbb{R}^n)} \leq A'', \text{ and } J_x(F_f) \in f(x) + A''\sigma(x) \text{ for all } x \in E,$$

where A'' depends only on A, m, n.

From (4), (5), (7), we see that

(9) $f \mapsto F_f$ has depth k^{**},

where k^{**} depends only on m and n.

Our results (8), (9) for the linear map $f \mapsto F_f$ are precisely the conclusions of the refinement of Theorem 3 asserted in Theorem 8.

Thus, we have proven that refinement of Theorem 3. The corresponding refinements of Theorems 2,..., 5 then follow from that of Theorem 3, just as in the Introduction.

The proof of Theorem 8 is complete. ∎

See [9] for a similar discussion in an easier case. It would be interesting to prove an analogue of Theorem 8 for infinite E.

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